AREAS OF TRIANGLES AND QUADRILATERALS FOR AMC 8

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1. Rectangles and Triangles

This section defines area and finds the areas of rectangles and triangles. Most other areas in this handout are found by adding or subtracting the areas of these two shapes.

1.1. Rectangles and Squares.

Definition 1.1.1. A **unit square** is a square with side 1. The **area** of a region is the number of unit squares needed to cover it without overlapping. When lengths are measured in inches, areas are measured in **square inches**, and in the same way for other units.

Proposition 1.1.2 (Rectangles)**.** A rectangle with sides a and b has area ab. A square with side s has area s^2.

When a and b are whole numbers, the rectangle is covered by a rows of b unit squares each, so its area is ab. The formula holds for all positive lengths.

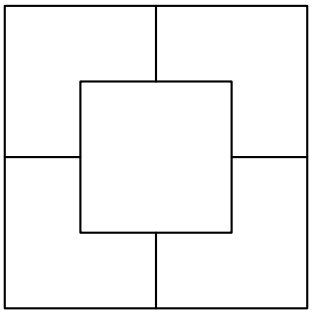
Proposition 1.1.3 (Adding and Subtracting Areas)**.**

(a) If a region is cut into pieces that do not overlap, its area is the sum of the areas of the pieces.
(b) If a region is removed from inside a larger region, the area of what remains is the area of the larger region minus the area of the removed region.

A figure whose sides run in two perpendicular directions can be cut into rectangles, or completed to a larger rectangle from which rectangles are removed.

Examples.

Problem 1.1.4 (1995 AJHSME, Problem #18)**.** The area of each of the four congruent L-shaped regions of this 100-inch by 100-inch square is $\frac{3}{16}$ of the total area. How many inches long is the side of the center square?



(A) 25 **(B)** 44 **(C)** 50 **(D)** 62 **(E)** 75

Strategy. As the figure shows, the four L-shaped regions and the center square fill the large square without overlapping. Find the area that is left for the center square.

Solution. The large square has area $100 \cdot 100 = 10000$ square inches. Each L-shaped region has area $\frac{3}{16} \cdot 10000 = 1875$, so the four of them have area $4 \cdot 1875 = 7500$, and the center square has area $10000 - 7500 = 2500$. A square with area 2500 has side 50, since $50^2 = 2500$.

Answer: (C) 50. □

Problem 1.1.5 (2012 AMC 8, Problem #6)**.** A rectangular photograph is placed in a frame that forms a border two inches wide on all sides of the photograph. The photograph measures 8 inches high and 10 inches wide. What is the area of the border, in square inches?

(A) 36 **(B)** 40 **(C)** 64 **(D)** 72 **(E)** 88

Strategy. The border is the outer rectangle of the frame with the photograph removed.

Solution. The border adds 2 inches on each side, so the outer rectangle is $8 + 2 + 2 = 12$ inches high and $10 + 2 + 2 = 14$ inches wide, with area $12 \cdot 14 = 168$ square inches. The photograph has area $8 \cdot 10 = 80$ square inches. So the border has area $168 - 80 = 88$ square inches. As a check, the border splits into two 14×2 strips along the top and bottom and two 2×8 strips along the sides, with total area $56 + 32 = 88$.

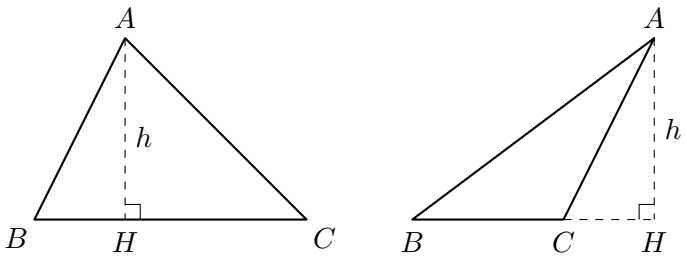
Answer: (E) 88. □

1.2. **The Area of a Triangle.** An altitude of a triangle is the segment from a vertex perpendicular to the line through the opposite side, and its length is the height of the triangle for that side as the base. Any of the three sides can be the base. The endpoint of the altitude on the line through the base is its **foot**; when the triangle has an obtuse angle at an end of the base, the foot lies outside the base. The area of a polygon is written with its vertices in **square brackets**: $[ABC]$ is the area of triangle ABC, and $[ABCD]$ is the area of quadrilateral $ABCD$.

Theorem 1.2.1 (Area of a Triangle)**.** *For every side of a triangle,*

$$\text{area} = \frac{1}{2} \cdot \text{base} \cdot \text{height},$$

where the base is the length of that side and the height is the distance from the opposite vertex to the line through that side.



Proof. Take BC as the base, let H be the foot of the perpendicular from A to the line BC, and let $h = AH$.

