

Session 2: Algebra and Number Theory I

Andy Jo

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Remainders and Divisibility

Division with remainder

Dividing a whole number n by a positive integer m gives a quotient q and a remainder r :

$$n = qm + r, \quad 0 \leq r < m.$$

The number n is **divisible** by m when $r = 0$.

Divisibility rules

- Each of 9, 99, 999, ... is divisible by 9, so a number and the sum of its digits leave the same remainder when divided by 9. In particular, a number is divisible by 9 exactly when its digit sum is.
- Since $1000 = 8 \cdot 125$, a number and its last three digits leave the same remainder when divided by 8.

Example 1 (2018 AMC 8, Problem #7)

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The 5-digit number $\underline{2} \underline{0} \underline{1} \underline{8} \underline{U}$ is divisible by 9. What is the remainder when this number is divided by 8?

- (A) 1 (B) 3 (C) 5 (D) 6 (E) 7

Strategy

Use the digit sum to find U . Then only the last three digits matter for the remainder on division by 8.

Example 1: Solution

Solution

The digit sum is $2 + 0 + 1 + 8 + U = 11 + U$, and it must be a multiple of 9. Since U is a digit, $11 \leq 11 + U \leq 20$, and the only multiple of 9 in this range is 18. So $U = 7$ and the number is 20187.

Since $20187 = 20 \cdot 1000 + 187$ and 1000 is divisible by 8, the numbers 20187 and 187 leave the same remainder when divided by 8. As $187 = 8 \cdot 23 + 3$, the remainder is 3.

Answer: (B) 3.

Prime Factorization, GCF and LCM

Definitions

- Every integer greater than 1 is a product of primes in exactly one way, apart from order: its **prime factorization**. For example, $60 = 2^2 \cdot 3 \cdot 5$.
- The **GCF** (greatest common factor) of positive integers is the largest integer that divides all of them. It uses each prime common to all of them, with the smallest exponent.
- The **LCM** (least common multiple) is the smallest positive integer divisible by all of them. It uses each prime that appears in any of them, with the largest exponent. Every common multiple is a multiple of the LCM.

Example

$12 = 2^2 \cdot 3$ and $18 = 2 \cdot 3^2$, so $\text{GCF}(12, 18) = 2 \cdot 3 = 6$ and $\text{LCM}(12, 18) = 2^2 \cdot 3^2 = 36$.

Example 2 (2012 AMC 8, Problem #15)

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The smallest number greater than 2 that leaves a remainder of 2 when divided by 3, 4, 5, or 6 lies between what numbers?

- (A) 40 and 50 (B) 51 and 55 (C) 56 and 60 (D) 61 and 65
(E) 66 and 99

Strategy

If N leaves a remainder of 2 on each division, then $N - 2$ is divisible by 3, 4, 5 and 6.

Example 2: Solution

Solution

Let N be the number. Each division leaves a remainder of 2, so $N - 2$ is divisible by 3, 4, 5 and 6. Since $N > 2$, the number $N - 2$ is a positive common multiple of 3, 4, 5 and 6, so it is a multiple of their LCM. With $4 = 2^2$ and $6 = 2 \cdot 3$,

$$\text{LCM}(3, 4, 5, 6) = 2^2 \cdot 3 \cdot 5 = 60.$$

So the smallest possible value is $N - 2 = 60$, that is, $N = 62$. Indeed $62 = 60 + 2$ leaves a remainder of 2 when divided by each of 3, 4, 5 and 6, since 2 is less than each divisor. The number 62 lies between 61 and 65.

Answer: (D) 61 and 65.

Units Digits

Units digit of a product

The units digit of a product depends only on the units digits of the factors. For example, 13^n has the same units digit as 3^n .

Powers repeat in a cycle

Each power of 3 is 3 times the previous one, so its units digit is the units digit of 3 times the previous units digit:

$$3^1, 3^2, 3^3, 3^4, 3^5, \dots \quad \text{end in} \quad 3, 9, 7, 1, 3, \dots$$

Once the units digit returns to 3, the digits repeat with cycle length 4. To find the units digit of 3^k , divide k by 4. A remainder of 1, 2, 3 gives 3, 9, 7, and a remainder of 0 gives 1.

Example 3 (2012 AMC 8, Problem #12)

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What is the units digit of 13^{2012} ?

- (A) 1 (B) 3 (C) 5 (D) 7 (E) 9

Strategy

Only the units digit 3 of the base matters. Find where 2012 falls in the cycle of units digits of powers of 3.

Example 3: Solution

Solution

The units digit of a product depends only on the units digits of the factors, so 13^{2012} has the same units digit as 3^{2012} . The units digits of the powers of 3 repeat in the cycle 3, 9, 7, 1 of length 4. Since $2012 = 4 \cdot 503$, the exponent is a multiple of 4, so 3^{2012} has the same units digit as $3^4 = 81$, namely 1.

Answer: (A) 1.

Successive Percent Changes

Percent changes as multipliers

- A loss of $p\%$ multiplies an amount by $1 - \frac{p}{100}$. For example, a 25% loss multiplies by 0.75.
- A gain of $p\%$ multiplies an amount by $1 + \frac{p}{100}$. For example, a 20% gain multiplies by 1.2.
- Several changes in a row multiply the amount by the product of their multipliers.

Percents do not add

Each percent is taken of the amount at that time, not of the original amount. A 20% gain followed by a 20% loss gives $1.2 \cdot 0.8 = 0.96$ times the original amount, a 4% loss. Starting from 100 makes the final amount equal to the percent of the original.

Example 4 (2008 AMC 8, Problem #9)

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In 2005 Tycoon Tammy invested \$100 for two years. During the first year her investment suffered a 15% loss, but during the second year the remaining investment showed a 20% gain. Over the two-year period, what was the change in Tammy's investment?

