AREAS OF TRIANGLES AND QUADRILATERALS FOR AMC 8: SOLUTIONS

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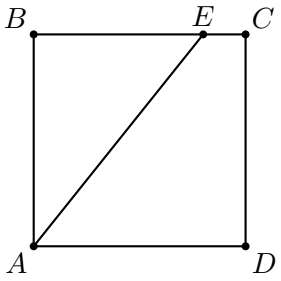
1. Solutions

Beginner.

1. (2013 AMC 10A, Problem #3) Square $ABCD$ has side length 10. Point E is on $\overline{BC}$, and the area of $\triangle ABE$ is 40. What is BE?



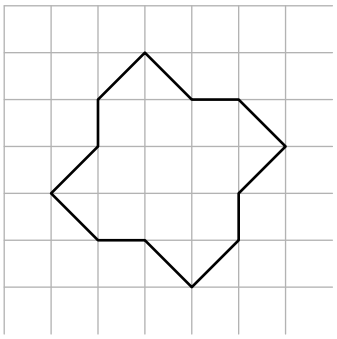
(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Strategy. Triangle ABE has a right angle at B. Its legs are a base and a height.

Solution. The angle ABE is an angle of the square, so it is a right angle, and triangle ABE has legs $AB = 10$ and BE. Hence $\frac{1}{2} \cdot 10 \cdot BE = 40$, so $5 \cdot BE = 40$ and $BE = 8$.

Answer: (E) 8. □

2. (2018 AMC 8, Problem #4) The twelve-sided figure shown has been drawn on 1 cm $\times$ 1 cm graph paper. What is the area of the figure in cm^2?



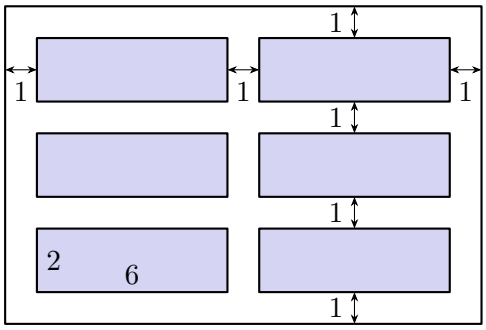
(A) 12 **(B)** 12.5 **(C)** 13 **(D)** 13.5 **(E)** 14

Strategy. Count the whole unit squares, then the half squares cut off by the slanted sides.

Solution. The figure contains a 3×3 block of whole unit squares, with area 9. Outside the block it has four triangles, one on each side of the block. A grid line cuts each of these triangles into two pieces, and each piece is half of a unit square, cut off by a diagonal of that square. So the rest of the figure consists of 8 half squares, with total area $8 \cdot \frac{1}{2} = 4$, and the area of the figure is $9 + 4 = 13$ square centimeters.

Answer: (C) 13. □

3. (2017 AMC 10A, Problem #3) Tamara has three rows of two 6-feet by 2-feet flower beds in her garden. The beds are separated and also surrounded by 1-foot-wide walkways, as shown on the diagram. What is the total area of the walkways, in square feet?



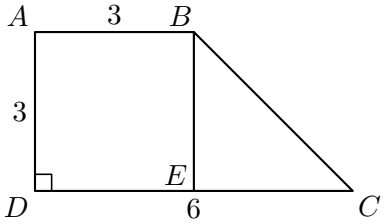
(A) 72 **(B)** 78 **(C)** 90 **(D)** 120 **(E)** 150

Strategy. Find the size of the whole garden, then subtract the beds.

Solution. As the diagram shows, each row has two beds side by side, with their 6-foot sides along the row, and the three rows are stacked. Across the garden there are three walkways and two beds: $1 + 6 + 1 + 6 + 1 = 15$ feet. From top to bottom there are four walkways and three beds: $1 + 2 + 1 + 2 + 1 + 2 + 1 = 10$ feet. The garden has area $15 \cdot 10 = 150$ square feet and the six beds have area $6 \cdot (6 \cdot 2) = 72$ square feet, so the walkways cover $150 - 72 = 78$ square feet.