- If $H = C$, the angle at C is a right angle. Complete the rectangle with sides CB and CA. Its diagonal AB cuts it into two right triangles with the same three sides. Such triangles are congruent, so they have equal areas, and $[ABC]$ is half of the rectangle: $[ABC] = \frac{1}{2} \cdot BC \cdot h$. The case $H = B$ is the same.
- If H lies between B and C, as on the left, the segment AH cuts the triangle into the right triangles ABH and AHC, so
$$[ABC] = \tfrac{1}{2} \cdot BH \cdot h + \tfrac{1}{2} \cdot HC \cdot h = \tfrac{1}{2}(BH + HC)h = \tfrac{1}{2} \cdot BC \cdot h.$$
- If H lies on the line BC beyond C, as on the right, triangle ABC is triangle ABH with triangle ACH removed, so
$$[ABC] = \tfrac{1}{2} \cdot BH \cdot h - \tfrac{1}{2} \cdot CH \cdot h = \tfrac{1}{2}(BH - CH)h = \tfrac{1}{2} \cdot BC \cdot h.$$
The case of H beyond B is the same, with B and C exchanged. □

Corollary 1.2.2 (Right Triangles)**.** *The area of a right triangle is half the product of its legs.*

Proof. Each leg is the height for the other leg as the base. □

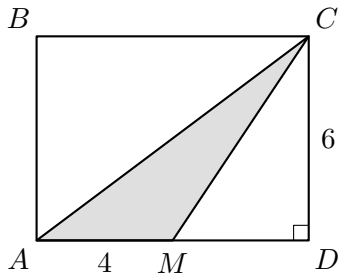
Examples.

Problem 1.2.3 (2016 AMC 8, Problem #2)**.** In rectangle $ABCD$, $AB = 6$ and $AD = 8$. Point M is the midpoint of $\overline{AD}$. What is the area of $\triangle AMC$?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 20 **(E)** 24

Strategy. Take AM as the base. The height is the distance from C to the line AD, and the side CD of the rectangle is perpendicular to that line.

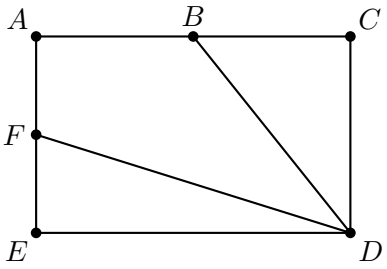
Solution. Since M is the midpoint of $\overline{AD}$, $AM = 4$. Take AM as the base of triangle AMC. Its height is the distance from C to the line AD. The side CD of the rectangle is perpendicular to AD, so this distance is $CD = AB = 6$, and the foot of the altitude is D, outside the segment AM.



Hence $[AMC] = \frac{1}{2} \cdot 4 \cdot 6 = 12$. As a check, triangle AMC is triangle ACD with triangle MCD removed: $\frac{1}{2} \cdot 8 \cdot 6 - \frac{1}{2} \cdot 4 \cdot 6 = 24 - 12 = 12$.

Answer: (A) 12. □

Problem 1.2.4 (1993 AJHSME, Problem #18)**.** The rectangle shown has length $AC = 32$, width $AE = 20$, and B and F are midpoints of $\overline{AC}$ and $\overline{AE}$, respectively. The area of the quadrilateral $ABDF$ is



(A) 320 **(B)** 325 **(C)** 330 **(D)** 335 **(E)** 340

Strategy. The segment AD cuts $ABDF$ into two triangles. For each, take the side that lies along a side of the rectangle as the base.

Solution. In the figure, D is the vertex opposite A, so $CD = AE = 20$ and $ED = AC = 32$. Since B and F are midpoints, $AB = 16$ and $AF = 10$. The segment AD cuts the quadrilateral into triangles ABD and AFD. In triangle ABD, take AB as the base: the height is the distance from D to the line AC, which is $CD = 20$, and the foot of this altitude is C, outside the segment AB. So $[ABD] = \frac{1}{2} \cdot 16 \cdot 20 = 160$. In triangle AFD, take AF as the base: the height is the distance from D to the line AE, which is $ED = 32$, and the foot is E. So $[AFD] = \frac{1}{2} \cdot 10 \cdot 32 = 160$. Hence $[ABDF] = 160 + 160 = 320$.

Answer: (A) 320. □

2. Quadrilaterals

This section finds the areas of parallelograms, trapezoids, rhombi and kites by cutting each of them into two triangles along a diagonal.

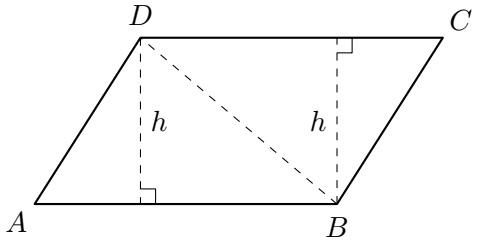
2.1. **Parallelograms.**

Definition 2.1.1. A **parallelogram** is a quadrilateral whose opposite sides are parallel. A rectangle is a parallelogram with four right angles. A **rhombus** is a quadrilateral whose four sides are equal.

Two basic properties are used without proof: the opposite sides of a parallelogram are equal, and every rhombus is a parallelogram.

Either pair of opposite sides of a parallelogram can serve as the bases. For a chosen base, the **height** is the distance between the line through that side and the line through the opposite side. A parallelogram therefore has two heights, one for each pair of sides.

