

Session 3: Geometry

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Area on a Grid

Pieces with known areas

- A unit square has area 1. A diagonal cuts it into two triangles of area $\frac{1}{2}$ each, called **half-squares**.
- A triangle with base b and height h has area $\frac{1}{2}bh$. On a grid, choose a base along a grid line; the height is then a horizontal or vertical distance.

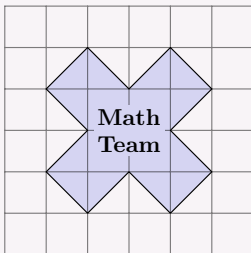
Two ways to find an area

- **Add:** cut the figure into pieces of known area and add their areas.
- **Subtract:** take a rectangle that contains the figure and subtract the areas of the parts of the rectangle outside the figure.

Example 1 (2022 AMC 8, Problem #1)

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The Math Team designed a logo shaped like a multiplication symbol, shown below on a grid of 1-inch squares. What is the area of the logo in square inches?



- (A) 10 (B) 12 (C) 13 (D) 14 (E) 15

Example 1: Solution

Strategy

Compare the logo with the 4×4 square around it, one unit square at a time.

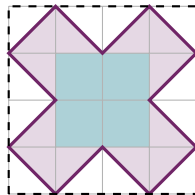
Solution

The logo fits in a 4×4 square. The four middle unit squares (teal) lie inside the logo. The edge of the logo cuts each of the other 12 unit squares along a diagonal, so each of them is half inside the logo.

Add: the area is $4 + 12 \cdot \frac{1}{2} = 4 + 6 = 10$.

Subtract: the 12 half-squares outside the logo have total area 6, so the area is $16 - 6 = 10$.

Answer: (A) 10.



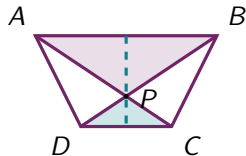
Similar Triangles

Similar triangles

Two triangles are **similar** when their angles are equal in pairs. Then one is a scaled copy of the other: all matching lengths, including heights, are in the same ratio.

Diagonals of a trapezoid

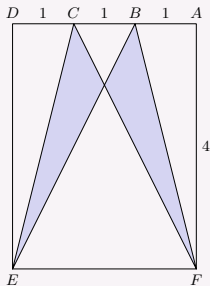
Let the diagonals of trapezoid $ABCD$ with $AB \parallel CD$ cross at P . The angles at P are vertical angles, and $\angle PAB = \angle PCD$ and $\angle PBA = \angle PDC$ are alternate interior angles. So triangles PAB and PCD are similar with ratio $AB : CD$. Their heights from P are in this ratio and add up to the distance between the lines AB and CD .



Example 2 (2016 AMC 8, Problem #22)

Problem 2 (2016 AMC 8, Problem #22)

Rectangle $DEFA$ below is a 3×4 rectangle with $DC = CB = BA = 1$. The area of the “bat wings” (shaded area) is



- (A) 2 (B) $2\frac{1}{2}$ (C) 3 (D) $3\frac{1}{2}$ (E) 4

Example 2: Solution

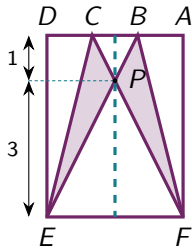
Strategy

Use similar triangles to find where $\overline{EB}$ and $\overline{FC}$ cross.

Solution

Let P be the point where $\overline{EB}$ and $\overline{FC}$ cross. Since $CB \parallel EF$, triangles PCB and PFE are similar with ratio $CB : FE = 1 : 3$. Their heights from P add up to 4, so the height of triangle PCB is 1. The left wing is triangle ECB without triangle PCB , and the right wing is triangle FBC without triangle PCB . Triangles ECB and FBC have area $\frac{1}{2} \cdot 1 \cdot 4 = 2$ and triangle PCB has area $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$, so each wing has area $\frac{3}{2}$ and the shaded area is 3.

Answer: (C) 3.



Radii and Right Triangles

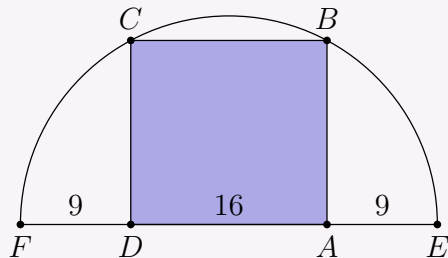
Radius first

- Every point of a circle is at the same distance, the **radius**, from the center. The center of a semicircle is the midpoint of its diameter.
- To find a length inside a circle, join the center to a point on the circle. This radius is often the hypotenuse of a right triangle, and the Pythagorean theorem $a^2 + b^2 = c^2$ gives the missing side.

Example 3 (2020 AMC 8, Problem #18)

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Rectangle $ABCD$ is inscribed in a semicircle with diameter $\overline{FE}$, as shown in the figure. Let $DA = 16$, and let $FD = AE = 9$. What is the area of $ABCD$?



- (A) 240 (B) 248 (C) 256 (D) 264 (E) 272

Example 3: Solution

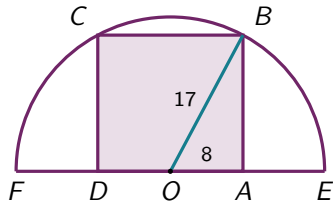
Strategy

Find the radius. Then use the right triangle formed by the center, A and B .

Solution

The diameter is $FE = 9 + 16 + 9 = 34$, so the radius is 17. Let O be the center, the midpoint of $\overline{FE}$. Since $FD = AE$, the point O is also the midpoint of $\overline{DA}$, so $OA = 8$. The point B is on the semicircle, so $OB = 17$. Triangle OAB has a right angle at A , so $AB^2 = 17^2 - 8^2 = 225$ and $AB = 15$. The area of $ABCD$ is $16 \cdot 15 = 240$.