Answer: (B) 78. □

4. (2007 AMC 8, Problem #8) In trapezoid $ABCD$, $\overline{AD}$ is perpendicular to $\overline{DC}$, $AD = AB = 3$, and $DC = 6$. In addition, E is on $\overline{DC}$, and $\overline{BE}$ is parallel to $\overline{AD}$. Find the area of $\triangle BEC$.



(A) 3 **(B)** 4.5 **(C)** 6 **(D)** 9 **(E)** 18

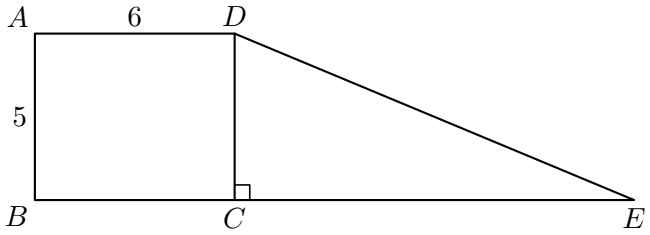
Strategy. The quadrilateral $ABED$ is a rectangle, which gives BE and EC.

Solution. In the figure, $AB \parallel DC$. Since $\overline{BE} \parallel \overline{AD}$, the quadrilateral $ABED$ is a parallelogram, and its angle at D is a right angle. Same-side interior angles along parallel lines are supplementary, so its other three angles are right angles too, and $ABED$ is a rectangle. Hence $BE = AD = 3$ and $DE = AB = 3$, so $EC = DC - DE = 6 - 3 = 3$. Triangle BEC has a right angle at E and legs $BE = 3$ and $EC = 3$, so its area is $\frac{1}{2} \cdot 3 \cdot 3 = 4.5$.

Answer: (B) 4.5. □

Intermediate.

5. (2014 AMC 8, Problem #14) Rectangle $ABCD$ and right triangle DCE have the same area. They are joined to form a trapezoid, as shown. What is DE?



(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Strategy. The rectangle's area is also the triangle's area. One leg of the triangle is a side of the rectangle, which gives the other leg.

Solution. In the figure, $AB = 5$ and $AD = 6$, so the rectangle has area $5 \cdot 6 = 30$. Triangle DCE has its right angle at C and legs $DC = AB = 5$ and CE, and its area is also 30: $\frac{1}{2} \cdot 5 \cdot CE = 30$, so $CE = 12$. By the Pythagorean theorem,

$$DE = \sqrt{5^2 + 12^2} = \sqrt{169} = 13.$$

Answer: (B) 13. □

6. (2007 AMC 10B, Problem #10) Two points B and C are in a plane. Let S be the set of all points A in the plane for which $\triangle ABC$ has area 1. Which of the following describes S?

(A) two parallel lines **(B)** a parabola **(C)** a circle **(D)** a line segment **(E)** two points

Strategy. The segment BC is fixed, so the area of triangle ABC is determined by how far A is from the line BC.

Solution. For a point A, let h be its distance from the line BC. With BC as the base, $[ABC] = \frac{1}{2} \cdot BC \cdot h$, and this equals 1 exactly when $h = \frac{2}{BC}$, a fixed positive length. Draw the two lines that are parallel to BC at distance $\frac{2}{BC}$ from it, one on either side. A point is at distance $\frac{2}{BC}$ from the line BC exactly when it lies on one of these two lines. Hence S is a pair of parallel lines.

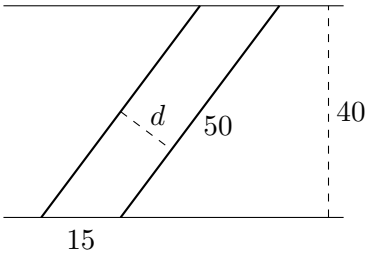
Answer: (A) two parallel lines. □

7. (2001 AMC 10, Problem #15) A street has parallel curbs 40 feet apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet and each stripe is 50 feet long. Find the distance, in feet, between the stripes.

(A) 9 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 25

Strategy. The crosswalk has two pairs of parallel sides. Find its area once with a curb segment as the base and once with a stripe as the base.

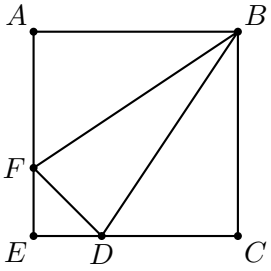
Solution. The two stripes are parallel, and so are the two curbs, so the crosswalk is a parallelogram with sides 15 along the curbs and 50 along the stripes, as in the figure.



