

Session 1: Counting I

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Multiplication Principle

Multiplication principle

Suppose a task is done in two steps. If the first step can be done in a ways and, whatever happens in the first step, the second step can be done in b ways, then the task can be done in ab ways. The same holds for any number of steps.

Conditions

The options at a step may depend on the earlier choices, but their number must not. Different sequences of choices must give different outcomes; otherwise some outcomes are counted more than once.

Example 1 (2007 AMC 8, Problem #4)

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A haunted house has six windows. In how many ways can Georgie the Ghost enter the house by one window and leave by a different window?

(A) 12 (B) 15 (C) 18 (D) 30 (E) 36

Strategy

Choose the entry window first, then the exit window.

Example 1: Solution

Solution

The entry window can be any of the 6 windows. Whatever the entry window is, the exit window can be any of the other 5. By the multiplication principle there are $6 \cdot 5 = 30$ ways. The order of the two windows matters: entering by window 1 and leaving by window 2 is different from entering by window 2 and leaving by window 1.

Answer: (D) 30.

Permutations

Factorials

For a positive integer n , $n! = n(n-1) \cdots 2 \cdot 1$, and $0! = 1$. By the multiplication principle, n distinct objects can be put in a row in $n!$ ways: there are n choices for the first place, $n-1$ for the second, and so on.

Permutations

A **permutation** of k objects from n distinct objects is an ordered selection: k different positions are filled with k of the objects. The number of permutations is

$$P(n, k) = n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

Combinations

Combinations

A **combination** of k objects from n distinct objects is an unordered selection: a group of k of the objects. Each group can be ordered in $k!$ ways, so $P(n, k)$ counts each group $k!$ times. The number of groups, read “ n choose k ”, is

$$\binom{n}{k} = \frac{P(n, k)}{k!} = \frac{n!}{k! (n - k)!}.$$

For example, $\binom{8}{3} = (8 \cdot 7 \cdot 6)/(3 \cdot 2 \cdot 1) = 56$. Choosing the k objects in a group is the same as choosing the $n - k$ objects left out, so $\binom{n}{k} = \binom{n}{n-k}$.

Order or no order

If swapping two of the selected objects gives a different outcome, count permutations. If it gives the same outcome, count combinations.

Example 2 (2004 AMC 8, Problem #4)

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Ms. Hamilton's eighth-grade class wants to participate in the annual three-person-team basketball tournament. Lance, Sally, Joy, and Fred are chosen for the team. In how many ways can the three starters be chosen?

- (A) 2 (B) 4 (C) 6 (D) 8 (E) 10

Strategy

The starters form a group, so their order does not matter.

Example 2: Solution

Solution

Swapping two starters gives the same set of starters, so count combinations. The number of groups of 3 from the 4 players is

$$\binom{4}{3} = \frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} = 4.$$

Equivalently, the starters are determined by the one player who does not start, and there are 4 choices for that player.

Answer: (B) 4.

Restricted Positions

Method

When some objects must go in certain positions, place those objects first. Then fill the remaining positions with the remaining objects, and multiply the counts.

Why it works

Each allowed arrangement comes from exactly one choice at each step, and every arrangement built this way satisfies the restriction.

Example 3 (2015 AMC 8, Problem #4)

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The Centerville Middle School chess team consists of two boys and three girls. A photographer wants to take a picture of the team to appear in the local newspaper. She decides to have them sit in a row with a boy at each end and the three girls in the middle. How many such arrangements are possible?

- (A) 2 (B) 4 (C) 5 (D) 6 (E) 12

Strategy

Seat the two boys at the ends first, then the three girls in the middle.

Example 3: Solution



Solution

The two boys fill the two end seats in $2! = 2$ orders. Whatever their order, the three girls fill the three middle seats in $3! = 6$ orders. By the multiplication principle there are $2 \cdot 6 = 12$ arrangements.

Answer: (E) 12.

Method

Split the outcomes into cases so that every outcome is in exactly one case. Count each case, then add the counts.

Choosing the cases

Choose cases that are easy to count, for example by the first digit of a number. If two cases overlap, the outcomes in both are counted twice. If an outcome is in no case, it is not counted.

Example 4 (2012 AMC 8, Problem #10)

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How many 4-digit numbers greater than 1000 are there that use the four digits of 2012?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 12

Strategy

Split by the first digit. It cannot be 0, so it is 1 or 2.

Example 4: Solution

Solution

The digits are 0, 1, 2, 2. The first digit cannot be 0. Every arrangement that starts with 1 or 2 is at least 1022, so it is greater than 1000.

Case 1: the first digit is 1. The other three places hold 0, 2, 2. The 0 can go in any of the 3 places, and the two 2s fill the rest. This gives 3 numbers: 1022, 1202, 1220.

Case 2: the first digit is 2. The other three places hold the different digits 0, 1, 2, in $3! = 6$ orders.