Theorem 2.1.2 (Area of a Parallelogram)**.** *The area of a parallelogram is base times height.*



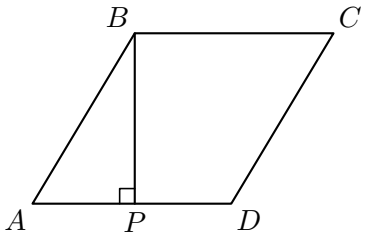
Proof. Let $ABCD$ be a parallelogram with base AB and height h. The diagonal BD cuts it into triangles ABD and CDB. In triangle ABD the vertex D lies on the line CD, at distance h from the line AB, so $[ABD] = \frac{1}{2} \cdot AB \cdot h$. In triangle CDB the base CD equals AB, and the vertex B lies on the line AB, at distance h from the line CD, so $[CDB] = \frac{1}{2} \cdot AB \cdot h$. Adding, $[ABCD] = AB \cdot h$. □

Corollary 2.1.3. *Each diagonal cuts a parallelogram into two triangles of equal area, each half of the parallelogram.*

Proof. The proof of the theorem shows this for the diagonal BD, and the same argument works for AC. □

Example.

Problem 2.1.4 (2022 AMC 10B, Problem #2)**.** In rhombus $ABCD$, point P lies on segment $\overline{AD}$ so that $\overline{BP} \perp \overline{AD}$, $AP = 3$, and $PD = 2$. What is the area of $ABCD$?



(A) $3\sqrt{5}$ **(B)** 10 **(C)** $6\sqrt{5}$ **(D)** 20 **(E)** 25

Strategy. A rhombus is a parallelogram. With AD as the base, BP is the height.

Solution. A rhombus has four equal sides, and $AD = AP + PD = 3 + 2 = 5$, so $AB = 5$. In the right triangle APB,

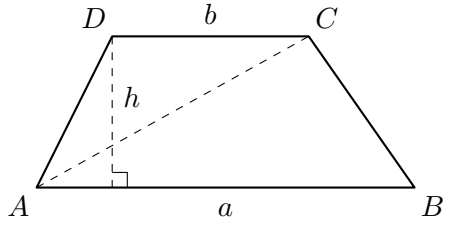
$$BP = \sqrt{AB^2 - AP^2} = \sqrt{25 - 9} = 4.$$

The rhombus is a parallelogram, so its height for the base AD is the distance between the lines AD and BC. The segment BP is perpendicular to AD and B lies on the line BC, so this height is BP. The area is $AD \cdot BP = 5 \cdot 4 = 20$.

Answer: (D) 20. □

2.2. **Trapezoids.** A trapezoid is a quadrilateral with a pair of parallel sides, called its bases, and its height, or altitude, is the distance between the bases.

Theorem 2.2.1 (Area of a Trapezoid)**.** *A trapezoid with bases a and b and height h has area $\frac{1}{2}(a+b)h$.*



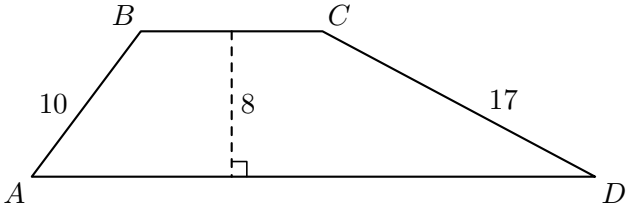
Proof. Let $ABCD$ have bases $AB = a$ and $DC = b$. The diagonal AC cuts it into triangles ABC and ACD. Triangle ABC has base AB and height h, because C lies on the line DC, at distance h from the line AB. Triangle ACD has base DC and height h, because A lies on the line AB. So

$$[ABCD] = \tfrac{1}{2}ah + \tfrac{1}{2}bh = \tfrac{1}{2}(a+b)h. \qquad \square$$

So the area of a trapezoid is the average of the two bases times the height. When the height or a base is missing, the perpendiculars from the ends of the shorter base often give it.

Example.

Problem 2.2.2 (2003 AMC 8, Problem #21)**.** The area of trapezoid $ABCD$ is 164 cm^2. The altitude is 8 cm, AB is 10 cm, and CD is 17 cm. What is BC, in centimeters?



(A) 9 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 20

Strategy. Drop perpendiculars from B and C to AD. The right triangles they cut off give the two end pieces of AD. Then use the trapezoid formula.

Solution. In the figure, BC and AD are the bases. Let the perpendiculars from B and C meet AD at P and Q. Each has length 8, and $BCQP$ is a rectangle, so $PQ = BC$. In the right triangles APB and DQC,

$$AP = \sqrt{10^2 - 8^2} = \sqrt{36} = 6, \qquad QD = \sqrt{17^2 - 8^2} = \sqrt{225} = 15.$$

In the figure, the angles at A and D are acute, so P and Q lie between A and D, and $AD = AP + PQ + QD = 6 + BC + 15 = BC + 21$. By the trapezoid formula,

$$164 = \tfrac{1}{2}(BC + AD) \cdot 8 = 4(2\,BC + 21),$$

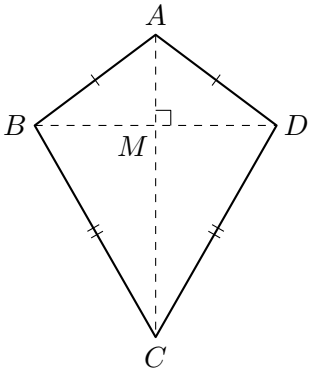
so $2\,BC + 21 = 41$ and $BC = 10$. As a check, $AD = 31$ and $\frac{1}{2}(10 + 31) \cdot 8 = 164$.