- (A)** 5% loss **(B)** 2% loss **(C)** 1% gain **(D)** 2% gain **(E)** 5% gain

Strategy

Multiply the starting amount by one multiplier for each year.

Example 4: Solution

Solution

The 15% loss multiplies the investment by 0.85, and the 20% gain multiplies the new amount by 1.2. After two years Tammy has

$$100 \cdot 0.85 \cdot 1.2 = 85 \cdot 1.2 = 102$$

dollars. This is \$2 more than the original \$100, a 2% gain.

Answer: (D) 2% gain.

Problem 1 (2019 AMC 8, Problem #1)

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Ike and Mike go into a sandwich shop with a total of \$30.00 to spend. Sandwiches cost \$4.50 each and soft drinks cost \$1.00 each. Ike and Mike plan to buy as many sandwiches as they can, and use any remaining money to buy soft drinks. Counting both soft drinks and sandwiches, how many items will they buy?

- (A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Problem 1: Solution

Strategy

Find the largest number of sandwiches that costs at most \$30.00. The money left over buys drinks.

Solution

Six sandwiches cost $6 \cdot \$4.50 = \27.00 , and seven would cost \$31.50, which is more than \$30.00. So they buy 6 sandwiches and have \$3.00 left, which buys exactly 3 soft drinks. In total they buy $6 + 3 = 9$ items.

Answer: (D) 9.

Problem 2 (2020 AMC 8, Problem #8)

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Ricardo has 2020 coins, some of which are pennies (1-cent coins) and the rest of which are nickels (5-cent coins). He has at least one penny and at least one nickel. What is the difference in cents between the greatest possible and least possible amounts of money that Ricardo can have?

- (A) 8062 (B) 8068 (C) 8072 (D) 8076 (E) 8082

Problem 2: Solution

Strategy

Write the amount of money in terms of the number of pennies.

Solution

Let p be the number of pennies, so there are $2020 - p$ nickels and $1 \leq p \leq 2019$. The amount in cents is

$$p + 5(2020 - p) = 10100 - 4p.$$

This decreases as p increases, so it is greatest when $p = 1$, giving 10096, and least when $p = 2019$, giving 2024. The difference is $10096 - 2024 = 8072$.

Answer: (C) 8072.

Problem 3 (2019 AMC 8, Problem #8)

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Gilda has a bag of marbles. She gives 20% of them to her friend Pedro. Then Gilda gives 10% of what is left to another friend, Ebony. Finally, Gilda gives 25% of what is now left in the bag to her brother Jimmy. What percentage of her original bag of marbles does Gilda have left for herself?

- (A) 20 (B) $33\frac{1}{3}$ (C) 38 (D) 45 (E) 54

Problem 3: Solution

Strategy

Each gift is a loss of a percent of the marbles left at that time. Start from 100 marbles and apply one multiplier per gift.

Solution

Giving away 20%, then 10%, then 25% multiplies the number of marbles by 0.8, 0.9 and 0.75 in turn. Starting from 100,

$$100 \cdot 0.8 = 80, \quad 80 \cdot 0.9 = 72, \quad 72 \cdot 0.75 = 54.$$

So Gilda keeps 54% of her original marbles.

Answer: (E) 54.

Problem 4 (2020 AMC 8, Problem #12)

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For a positive integer n , the factorial notation $n!$ represents the product of the integers from n to 1. For example:

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

What value of N satisfies the following equation?

$$5! \times 9! = 12 \times N!$$

- (A) 10 (B) 11 (C) 12 (D) 13 (E) 14

Problem 4: Solution

Strategy

Split $5!$ into 12 times another factor, and combine that factor with $9!$.

Solution

Since $5! = 120 = 12 \cdot 10$,

$$5! \times 9! = 12 \times 10 \times 9! = 12 \times 10!.$$

So $N! = 10!$. The factorials $1!, 2!, 3!, \dots$ increase, so $N = 10$.

Answer: (A) 10.

Problem 5 (2020 AMC 8, Problem #15)

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Suppose 15% of x equals 20% of y . What percentage of x is y ?

- (A) 5 (B) 35 (C) 75 (D) $133\frac{1}{3}$ (E) 300

Problem 5: Solution

Strategy

Write the condition as an equation and solve for y in terms of x .

Solution

The condition says $0.15x = 0.20y$. Dividing both sides by 0.20 gives

$$y = \frac{0.15}{0.20}x = \frac{3}{4}x = 0.75x.$$

So y is 75% of x .

Answer: (C) 75.

Problem 6 (2016 AMC 8, Problem #20)

Problem 6 (2016 AMC 8, Problem #20)

The least common multiple of a and b is 12, and the least common multiple of b and c is 15. What is the least possible value of the least common multiple of a and c ?

- (A) 20 (B) 30 (C) 60 (D) 120 (E) 180

Problem 6: Solution

Strategy

Find the possible values of b first. Then use prime factorizations to see which factors a and c must have.

Solution

The number b divides $\text{LCM}(a, b) = 12$ and $\text{LCM}(b, c) = 15$, so $b = 1$ or $b = 3$. An LCM takes the larger power of each prime. Since $12 = 2^2 \cdot 3$ and b has no factor 2, the factor 2^2 comes from a , so 4 divides a . Since $15 = 3 \cdot 5$ and b has no factor 5, the factor 5 comes from c , so 5 divides c .

So $\text{LCM}(a, c)$ is divisible by 4 and by 5, hence by $\text{LCM}(4, 5) = 20$. The value 20 occurs for $a = 4$, $b = 3$, $c = 5$: then $\text{LCM}(4, 3) = 12$, $\text{LCM}(3, 5) = 15$ and $\text{LCM}(4, 5) = 20$.

Answer: (A) 20.