The cases do not overlap, so there are $3 + 6 = 9$ numbers.

Answer: (D) 9.

Complementary Counting

Method

Count the outcomes that do not have a property, and subtract from the total:

$$(\text{outcomes with the property}) = (\text{all outcomes}) - (\text{outcomes without the property}).$$

This is shorter when the outcomes without the property are easier to count. For example, the outcomes without “at least one die shows 5” are the outcomes with “neither die shows 5”.

Equally likely outcomes

When all outcomes are equally likely, the probability of an event is

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes}}.$$

For two fair dice, record the first die and then the second. By the multiplication principle there are $6 \cdot 6 = 36$ ordered outcomes, all equally likely.

Complement

The **complement** of an event is the event that it does not happen. Each outcome is in exactly one of the two, so

$$\text{probability of an event} = 1 - \text{probability of its complement}.$$

Example 5 (2001 AMC 8, Problem #18)

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Two dice are thrown. What is the probability that the product of the two numbers is a multiple of 5?

- (A) $\frac{1}{36}$ (B) $\frac{1}{18}$ (C) $\frac{1}{6}$ (D) $\frac{11}{36}$ (E) $\frac{1}{3}$

Strategy

Count the outcomes in which the product is not a multiple of 5, then subtract.

Example 5: Solution

Solution

Record the two dice in order. There are 36 equally likely outcomes. Since 5 is prime, the product is a multiple of 5 exactly when at least one of the two numbers is a multiple of 5, and the only multiple of 5 from 1 to 6 is 5. So the product is not a multiple of 5 exactly when neither die shows 5. Then each die has 5 possible results, which gives $5 \cdot 5 = 25$ outcomes. The other $36 - 25 = 11$ outcomes have a product that is a multiple of 5, so the probability is $\frac{11}{36}$.

Answer: (D) $\frac{11}{36}$.

Inclusion–Exclusion

Notation

For a set A , $|A|$ is the number of elements of A . The **union** $A \cup B$ is the set of elements in A or B (or both). The **intersection** $A \cap B$ is the set of elements in both A and B .

Two sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The sum $|A| + |B|$ counts each element of $A \cap B$ twice and every other element of $A \cup B$ once. Subtracting $|A \cap B|$ counts each element once.

Example 6 (2008 AMC 8, Problem #11)

Problem 6 (2008 AMC 8, Problem #11)

Each of the 39 students in the eighth grade at Lincoln Middle School has one dog or one cat or both a dog and a cat. Twenty students have a dog and 26 students have a cat. How many students have both a dog and a cat?

(A) 7 (B) 13 (C) 19 (D) 39 (E) 46

Strategy

Use inclusion–exclusion with the set of students who have a dog and the set of students who have a cat.

Example 6: Solution

Solution

Let D be the set of students with a dog and C the set of students with a cat. Every student has a dog or a cat or both, so $|D \cup C| = 39$. Also $|D| = 20$ and $|C| = 26$.

Inclusion–exclusion gives

$$39 = 20 + 26 - |D \cap C|,$$

so $|D \cap C| = 46 - 39 = 7$.

Answer: (A) 7.

Problem 1 (2017 AMC 8, Problem #10)

Problem 1 (2017 AMC 8, Problem #10)

A box contains five cards, numbered 1, 2, 3, 4, and 5. Three cards are selected randomly without replacement from the box. What is the probability that 4 is the largest value selected?

- (A) $\frac{1}{10}$ (B) $\frac{1}{5}$ (C) $\frac{3}{10}$ (D) $\frac{2}{5}$ (E) $\frac{1}{2}$

Problem 1: Solution

Strategy

Count the groups of three cards whose largest card is 4.

Solution

The order in which the cards are drawn does not matter, so the outcomes are the $\binom{5}{3} = 10$ groups of three cards, all equally likely. The largest card is 4 exactly when the group contains 4, does not contain 5, and contains two of the cards 1, 2, 3. There are $\binom{3}{2} = 3$ such groups, so the probability is $\frac{3}{10}$.

Answer: (C) $\frac{3}{10}$.

Problem 2 (2020 AMC 8, Problem #10)

Problem 2 (2020 AMC 8, Problem #10)

Zara has a collection of 4 marbles: an Aggie, a Bumblebee, a Steelie, and a Tiger. She wants to display them in a row on a shelf, but does not want to put the Steelie and the Tiger next to one another. In how many ways can she do this?

- (A)** 6 **(B)** 8 **(C)** 12 **(D)** 18 **(E)** 24

Problem 2: Solution

Strategy

Count all the orders, then subtract the orders in which the Steelie and the Tiger are next to each other.

Solution

The 4 marbles can be put in a row in $4! = 24$ orders. To count the orders with the Steelie and the Tiger next to each other, treat those two marbles as one block. The block, the Aggie and the Bumblebee can be arranged in $3! = 6$ orders, and inside the block the Steelie can be on the left or on the right. So $6 \cdot 2 = 12$ orders put them next to each other, and the other $24 - 12 = 12$ orders keep them apart.