With a curb segment as the base, the height is the distance between the curbs, 40, so the area is $15 \cdot 40 = 600$ square feet. With a stripe as the base, the height is the distance d between the stripes, so $50d = 600$ and $d = 12$.

Answer: (C) 12. □

8. (2008 AMC 8, Problem #23) In square $ABCE$, $AF = 2FE$ and $CD = 2DE$. What is the ratio of the area of $\triangle BFD$ to the area of square $ABCE$?



(A) $\frac{1}{6}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{5}{18}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{7}{20}$

Strategy. Subtract the three right triangles at the corners from the square. Call the side $3x$, so that every piece has sides that are multiples of x.

Solution. Let the side of the square be $3x$. Then $AF = 2x$, $FE = x$, $CD = 2x$ and $DE = x$. Outside triangle BFD the square contains three right triangles, with right angles at A, C and E:

$$[ABF] = \tfrac{1}{2} \cdot 3x \cdot 2x = 3x^2, \qquad [BCD] = \tfrac{1}{2} \cdot 3x \cdot 2x = 3x^2,$$
$$[FED] = \tfrac{1}{2} \cdot x \cdot x = \tfrac{1}{2}x^2.$$

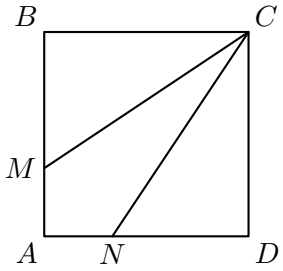
So $[BFD] = 9x^2 - 3x^2 - 3x^2 - \frac{1}{2}x^2 = \frac{5}{2}x^2$, and the ratio is

$$\frac{\frac{5}{2}x^2}{9x^2} = \frac{5}{18}.$$

Answer: (C) $\frac{5}{18}$. □

Advanced.

9. (1999 AMC 8, Problem #23) Square $ABCD$ has sides of length 3. Segments CM and CN divide the square's area into three equal parts. How long is segment CM?



(A) $\sqrt{10}$ **(B)** $\sqrt{12}$ **(C)** $\sqrt{13}$ **(D)** $\sqrt{14}$ **(E)** $\sqrt{15}$

Strategy. Each part has area 3. The part next to the side CB is a right triangle with one leg known.

Solution. The square has area 9, so each part has area 3. In the figure, M lies on $\overline{AB}$, and triangle CBM is one of the parts. It has a right angle at B and legs $CB = 3$ and BM, so $\frac{1}{2} \cdot 3 \cdot BM = 3$ and $BM = 2$. By the Pythagorean theorem,

$$CM = \sqrt{CB^2 + BM^2} = \sqrt{9 + 4} = \sqrt{13}.$$

Answer: (C) $\sqrt{13}$. □

10. (1961 AHSME, Problem #14) A rhombus is given with one diagonal twice the length of the other diagonal. Express the side of the rhombus in terms of K, where K is the area of the rhombus in square inches.

(A) $\sqrt{K}$ **(B)** $\frac{1}{2}\sqrt{2K}$ **(C)** $\frac{1}{3}\sqrt{3K}$ **(D)** $\frac{1}{4}\sqrt{4K}$
(E) none of these are correct

Strategy. Write both diagonals in terms of the shorter one. The area formula gives it in terms of K, and a right triangle gives the side.

Solution. Let the diagonals have lengths d and $2d$. The area of a rhombus is half the product of its diagonals, so $K = \frac{1}{2} \cdot d \cdot 2d = d^2$. The diagonals bisect each other at right angles, so each side s is the hypotenuse of a right triangle with legs $\frac{d}{2}$ and d:

$$s^2 = \left(\tfrac{d}{2}\right)^2 + d^2 = \tfrac{5}{4}d^2 = \tfrac{5}{4}K, \qquad s = \tfrac{1}{2}\sqrt{5K}.$$

None of the choices (A) to (D) equals $\frac{1}{2}\sqrt{5K}$: they are $\sqrt{K} = \frac{1}{2}\sqrt{4K}$, $\frac{1}{2}\sqrt{2K}$, $\frac{1}{3}\sqrt{3K} = \frac{1}{2}\sqrt{\frac{4}{3}K}$ and $\frac{1}{4}\sqrt{4K} = \frac{1}{2}\sqrt{K}$.

