

Session 4: Counting II

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Circular Arrangements

Rotations

In a **circular arrangement**, n people sit in n equally spaced seats around a round table. Two seatings are the same when one is a rotation of the other, that is, when every person has the same left neighbor and the same right neighbor in both.

Counting

Fix the seat of one person: every seating can be rotated in exactly one way so that this person sits there. The other $n - 1$ people then fill the remaining $n - 1$ seats in order, so the number of circular arrangements of n people is

$$(n - 1)!.$$

Equivalently, the $n!$ seatings in numbered seats fall into groups of n rotations of each other, and $\frac{n!}{n} = (n - 1)!.$

Example 1

Problem 1

Eight people sit around a round table. Seatings that differ only by a rotation are the same. In how many seatings do Maya and Noah sit next to each other?

Strategy

Treat Maya and Noah as one block. Arrange the block and the other six people around the table, then order the two people inside the block.

Example 1: Solution

Solution

Put Maya and Noah together in one block. The block and the other 6 people are 7 objects around the table, and they can be arranged in $(7 - 1)! = 6! = 720$ ways. Inside the block, Maya can sit on Noah's left or on his right, which gives 2 orders. Each seating with Maya and Noah next to each other comes from exactly one arrangement of the 7 objects and one order inside the block, so the number of seatings is

$$2 \cdot 6! = 2 \cdot 720 = 1440.$$

Answer: 1440.

Combinations and Permutations

Permutations

By the multiplication principle, the number of ways to fill k different positions with k of n distinct objects is

$$P(n, k) = n(n-1) \cdots (n-k+1).$$

Combinations

Each group of k objects can be put in order in $k!$ ways, so the number of groups of k objects from n distinct objects is

$$\binom{n}{k} = \frac{P(n, k)}{k!}.$$

Order or no order

Swap two of the selected objects. If the outcome changes, count permutations. If it does not, count combinations.

Example 2

Problem 2

A class has 10 students.

- (a) In how many ways can a team of 4 students be chosen?
- (b) In how many ways can a president, a vice-president, a secretary and a treasurer be chosen, if no student holds two offices?

Strategy

In (a), swapping two team members gives the same team. In (b), swapping two officers gives a different outcome.

Example 2: Solution

Solution

(a) The team is a group, so the number of teams is

$$\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210.$$

(b) The four offices are different positions, so the number of ways is

$$P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040.$$

The two counts agree: each group of 4 students can fill the four offices in $4! = 24$ ways, and $210 \cdot 24 = 5040$.

Answer: (a) 210; (b) 5040.

Stars and Bars

Identical objects for different people

Giving k identical objects to n different people, where a person may get none, is the same as choosing a solution of $x_1 + x_2 + \cdots + x_n = k$ in nonnegative integers.

Stars and bars

Write the objects as k stars in a row and separate the people's shares with $n - 1$ bars. For $k = 5$ and $n = 3$, the row $\star\star || \star\star\star$ means the shares 2, 0, 3. Each distribution is exactly one row of k stars and $n - 1$ bars, and a row is determined by which $n - 1$ of its $n + k - 1$ places hold bars, or equally by which k places hold stars. So the number of distributions is $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

At least m each

Give m objects to each person first, then distribute the other $k - nm$ with no restriction.

Example 3

Problem 3

In how many ways can 14 identical stickers be given to 4 students so that each student gets at least 2 stickers?

Strategy

Give each student 2 stickers first, then distribute the rest with stars and bars.

Example 3: Solution

Solution

Giving 2 stickers to each student uses 8 stickers and leaves 6. A student who gets y of these 6 stickers has $y + 2$ stickers in all, so the distributions with at least 2 stickers each correspond exactly to the distributions of the 6 remaining stickers with no restriction.

Stars and bars with $k = 6$ and $n = 4$ gives

$$\binom{6 + 4 - 1}{4 - 1} = \binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84.$$

Answer: 84.

Restricted Arrangements

Two objects together

To count the arrangements in a row in which two given objects are next to each other, treat the two as one block. Arrange the block with the other objects, then multiply by 2 for the two orders inside the block.

Complement

The number of arrangements in which two objects are not next to each other is the number of all arrangements minus the number in which they are next to each other.

Gap method

Arrange the other objects first. A row of m objects has $m + 1$ gaps: one before the first object, one between each two neighbors, and one after the last. Put each restricted object in a different gap. Objects in different gaps are never next to each other.

Example 4

Problem 4

How many arrangements of the letters A, B, C, D, E in a row have A not next to B ?

Strategy

Count all arrangements, then subtract the arrangements in which A is next to B .

Example 4: Solution

Solution

The five letters can be arranged in $5! = 120$ ways. To count the arrangements with A next to B , treat A and B as one block. The block, C , D and E can be arranged in $4! = 24$ ways, and the block is AB or BA , so there are $2 \cdot 24 = 48$ such arrangements. Hence

$$120 - 48 = 72$$

arrangements have A not next to B .