Answer: (C) 12.

Problem 3 (2020 AMC 8, Problem #7)

Problem 3 (2020 AMC 8, Problem #7)

How many integers between 2020 and 2400 have four distinct digits arranged in increasing order? (For example, 2357 is one such integer.)

- (A) 9 (B) 10 (C) 15 (D) 21 (E) 28

Problem 3: Solution

Strategy

Find the first two digits, then choose the last two.

Solution

The first digit is 2. The second digit is larger than 2, and it is less than 4 because the integer is less than 2400, so it is 3. The last two digits are two different digits from 4, 5, 6, 7, 8, 9. Two different digits can be written in increasing order in exactly one way, so the count is the number of groups of two of these six digits, $\binom{6}{2} = 15$. All of these integers, from 2345 to 2389, lie between 2020 and 2400.

Answer: (C) 15.

Problem 4 (2018 AMC 8, Problem #11)

Problem 4 (2018 AMC 8, Problem #11)

Abby, Bridget, and four of their classmates will be seated in two rows of three for a group picture, as shown.

$$\begin{array}{ccc} \text{X} & \text{X} & \text{X} \\ \text{X} & \text{X} & \text{X} \end{array}$$

If the seating positions are assigned randomly, what is the probability that Abby and Bridget are adjacent to each other in the same row or the same column?

- (A) $\frac{1}{3}$ (B) $\frac{2}{5}$ (C) $\frac{7}{15}$ (D) $\frac{1}{2}$ (E) $\frac{2}{3}$

Problem 4: Solution

Strategy

Count the pairs of adjacent seats, then the seatings that put Abby and Bridget in one of them.

Solution

The six people can be seated in $6! = 720$ equally likely ways. Each row has 2 pairs of adjacent seats and each column has 1, so there are $2 \cdot 2 + 3 \cdot 1 = 7$ pairs of adjacent seats. A seating with Abby and Bridget adjacent is given by their pair of seats (7 ways), which of the two seats Abby takes (2 ways), and the order of the other four people ($4! = 24$ ways). This gives $7 \cdot 2 \cdot 24 = 336$ seatings, so the probability is

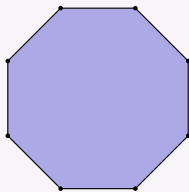
$$\frac{336}{720} = \frac{7}{15}.$$

Answer: (C) $\frac{7}{15}$.

Problem 5 (2018 AMC 8, Problem #23)

Problem 5 (2018 AMC 8, Problem #23)

From a regular octagon, a triangle is formed by connecting three randomly chosen vertices of the octagon. What is the probability that at least one of the sides of the triangle is also a side of the octagon?



- (A) $\frac{2}{7}$ (B) $\frac{5}{42}$ (C) $\frac{11}{14}$ (D) $\frac{5}{7}$ (E) $\frac{6}{7}$

Problem 5: Solution

Strategy

Count the triangles with no side on the octagon, then subtract.

Solution

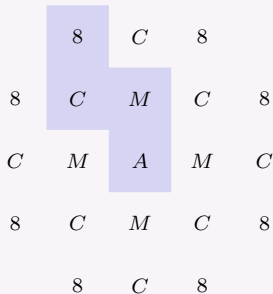
A triangle is determined by its three vertices, so there are $\binom{8}{3} = 56$ equally likely triangles. A triangle has no side on the octagon exactly when no two of its vertices are adjacent. To count these, choose one vertex (8 ways). The other two vertices come from the 5 vertices that are neither this vertex nor its neighbors. These 5 vertices lie along the octagon in a path with 4 adjacent pairs, so there are $\binom{5}{2} - 4 = 6$ ways to choose two that are not adjacent. This counts each triangle 3 times, once for each of its vertices, so $8 \cdot 6/3 = 16$ triangles have no side on the octagon. The other $56 - 16 = 40$ triangles have at least one, so the probability is $\frac{40}{56} = \frac{5}{7}$.

Answer: (D) $\frac{5}{7}$.

Problem 6 (2017 AMC 8, Problem #15)

Problem 6 (2017 AMC 8, Problem #15)

In the arrangement of letters and numerals below, by how many different paths can one spell AMC8? Beginning at the *A* in the middle, a path allows only moves from one letter to an adjacent (above, below, left, or right, but not diagonal) letter. One example of such a path is traced in the picture.



Problem 6: Solution

Strategy

Build a path one letter at a time and multiply the numbers of choices.

Solution

From the A , the path moves to one of the 4 adjacent copies of M . Each M is adjacent to exactly 3 copies of C , and each C is adjacent to exactly 2 copies of 8 . A path is determined by these three choices, and different choices give different paths, so by the multiplication principle there are $4 \cdot 3 \cdot 2 = 24$ paths.

Answer: 24.