Answer: (B) 10. $\square$

2.3. **Rhombi and Kites.**

Definition 2.3.1. A **kite** is a quadrilateral with two pairs of equal adjacent sides: in kite $ABCD$, $AB = AD$ and $CB = CD$. All kites in this handout are convex. A rhombus $ABCD$ is a kite in two ways, since $AB = AD$ and $CB = CD$, and also $BA = BC$ and $DA = DC$.

Theorem 2.3.2 (Diagonals of a Kite)**.** *In a kite $ABCD$ with $AB = AD$ and $CB = CD$, the diagonal AC is perpendicular to BD and passes through the midpoint of BD. The diagonals of a rhombus are perpendicular and bisect each other.*



Proof. Let M be the midpoint of BD. Triangle ABD is isosceles with $AB = AD$, so by the Isosceles Triangle Theorem the bisector of $\angle BAD$ meets BD at its midpoint, which is M, and is perpendicular to BD; that is, $AM \perp BD$. In the same way, in the isosceles triangle CBD, $CM \perp BD$. Only one line through M is perpendicular to BD, so A, M and C lie on it: the diagonal AC passes through M and is perpendicular to BD. In a rhombus the same argument applies to both pairs of equal sides, so each diagonal passes through the midpoint of the other. □

Theorem 2.3.3 (Area from the Diagonals)**.** *The area of a kite or a rhombus is half the product of its diagonals.*

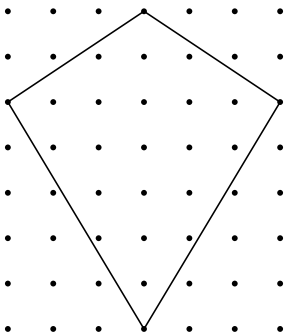
Proof. The diagonals meet at M, which lies between A and C because the kite is convex. The diagonal BD cuts the kite into triangles ABD and CBD. Since $AC \perp BD$, their heights for the base BD are AM and CM, so

$$[ABCD] = \tfrac{1}{2} \cdot BD \cdot AM + \tfrac{1}{2} \cdot BD \cdot CM = \tfrac{1}{2} \cdot BD \cdot AC.$$ □

Examples.

Problem 2.3.4 (Adapted from 2001 AMC 8, Problem #7)**.** To promote her school's annual Kite Olympics, Genevieve makes a small kite and a large kite for a bulletin board display. The kites look like the one in the diagram. For her small kite Genevieve draws the kite on a one-inch grid. For the large kite she triples both the height and width of the entire grid.

What is the number of square inches in the area of the small kite?



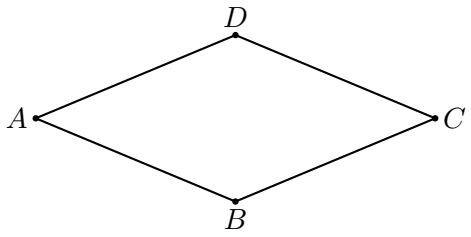
(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

Strategy. The diagonals of the kite run along a row and a column of the grid. Count their lengths on the grid.

Solution. In the figure, the dots of the one-inch grid are 1 inch apart. The left and right vertices of the small kite lie in the same row of dots, 6 inches apart, and its top and bottom vertices lie in the same column, 7 inches apart. So the diagonals of the small kite are horizontal and vertical, hence perpendicular, and have lengths 6 and 7 inches. The area of a kite is half the product of its diagonals, so the small kite has area $\frac{1}{2} \cdot 6 \cdot 7 = 21$ square inches.

Answer: (A) 21. □

Problem 2.3.5 (2019 AMC 8, Problem #4)**.** Quadrilateral $ABCD$ is a rhombus with perimeter 52 meters. The length of diagonal $\overline{AC}$ is 24 meters. What is the area in square meters of rhombus $ABCD$?



(A) 60 **(B)** 90 **(C)** 105 **(D)** 120 **(E)** 144

Strategy. The diagonals of a rhombus are perpendicular and bisect each other, so half of each diagonal and a side form a right triangle.

Solution. The four sides are equal, so each has length $\frac{52}{4} = 13$. Let the diagonals meet at O. They bisect each other at right angles, so $AO = 12$ and triangle AOB has a right angle at O. Then

$$BO = \sqrt{13^2 - 12^2} = \sqrt{25} = 5,$$

so $BD = 10$, and the area is $\frac{1}{2} \cdot AC \cdot BD = \frac{1}{2} \cdot 24 \cdot 10 = 120$.

