

Session 5: Algebra and Number Theory II

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Arithmetic Sequences

Definition

An **arithmetic sequence** is a sequence in which each term after the first is the previous term plus a fixed number d , the **common difference**. For example, 2, 5, 8, 11, 14 has $d = 3$. The n th term is $a_n = a_1 + (n - 1)d$.

Sum and middle term

Moving one place in from each end adds d to one term and subtracts d from the other, so $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \dots$. Hence the sum of the n terms is

$$a_1 + a_2 + \dots + a_n = n \cdot \frac{a_1 + a_n}{2}.$$

When n is odd, the middle term pairs with itself, so the middle term equals $\frac{a_1 + a_n}{2}$, the average of the first and last terms. For example, $1 + 2 + 3 + 4 + 5 = 5 \cdot \frac{1+5}{2} = 15$, and the middle term is 3.

Example 1 (2015 AMC 8, Problem #18)

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An arithmetic sequence is a sequence in which each term after the first is obtained by adding a constant to the previous term. For example, 2, 5, 8, 11, 14 is an arithmetic sequence with five terms, in which the first term is 2 and the constant 3 is added. Each row and each column in this 5×5 array is an arithmetic sequence with five terms. What is the value of X ?

1				25
		X		
17				81

Example 1: Solution

Strategy

Find the middle entries of the top and bottom rows, then use the middle column.

Solution

Each row and each column has five terms, so its middle term is the average of its first and last terms. The top row starts with 1 and ends with 25, so its middle term is $\frac{1+25}{2} = 13$. The bottom row starts with 17 and ends with 81, so its middle term is $\frac{17+81}{2} = 49$. These are the first and last terms of the middle column, and X is the middle term of that column, so

$$X = \frac{13 + 49}{2} = 31.$$

Answer: 31.

Prime Factorization and Divisors

Prime factorization

Every integer greater than 1 is a product of primes in exactly one way, apart from the order of the factors. For example, $360 = 2^3 \cdot 3^2 \cdot 5$.

Number of divisors

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where $p_1, \dots, p_k$ are different primes. Each positive divisor of n is $p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}$ with $0 \leq f_i \leq e_i$ for every i , and different choices of the exponents give different divisors. The exponent f_i has $e_i + 1$ possible values, so by the multiplication principle n has

$$(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$$

positive divisors. For example, $12 = 2^2 \cdot 3$ has $(2 + 1)(1 + 1) = 6$ divisors: 1, 2, 3, 4, 6, 12.

Example 2 (1974 AHSME, Problem #26)

Problem 2 (1974 AHSME, Problem #26)

The number of distinct positive integral divisors of $(30)^4$ excluding 1 and $(30)^4$ is

- (A)** 100 **(B)** 125 **(C)** 123 **(D)** 30 **(E)** none of these

Example 2: Solution

Strategy

Write 30^4 as a product of primes and use the divisor formula.

Solution

Since $30 = 2 \cdot 3 \cdot 5$, the prime factorization is $30^4 = 2^4 \cdot 3^4 \cdot 5^4$. By the divisor formula, 30^4 has

$$(4 + 1)(4 + 1)(4 + 1) = 125$$

positive divisors. Excluding 1 and 30^4 leaves $125 - 2 = 123$.

Answer: (C) 123.

Congruences

Definition

Integers a and b are **congruent modulo** m , written $a \equiv b \pmod{m}$, if they leave the same remainder when divided by m . Equivalently, $a - b$ is a multiple of m . For example, $17 \equiv 2 \pmod{5}$ because $17 = 3 \cdot 5 + 2$.

Sums, products and powers

If $a \equiv a' \pmod{m}$ and $b \equiv b' \pmod{m}$, then $a + b \equiv a' + b' \pmod{m}$ and $ab \equiv a'b' \pmod{m}$. To see this, write $a = a' + km$ and $b = b' + lm$; then $ab = a'b' + (a'l + b'k + klm)m$. So each number in a sum or product can be replaced by its remainder, and $a^n \equiv a'^n \pmod{m}$. For example, $12 \cdot 13 \equiv 2 \cdot 3 = 6 \pmod{10}$, and indeed $12 \cdot 13 = 156$.