Answer: (A) 240.



Nets and Prisms

Nets

A **net** is a flat figure that folds into a solid. When the net is folded, its edges are joined in pairs, and two joined edges have the same length. The area of a net is the surface area of the solid.

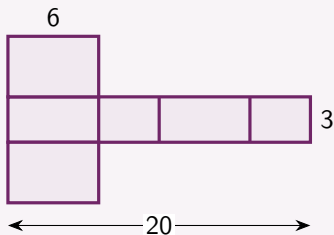
Prisms

The volume of a prism is the area of its base times its height. A box that is ℓ by w by h has volume ℓwh .

Example 4

Problem 4

The net below folds into a box. The four faces in the row have height 3 and total length 20, and the face on top is 6 units wide. What is the volume of the box?



Example 4: Solution

Strategy

The row of four faces wraps once around the box. Compare its length with the perimeter of the top face.

Solution

Let the top face be 6 by w . When the net is folded, the four faces in the row become the four sides of the box, and the top edge of each of them is joined to one edge of the top face. So the length of the row equals the perimeter of the top face:

$$2(6 + w) = 20,$$

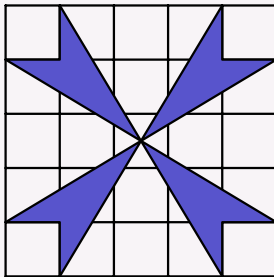
and $w = 4$. The box is 6 by 4 by 3, so its volume is $6 \cdot 4 \cdot 3 = 72$.

Answer: 72.

Problem 1 (2007 AMC 8, Problem #23)

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What is the area of the shaded pinwheel shown in the 5×5 grid?



- (A) 4 (B) 6 (C) 8 (D) 10 (E) 12

Problem 1: Solution

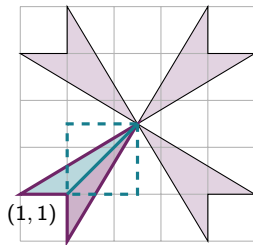
Strategy

Cut a dart into two triangles with bases on grid lines.

Solution

Put the lower-left corner of the grid at $(0,0)$. The center of the grid is $(\frac{5}{2}, \frac{5}{2})$. The segment from the center to $(1,1)$ cuts the lower-left dart into two triangles. One has base 1 on the line $y = 1$, the other has base 1 on the line $x = 1$, and the center is $\frac{3}{2}$ from each of these lines. So each triangle has area $\frac{1}{2} \cdot 1 \cdot \frac{3}{2} = \frac{3}{4}$, and the dart has area $\frac{3}{2}$. The four darts are congruent, so the pinwheel has area $4 \cdot \frac{3}{2} = 6$.

Answer: (B) 6.



Problem 2

Problem 2

In trapezoid $ABCD$, $AB \parallel CD$, $AB = 12$, $CD = 3$, and the distance between the lines AB and CD is 20. The diagonals $\overline{AC}$ and $\overline{BD}$ cross at P . What is the distance from P to the line AB ?

Problem 2: Solution

Strategy

Use the similar triangles PAB and PCD .

Solution

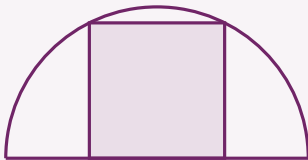
Since $AB \parallel CD$, triangles PAB and PCD are similar with ratio $AB : CD = 12 : 3 = 4 : 1$. So the height of triangle PAB from P is 4 times the height of triangle PCD from P . The two heights add up to 20, so they are 16 and 4. The distance from P to the line AB is the height of triangle PAB , which is 16.

Answer: 16.

Problem 3

Problem 3

A square has one side on the diameter of a semicircle of radius 5, and its other two vertices lie on the semicircle. What is the area of the square?



Problem 3: Solution

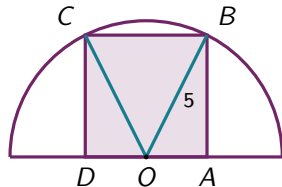
Strategy

Show that the center is the midpoint of the bottom side. Then use a right triangle with a radius as its hypotenuse.

Solution

Let the square $ABCD$ have side s , with D and A on the diameter, and let O be the center. Right triangles ODC and OAB have hypotenuses $OC = OB = 5$ and legs $DC = AB = s$, so by the Pythagorean theorem $OD = OA$. Hence O is the midpoint of $\overline{DA}$ and $OA = \frac{s}{2}$. In triangle OAB , $(\frac{s}{2})^2 + s^2 = 5^2$, so $\frac{5}{4}s^2 = 25$. The area of the square is $s^2 = 20$.

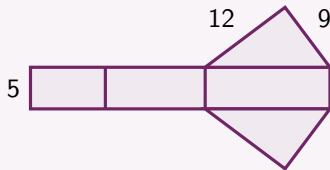
Answer: 20.



Problem 4

Problem 4

The net below folds into a prism. Its two triangular faces are right triangles with legs 9 and 12, and each rectangular face has a side of length 5. What is the area of the net?



Problem 4: Solution

Strategy

Each rectangle is joined to one side of a triangle, so it has that side as one of its sides.

Solution

The hypotenuse of each triangle is $\sqrt{9^2 + 12^2} = \sqrt{225} = 15$. When the net is folded, each rectangle is joined to one side of each triangle, so the rectangles are 9 by 5, 12 by 5 and 15 by 5. The two triangles have area $\frac{1}{2} \cdot 9 \cdot 12 = 54$ each. The area of the net is

$$2 \cdot 54 + 5 \cdot (9 + 12 + 15) = 108 + 180 = 288.$$

Answer: 288.