Answer: (D) 120. □

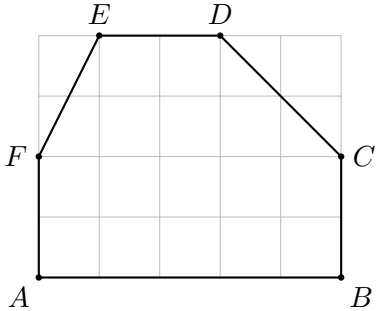
3. Areas on a Grid

On a grid of unit squares, lengths along grid lines can be counted, so a figure drawn on the grid can be measured by adding pieces or by subtracting them from a rectangle.

3.1. **Splitting into Pieces.** Cut the figure along grid lines into whole unit squares, rectangles and right triangles whose legs lie along grid lines, and add their areas. A right triangle with legs a and b along the grid has area $\frac{1}{2}ab$; in particular, a diagonal cuts a unit square into two triangles of area $\frac{1}{2}$ each.

Example.

Problem 3.1.1. Hexagon $ABCDEF$ is drawn on a grid of unit squares. Find its area.



Strategy. Cut the hexagon along grid lines into rectangles and right triangles whose legs lie along grid lines.

Solution. The horizontal grid line through F and C cuts the hexagon into the 5×2 rectangle $ABCF$ and the part above it. The vertical grid lines through E and D cut that part into a right triangle with legs 1 and 2 next to F, a 2×2 square, and a right triangle with legs 2 and 2 next to C. So

$$[ABCDEF] = 5 \cdot 2 + \tfrac{1}{2} \cdot 1 \cdot 2 + 2 \cdot 2 + \tfrac{1}{2} \cdot 2 \cdot 2 = 10 + 1 + 4 + 2 = 17.$$

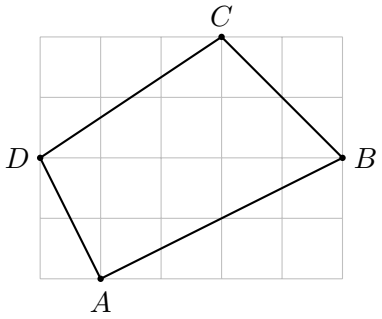
As a check, the part above FC is a trapezoid with bases $FC = 5$ and $ED = 2$ and height 2, with area $\frac{1}{2}(5 + 2) \cdot 2 = 7$, and $10 + 7 = 17$.

Answer: 17. □

3.2. **Subtracting from a Rectangle.** When most sides of a figure are slanted, draw the smallest rectangle with sides along grid lines that contains the figure. If every vertex of the figure lies on a side of this rectangle, the parts of the rectangle outside the figure are right triangles with legs along grid lines. The area of the figure is the area of the rectangle minus the areas of these parts. The same method works without a grid, whenever a figure lies inside a rectangle and the parts outside it are right triangles with legs along the sides of the rectangle.

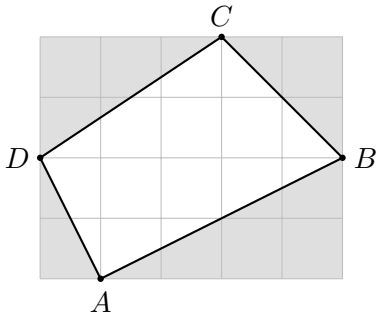
Examples.

Problem 3.2.1. The vertices of quadrilateral $ABCD$ are grid points of the grid of unit squares shown. Find the area of $ABCD$.



Strategy. The quadrilateral lies in the 5×4 rectangle formed by the grid. Subtract the four right triangles at the corners of this rectangle.

Solution. The rectangle formed by the grid has area $5 \cdot 4 = 20$, and each vertex of $ABCD$ lies on one of its sides. So the part of the rectangle outside $ABCD$ consists of four right triangles, one at each corner, with legs along grid lines. They are shaded below.

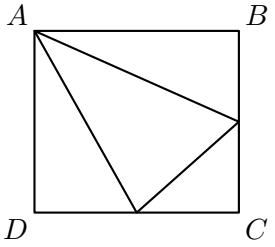


- At the lower left corner the legs are 1 and 2, and the area is 1.
- At the lower right corner the legs are 4 and 2, and the area is 4.
- At the upper right corner the legs are 2 and 2, and the area is 2.
- At the upper left corner the legs are 3 and 2, and the area is 3.

Hence $[ABCD] = 20 - (1 + 4 + 2 + 3) = 10$.

Answer: 10. □

Problem 3.2.2 (2000 AMC 8, Problem #25)**.** The area of rectangle $ABCD$ is 72 square units. If point A and the midpoints of $\overline{BC}$ and $\overline{CD}$ are joined to form a triangle, the area of that triangle is



(A) 21 **(B)** 27 **(C)** 30 **(D)** 36 **(E)** 40

Strategy. The triangle leaves three right triangles in the corners of the rectangle. Each is a fixed fraction of the rectangle, even though its sides are unknown.