Example 3 (1999 AMC 8, Problem #24)

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When 1999^{2000} is divided by 5, the remainder is

- (A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Example 3: Solution

Strategy

Replace 1999 by its remainder modulo 5, then find a power of that remainder that is congruent to 1.

Solution

Since $1999 = 399 \cdot 5 + 4$, the remainder of 1999 modulo 5 is 4, so $1999^{2000} \equiv 4^{2000} \pmod{5}$. Next, $4^2 = 16 \equiv 1 \pmod{5}$. Therefore

$$4^{2000} = (4^2)^{1000} \equiv 1^{1000} = 1 \pmod{5},$$

and the remainder is 1.

Answer: (D) 1.

Defined Operations

Method

A problem may define a new symbol by a rule, such as $a \heartsuit b = 2a + b$. To evaluate an expression with the symbol, substitute the given numbers for the letters in the rule. When one operation is inside another, evaluate the inner one first.

Order

The order of the two numbers can matter: $3 \heartsuit 5 = 2 \cdot 3 + 5 = 11$, but $5 \heartsuit 3 = 2 \cdot 5 + 3 = 13$.

Example 4 (2022 AMC 8, Problem #2)

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Consider these two operations:

$$a \blacklozenge b = a^2 - b^2$$

$$a \star b = (a - b)^2$$

Compute the value:

$$(5 \blacklozenge 3) \star 6$$

- (A) -20 (B) 4 (C) 16 (D) 100 (E) 220

Example 4: Solution

Strategy

Evaluate the operation in parentheses first.

Solution

By the rule for $\blacklozenge$, $5 \blacklozenge 3 = 5^2 - 3^2 = 25 - 9 = 16$. Then by the rule for $\star$,

$$16 \star 6 = (16 - 6)^2 = 10^2 = 100.$$

Answer: (D) 100.

Problem 1 (2018 AMC 8, Problem #5)

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What is the value of

$$1 + 3 + 5 + \cdots + 2017 + 2019 - 2 - 4 - 6 - \cdots - 2016 - 2018?$$

- (A) -1010 (B) -1009 (C) 1008 (D) 1009 (E) 1010

Problem 1: Solution

Strategy

Pair each odd number with the even number that follows it.

Solution

Group the terms as

$$(1 - 2) + (3 - 4) + \cdots + (2017 - 2018) + 2019.$$

The even numbers $2, 4, \dots, 2018$ give $2018/2 = 1009$ pairs, and each pair equals -1 . So the value is $-1009 + 2019 = 1010$.

Answer: (E) 1010.

Problem 2 (2017 AMC 8, Problem #14)

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Chloe and Zoe are both students in Ms. Demeanor's math class. Last night they each solved half of the problems in their homework assignment alone and then solved the other half together. Chloe had correct answers to only 80% of the problems she solved alone, but overall 88% of her answers were correct. Zoe had correct answers to 90% of the problems she solved alone. What was Zoe's overall percentage of correct answers?

- (A) 89 (B) 92 (C) 93 (D) 96 (E) 98

Problem 2: Solution

Strategy

Find how many of the problems solved together were correct. Chloe and Zoe have the same answers on those problems.

Solution

Let the assignment have $2n$ problems, so each student solved n problems alone and n together. Chloe answered $0.80n$ of her own problems correctly and $0.88 \cdot 2n = 1.76n$ problems correctly in all. So $1.76n - 0.80n = 0.96n$ of the problems solved together were answered correctly. Zoe answered $0.90n$ of her own problems correctly, so she answered $0.90n + 0.96n = 1.86n$ correctly in all. Her percentage is

$$\frac{1.86n}{2n} = 0.93 = 93\%.$$

Answer: (C) 93.