The gap method gives the same count. Arrange C , D , E in $3! = 6$ ways, which leaves 4 gaps. Put A in one gap and B in a different gap, in $4 \cdot 3 = 12$ ways. Letters in different gaps are not next to each other, and each arrangement with A not next to B arises exactly once, so the count is $6 \cdot 12 = 72$.

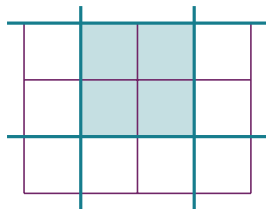
Answer: 72.

Rectangles in a Grid

Grid lines

A grid of unit squares with r rows and c columns has $r + 1$ horizontal lines and $c + 1$ vertical lines. A rectangle with its sides on grid lines is determined by its top and bottom sides, which lie on 2 of the horizontal lines, and its left and right sides, which lie on 2 of the vertical lines. Every choice of 2 horizontal and 2 vertical lines gives exactly one rectangle, so the number of rectangles, squares included, is

$$\binom{r+1}{2} \binom{c+1}{2}.$$



The shaded rectangle in a grid with 3 rows and 4 columns, and the two horizontal and two vertical lines that determine it.

Example 5

Problem 5

A rectangle is divided into a grid of unit squares with 4 rows and 6 columns. How many rectangles, squares included, have all four sides on lines of the grid?

Strategy

Choose two horizontal lines and two vertical lines.

Example 5: Solution

Solution

The grid has $4 + 1 = 5$ horizontal lines and $6 + 1 = 7$ vertical lines. A rectangle is given by a choice of 2 horizontal lines, in $\binom{5}{2} = 10$ ways, and a choice of 2 vertical lines, in $\binom{7}{2} = 21$ ways. So the number of rectangles is

$$\binom{5}{2} \binom{7}{2} = 10 \cdot 21 = 210.$$

Answer: 210.

Problem 1

Problem 1

Nine people sit around a round table. Seatings that differ only by a rotation are the same. How many different seatings are there?

Problem 1: Solution

Strategy

Fix the seat of one person.

Solution

Every seating can be rotated in exactly one way so that a chosen person sits in a fixed seat. The other 8 people then fill the remaining 8 seats in $8!$ ways, so the number of seatings is

$$(9 - 1)! = 8! = 40320.$$

Answer: 40320.

Problem 2

Problem 2

Seven people sit around a round table. Seatings that differ only by a rotation are the same. In how many seatings do two particular people sit next to each other?

Problem 2: Solution

Strategy

Treat the two people as one block.

Solution

Put the two people together in one block. The block and the other 5 people are 6 objects around the table, and they can be arranged in $(6 - 1)! = 5! = 120$ ways. The two people can sit in 2 orders inside the block, so the number of seatings is

$$2 \cdot 5! = 2 \cdot 120 = 240.$$

Answer: 240.

Problem 3

Problem 3

A club has 12 students.

- (a) In how many ways can a team of 5 students be chosen?
- (b) In how many ways can a team of 5 students be chosen and two different members of the team be named captain and co-captain?

Problem 3: Solution

Strategy

Choose the team as a group, then fill the two different roles in order.

Solution

(a) The team is a group, so the number of teams is

$$\binom{12}{5} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 792.$$

(b) For each team, the captain can be any of the 5 members and the co-captain any of the other 4, so there are $P(5, 2) = 5 \cdot 4 = 20$ ways to name them. By the multiplication principle, the total is $792 \cdot 20 = 15840$.

Answer: (a) 792; (b) 15840.

Problem 4

Problem 4

How many solutions does $x_1 + x_2 + x_3 + x_4 = 12$ have in nonnegative integers?

Problem 4: Solution

Strategy

Write each solution as a row of 12 stars and 3 bars.

Solution

A solution is a way to give 12 identical stars to the 4 variables, where a variable may get none. Stars and bars with $k = 12$ and $n = 4$ gives

$$\binom{12 + 4 - 1}{4 - 1} = \binom{15}{3} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 455.$$

Answer: 455.

Problem 5

Problem 5

How many arrangements of the letters A, B, C, D, E, F, G in a row have A not next to B ?

Problem 5: Solution

Strategy

Use the gap method: arrange the other five letters first.

Solution

Arrange C, D, E, F, G in $5! = 120$ ways. This leaves 6 gaps. Put A in one gap and B in a different gap, in $6 \cdot 5 = 30$ ways. Letters in different gaps are not next to each other, and every arrangement with A not next to B arises in exactly one way. So the number of arrangements is

$$120 \cdot 30 = 3600.$$

The complement gives the same count: $7! - 2 \cdot 6! = 5040 - 1440 = 3600$.

Answer: 3600.

Problem 6

Problem 6

A rectangle is divided into a grid of unit squares with 3 rows and 8 columns. How many rectangles, squares included, have all four sides on lines of the grid?

Problem 6: Solution

Strategy

Choose two horizontal lines and two vertical lines.

Solution

The grid has $3 + 1 = 4$ horizontal lines and $8 + 1 = 9$ vertical lines. Each rectangle is given by 2 of the horizontal lines and 2 of the vertical lines, so the number of rectangles is

$$\binom{4}{2} \binom{9}{2} = 6 \cdot 36 = 216.$$

Answer: 216.