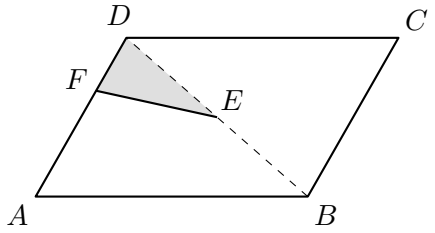
Answer: (E) none of these are correct. □

11. (1964 AHSME, Problem #22) Given parallelogram $ABCD$ with E the midpoint of diagonal BD. Point E is connected to a point F in DA so that $DF = \frac{1}{3}DA$. What is the ratio of the area of triangle DFE to the area of quadrilateral $ABEF$?

(A) 1 : 2 **(B)** 1 : 3 **(C)** 1 : 5 **(D)** 1 : 6 **(E)** 1 : 7

Strategy. Write every area as a fraction of the area P of the parallelogram. The diagonal BD halves the parallelogram; then use triangles with equal bases or with the same height.

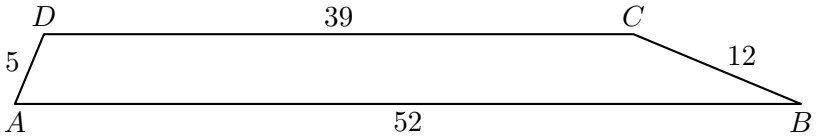
Solution. Let $P = [ABCD]$. The diagonal BD cuts the parallelogram into two triangles of equal area, so $[ABD] = \frac{1}{2}P$. Triangles ABE and ADE have equal bases $BE = ED$ on the line BD and the same height from A, so $[ADE] = \frac{1}{2}[ABD] = \frac{1}{4}P$. Triangles DFE and DAE have the same height from E to the line DA, and $DF = \frac{1}{3}DA$, so $[DFE] = \frac{1}{3}[DAE] = \frac{1}{12}P$.



The quadrilateral $ABEF$ is triangle ABD with triangle DFE removed, so $[ABEF] = \frac{1}{2}P - \frac{1}{12}P = \frac{5}{12}P$. The ratio is $\frac{1}{12}P : \frac{5}{12}P = 1 : 5$.

Answer: (C) 1 : 5. □

12. (2002 AMC 10A, Problem #25) In trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{CD}$, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$. The area of $ABCD$ is



(A) 182 **(B)** 195 **(C)** 210 **(D)** 234 **(E)** 260

Strategy. Draw through C the line parallel to DA. It cuts the trapezoid into a parallelogram and a triangle whose three sides are known.

Solution. Let the line through C parallel to $\overline{DA}$ meet $\overline{AB}$ at E. Then $AECD$ has two pairs of parallel sides, so it is a parallelogram, and $AE = DC = 39$, $EC = DA = 5$. Hence $EB = 52 - 39 = 13$. Triangle EBC has sides $EC = 5$, $CB = 12$ and $EB = 13$, and $5^2 + 12^2 = 13^2$, so by the converse of the Pythagorean theorem its angle at C is a right angle, and $[EBC] = \frac{1}{2} \cdot 5 \cdot 12 = 30$. With EB as the base, its height h satisfies $\frac{1}{2} \cdot 13 \cdot h = 30$, so $h = \frac{60}{13}$. This height is the distance from C to the line AB, which is the height of the trapezoid. Hence

$$[ABCD] = \tfrac{1}{2}(52 + 39) \cdot \tfrac{60}{13} = \tfrac{1}{2} \cdot 91 \cdot \tfrac{60}{13} = 210.$$

As a check, the parallelogram has area $39 \cdot \frac{60}{13} = 180$, and $180 + 30 = 210$.

Answer: (C) 210. □

2. Answer Key

Problem	Letter	Value
1	E	8
2	C	13
3	B	78
4	B	4.5
5	B	13
6	A	two parallel lines
7	C	12
8	C	$\frac{5}{18}$
9	C	$\sqrt{13}$
10	E	none of these are correct
11	C	$1:5$
12	C	210