Problem 3 (1996 AJHSME, Problem #15)

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The remainder when the product $1492 \cdot 1776 \cdot 1812 \cdot 1996$ is divided by 5 is

- (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 3: Solution

Strategy

Replace each factor by its remainder modulo 5.

Solution

Since $1492 = 1490 + 2$, $1776 = 1775 + 1$, $1812 = 1810 + 2$ and $1996 = 1995 + 1$, where 1490, 1775, 1810 and 1995 are multiples of 5,

$$1492 \cdot 1776 \cdot 1812 \cdot 1996 \equiv 2 \cdot 1 \cdot 2 \cdot 1 = 4 \pmod{5}.$$

The remainder is 4.

Answer: (E) 4.

Problem 4 (1990 AHSME, Problem #11)

Problem 4 (1990 AHSME, Problem #11)

How many positive integers less than 50 have an odd number of positive integer divisors?

- (A) 3 (B) 5 (C) 7 (D) 9 (E) 11

Problem 4: Solution

Strategy

Use the divisor formula to decide when the number of divisors is odd.

Solution

The number 1 has 1 divisor, an odd number. For $n = p_1^{e_1} \cdots p_k^{e_k} > 1$, the number of divisors $(e_1 + 1) \cdots (e_k + 1)$ is odd exactly when every factor $e_i + 1$ is odd, that is, when every exponent e_i is even. Squaring a number doubles every exponent in its prime factorization, and if every e_i is even, then n is the square of $p_1^{e_1/2} \cdots p_k^{e_k/2}$. So this happens exactly when n is a perfect square. The perfect squares less than 50 are 1, 4, 9, 16, 25, 36, 49, so there are 7 such integers.

Answer: (C) 7.

Problem 5 (2018 AMC 8, Problem #14)

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Let N be the greatest five-digit number whose digits have a product of 120. What is the sum of the digits of N ?

- (A) 15 (B) 16 (C) 17 (D) 18 (E) 20

Problem 5: Solution

Strategy

Choose the digits from left to right, each as large as possible.

Solution

Of two five-digit numbers, the one with the larger first digit is larger; if the first digits are equal, the second digits decide, and so on. So N has the largest possible first digit, then the largest possible second digit, and so on. Since $120 = 2^3 \cdot 3 \cdot 5$, the digits dividing 120 are 1, 2, 3, 4, 5, 6, 8, so the first digit is at most 8. With first digit 8, the other four digits have product $120/8 = 15$. The digits dividing 15 are 1, 3, 5, so the second digit is 5. The last three digits then have product 3, so they are 3, 1, 1 in that order. Hence $N = 85311$, and its digit sum is $8 + 5 + 3 + 1 + 1 = 18$.

Answer: (D) 18.

Problem 6 (2020 AMC 8, Problem #17)

Problem 6 (2020 AMC 8, Problem #17)

How many factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely 1, 2, 3, 4, 6, and 12.)

- (A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Problem 6: Solution

Strategy

Count the factors of 2020 that have at most 3 factors, then subtract from the total.

Solution

Since $2020 = 2^2 \cdot 5 \cdot 101$ and 101 is prime, 2020 has $(2 + 1)(1 + 1)(1 + 1) = 12$ factors. A number $p_1^{e_1} \cdots p_k^{e_k}$ has $(e_1 + 1) \cdots (e_k + 1)$ factors, and each $e_i + 1$ is at least 2. So a number with two or more different prime factors has at least $2 \cdot 2 = 4$ factors, and p^e with $e \geq 3$ has $e + 1 \geq 4$. The numbers with at most 3 factors are therefore 1, the primes p and their squares p^2 . The factors of 2020 of these kinds are 1, the primes 2, 5, 101, and $2^2 = 4$. That is 5 factors, so $12 - 5 = 7$ factors of 2020 have more than 3 factors.

Answer: (B) 7.