Solution. Let M and N be the midpoints of $\overline{BC}$ and $\overline{CD}$, and let $AB = CD = a$ and $BC = AD = b$, so $ab = 72$. The rectangle is triangle AMN together with three right triangles:

$$[ABM] = \tfrac{1}{2} \cdot a \cdot \tfrac{b}{2} = \tfrac{1}{4}ab = 18, \quad [MCN] = \tfrac{1}{2} \cdot \tfrac{b}{2} \cdot \tfrac{a}{2} = \tfrac{1}{8}ab = 9, \quad [NDA] = \tfrac{1}{2} \cdot \tfrac{a}{2} \cdot b = \tfrac{1}{4}ab = 18.$$

Hence $[AMN] = 72 - 18 - 9 - 18 = 27$.

Answer: (B) 27. □

4. Comparing Areas

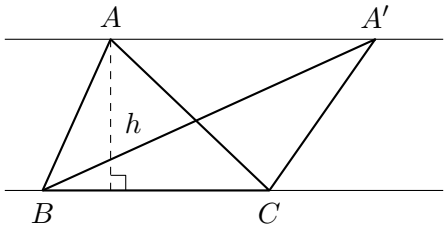
Two triangles that share a base or a height can be compared without computing either area. This section also finds lengths by computing one area in two ways.

4.1. **Equal Bases and Equal Heights.**

Theorem 4.1.1. *Two triangles with equal bases and equal heights have equal areas. In particular:*

(a) if A and A' lie on a line parallel to BC, then $[ABC] = [A'BC]$, since both heights equal the distance between the two parallel lines;

(b) if M is the midpoint of BC, then $[ABM] = [AMC]$, since the bases BM and MC are equal and both triangles have the height from A to the line BC.



Proof. In each case both areas equal $\frac{1}{2} \cdot \text{base} \cdot \text{height}$ with the same two numbers. □

In the figure, triangle $A'BC$ has an obtuse angle at C, so the foot of its altitude from A' lies outside BC. Its height is still h, the distance between the two parallel lines.

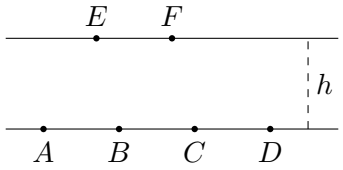
Example.

Problem 4.1.2 (2009 AMC 10B, Problem #12)**.** Distinct points A, B, C, and D lie on a line, with $AB = BC = CD = 1$. Points E and F lie on a second line, parallel to the first, with $EF = 1$. A triangle with positive area has three of the six points as its vertices. How many possible values are there for the area of the triangle?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Strategy. Each triangle has two vertices on one line and one on the other. With the side on that line as the base, all the triangles have the same height, so the number of different areas equals the number of different base lengths.

Solution. Since the points are distinct, B lies between A and C, and C lies between B and D, so the four points lie in the order A, B, C, D, one unit apart. Let h be the distance between the two lines.



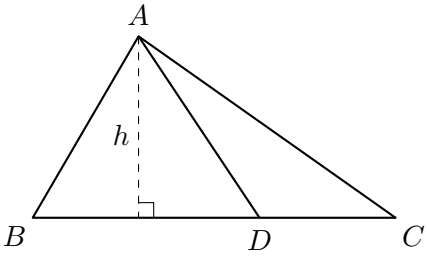
Three points on one line do not form a triangle with positive area, so every triangle has two vertices on one line and its third vertex on the other. Take the side joining the two vertices on the same line as the base. The height is h in every case, so the area is $\frac{1}{2}h$ times the base. On the first line the base is one of AB, BC, CD, AC, BD, AD, with length 1, 2 or 3; on the second line it is EF, with length 1. So the possible areas are $\frac{1}{2}h$, h and $\frac{3}{2}h$, three different values.

Answer: (A) 3. □

4.2. **Triangles with the Same Height.**

Theorem 4.2.1. *If two triangles have the same height, the ratio of their areas equals the ratio of their bases. In particular, if D lies on side BC of triangle ABC, then*

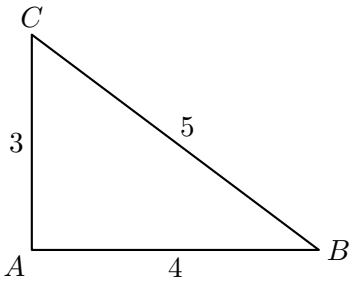
$$[ABD] : [ADC] = BD : DC \qquad \text{and} \qquad [ABD] = \frac{BD}{BC} \cdot [ABC].$$



Proof. Triangles ABD, ADC and ABC all have the height h from A to the line BC, so their areas are $\frac{1}{2} \cdot BD \cdot h$, $\frac{1}{2} \cdot DC \cdot h$ and $\frac{1}{2} \cdot BC \cdot h$, which are in the ratio of the bases. □

Example.

Problem 4.2.2 (2017 AMC 8, Problem #16)**.** In the figure shown below, choose point D on side BC so that $\triangle ACD$ and $\triangle ABD$ have equal perimeters. What is the area of $\triangle ABD$?



(A) $\frac{3}{4}$ **(B)** $\frac{3}{2}$ **(C)** 2 **(D)** $\frac{12}{5}$ **(E)** $\frac{5}{2}$

Strategy. Equal perimeters fix BD. Then compare triangle ABD with triangle ABC, which has the same height from A.

Solution. In the figure, $AB = 4$, $AC = 3$ and $BC = 5$. Since $3^2 + 4^2 = 5^2$, the angle at A is a right angle by the converse of the Pythagorean theorem, so $[ABC] = \frac{1}{2} \cdot 3 \cdot 4 = 6$. The two triangles share the side AD, so their perimeters are equal exactly when $AB + BD = AC + CD$, that is, $4 + BD = 3 + CD$. Since $BD + CD = 5$, this gives $4 + BD = 3 + (5 - BD)$, so $BD = 2$. Triangles ABD and ABC have the same height, the distance from A to the line BC, so

$$[ABD] = \frac{BD}{BC} \cdot [ABC] = \frac{2}{5} \cdot 6 = \frac{12}{5}.$$

Answer: (D) $\frac{12}{5}$. □

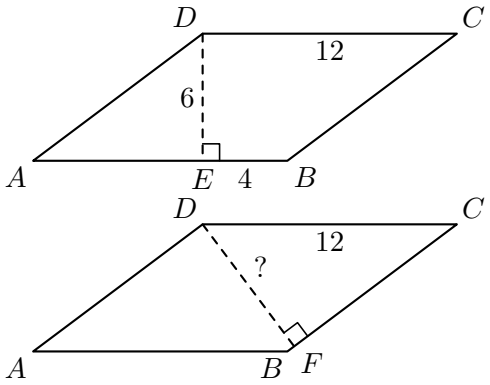
4.3. **Lengths from Areas.** An area formula can be solved for a length. A triangle with base b and area K has height $\frac{2K}{b}$, and a parallelogram with base b and area K has height $\frac{K}{b}$. When the same figure is measured with two different bases, the two expressions for its area are equal. For a triangle with sides a, b, c and heights h_a, h_b, h_c to those sides,

$$ah_a = bh_b = ch_c,$$

each equal to twice the area. For a parallelogram with sides a and b and heights h_a and h_b to them, $ah_a = bh_b$.

Example.

Problem 4.3.1 (1995 AJHSME, Problem #24)**.** In parallelogram $ABCD$, $\overline{DE}$ is the altitude to the base $\overline{AB}$ and $\overline{DF}$ is the altitude to the base $\overline{BC}$. (Both pictures represent the same parallelogram.) If $DC = 12$, $EB = 4$, and $DE = 6$, then $DF =$



(A) 6.4 **(B)** 7 **(C)** 7.2 **(D)** 8 **(E)** 10

Strategy. Compute the area of the parallelogram twice, once with base AB and once with base BC.

Solution. Opposite sides of a parallelogram are equal, so $AB = DC = 12$. In the figure, E lies between A and B, so $AE = AB - EB = 12 - 4 = 8$. In the right triangle AED,

$$AD = \sqrt{8^2 + 6^2} = \sqrt{100} = 10,$$

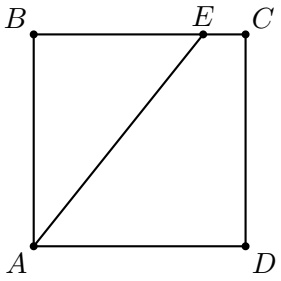
so $BC = AD = 10$. With base AB and height DE the area is $12 \cdot 6 = 72$; with base BC and height DF it is $10 \cdot DF$. Hence $10 \cdot DF = 72$ and $DF = 7.2$.

Answer: (C) 7.2. □

5. Problems

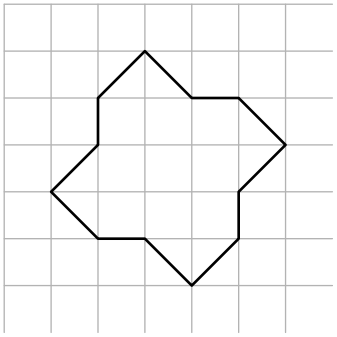
Beginner.

1. (2013 AMC 10A, Problem #3) Square $ABCD$ has side length 10. Point E is on $\overline{BC}$, and the area of $\triangle ABE$ is 40. What is BE?



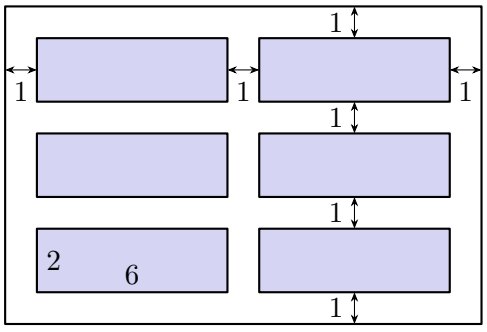
(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

2. (2018 AMC 8, Problem #4) The twelve-sided figure shown has been drawn on 1 cm $\times$ 1 cm graph paper. What is the area of the figure in cm^2?



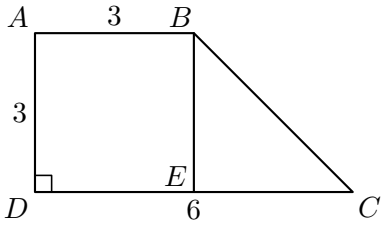
(A) 12 **(B)** 12.5 **(C)** 13 **(D)** 13.5 **(E)** 14

3. (2017 AMC 10A, Problem #3) Tamara has three rows of two 6-feet by 2-feet flower beds in her garden. The beds are separated and also surrounded by 1-foot-wide walkways, as shown on the diagram. What is the total area of the walkways, in square feet?



(A) 72 **(B)** 78 **(C)** 90 **(D)** 120 **(E)** 150

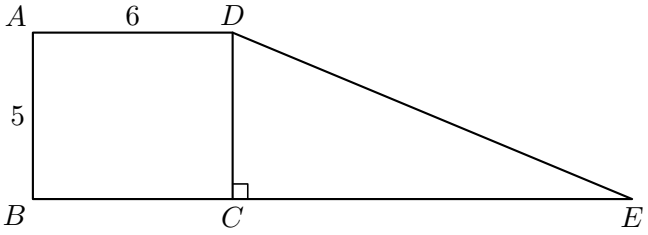
4. (2007 AMC 8, Problem #8) In trapezoid $ABCD$, $\overline{AD}$ is perpendicular to $\overline{DC}$, $AD = AB = 3$, and $DC = 6$. In addition, E is on $\overline{DC}$, and $\overline{BE}$ is parallel to $\overline{AD}$. Find the area of $\triangle BEC$.



(A) 3 **(B)** 4.5 **(C)** 6 **(D)** 9 **(E)** 18

Intermediate.

5. (2014 AMC 8, Problem #14) Rectangle $ABCD$ and right triangle DCE have the same area. They are joined to form a trapezoid, as shown. What is DE?



(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

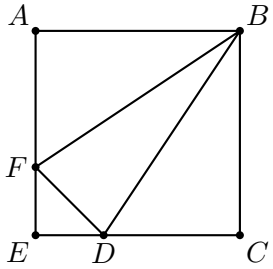
6. (2007 AMC 10B, Problem #10) Two points B and C are in a plane. Let S be the set of all points A in the plane for which $\triangle ABC$ has area 1. Which of the following describes S?

(A) two parallel lines **(B)** a parabola **(C)** a circle **(D)** a line segment **(E)** two points

7. (2001 AMC 10, Problem #15) A street has parallel curbs 40 feet apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet and each stripe is 50 feet long. Find the distance, in feet, between the stripes.

(A) 9 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 25

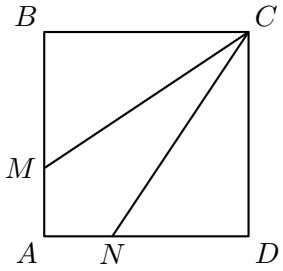
8. (2008 AMC 8, Problem #23) In square $ABCE$, $AF = 2FE$ and $CD = 2DE$. What is the ratio of the area of $\triangle BFD$ to the area of square $ABCE$?



(A) $\frac{1}{6}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{5}{18}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{7}{20}$

Advanced.

9. (1999 AMC 8, Problem #23) Square $ABCD$ has sides of length 3. Segments CM and CN divide the square's area into three equal parts. How long is segment CM?



(A) $\sqrt{10}$ **(B)** $\sqrt{12}$ **(C)** $\sqrt{13}$ **(D)** $\sqrt{14}$ **(E)** $\sqrt{15}$

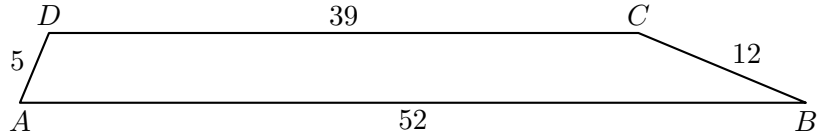
10. (1961 AHSME, Problem #14) A rhombus is given with one diagonal twice the length of the other diagonal. Express the side of the rhombus in terms of K, where K is the area of the rhombus in square inches.

(A) $\sqrt{K}$ **(B)** $\frac{1}{2}\sqrt{2K}$ **(C)** $\frac{1}{3}\sqrt{3K}$ **(D)** $\frac{1}{4}\sqrt{4K}$
(E) none of these are correct

11. (1964 AHSME, Problem #22) Given parallelogram $ABCD$ with E the midpoint of diagonal BD. Point E is connected to a point F in DA so that $DF = \frac{1}{3}DA$. What is the ratio of the area of triangle DFE to the area of quadrilateral $ABEF$?

(A) 1 : 2 **(B)** 1 : 3 **(C)** 1 : 5 **(D)** 1 : 6 **(E)** 1 : 7

12. (2002 AMC 10A, Problem #25) In trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{CD}$, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$. The area of $ABCD$ is



(A) 182 **(B)** 195 **(C)** 210 **(D)** 234 **(E)** 260