AREAS OF TRIANGLES AND QUADRILATERALS FOR AMC 8: PRACTICE SET SOLUTIONS

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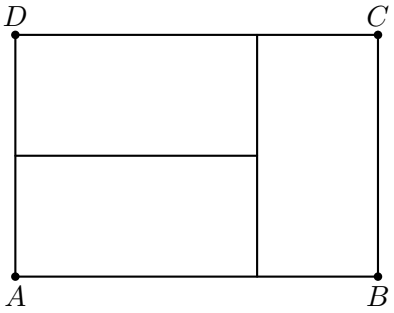
1. Solutions

Beginner.

1. (2019 AMC 8, Problem #2) Three identical rectangles are put together to form rectangle $ABCD$, as shown in the figure below. Given that the length of the shorter side of each of the smaller rectangles is 5 feet, what is the area in square feet of rectangle $ABCD$?



(A) 45 **(B)** 75 **(C)** 100 **(D)** 125 **(E)** 150

Strategy. The rectangle on the right is as tall as the two rectangles stacked on the left, so its long side is twice the short side. Then both sides of $ABCD$ are known.

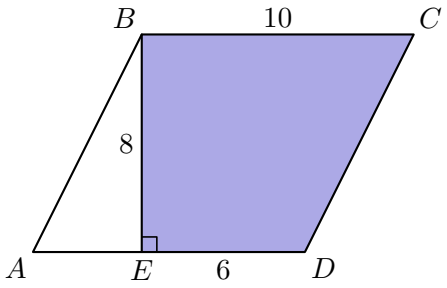
Solution. Let each small rectangle have shorter side 5 and longer side L. In the figure, the rectangle on the right stands on a short side, so it is L feet tall, and the two rectangles on the left lie on their long sides, one on top of the other, so together they are $5+5=10$ feet tall. The rectangle on the right and the stack of two rectangles on the left both reach from $\overline{AB}$ to $\overline{CD}$, so $L=10$. The side AB is made of a long side of a rectangle on the left and a short side of the rectangle on the right, so $AB=10+5=15$, while $BC=L=10$. The area of rectangle $ABCD$ is therefore

$$15\cdot 10=150$$

square feet. As a check, $ABCD$ consists of three rectangles that do not overlap, each with area $5\cdot 10=50$, and $3\cdot 50=150$.

Answer: (E) 150. □

2. (1989 AJHSME, Problem #15) In parallelogram $ABCD$ shown, what is the area of the shaded region $BEDC$?



(A) 24 **(B)** 48 **(C)** 60 **(D)** 64 **(E)** 80

Strategy. Find the area of the whole parallelogram, with base AD and height BE. Then remove the unshaded right triangle ABE.

Solution. In the figure, $BC=10$, $BE=8$, $ED=6$, and $\overline{BE}$ is perpendicular to $\overline{AD}$. Opposite sides of a parallelogram are equal, so $AD=BC=10$, and since E lies between A and D, $AE=10-6=4$. The height of the parallelogram for the base AD is the distance

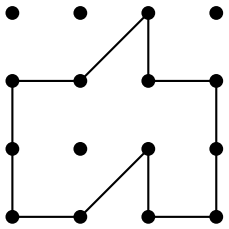
between the lines AD and BC. The segment BE is perpendicular to AD and B lies on the line BC, so this height is $BE = 8$. By the area of a parallelogram,

$$[ABCD] = AD \cdot BE = 10 \cdot 8 = 80.$$

The unshaded part is the right triangle ABE with legs $AE = 4$ and $BE = 8$, and its area is $\frac{1}{2} \cdot 4 \cdot 8 = 16$. Hence $[BEDC] = 80 - 16 = 64$. As a check, $BEDC$ is a trapezoid with parallel sides $BC = 10$ and $ED = 6$ and height $BE = 8$, and $\frac{1}{2}(10 + 6) \cdot 8 = 64$.

Answer: (D) 64. □

3. (1998 AJHSME, Problem #6) Dots are spaced one unit apart, horizontally and vertically. The number of square units enclosed by the polygon is



(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Strategy. Cut the polygon along the grid lines through the dots. Each piece is a whole unit square or half of a unit square, cut off by a diagonal.

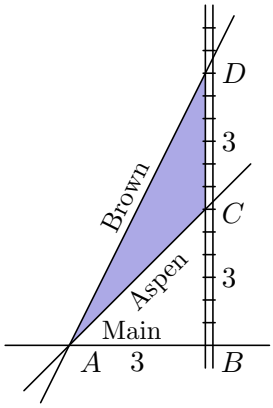
Solution. The dots mark out a 3×3 array of unit squares. Number its rows 1, 2, 3 from the bottom and its columns 1, 2, 3 from the left. The polygon contains five whole unit squares: those in columns 1 and 3 of rows 1 and 2, and the one in column 2 of row 2. Two of its sides are diagonals of unit squares. One of them cuts the square in column 2 of row 1, and the polygon contains the half above that diagonal; the other cuts the square in column 2 of row 3, and the polygon contains the half below it. The polygon meets no other unit square. A diagonal cuts a unit square into two triangles of area $\frac{1}{2}$, so the area of the polygon is

$$5 + \tfrac{1}{2} + \tfrac{1}{2} = 6$$

square units.

Answer: (B) 6. □

4. (2009 AMC 8, Problem #7) The triangular plot of land ACD lies between Aspen Road, Brown Road and a railroad. Main Street runs east and west, and the railroad runs north and south. The numbers in the diagram indicate distances in miles. The width of the railroad track can be ignored. How many square miles are in the plot of land ACD?



(A) 2 **(B)** 3 **(C)** 4.5 **(D)** 6 **(E)** 9

Strategy. Take CD, which lies along the railroad, as the base. The height is then the distance from A to the railroad, which is measured along Main Street.

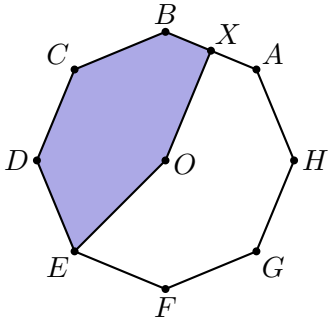
Solution. In the diagram, C and D lie on the railroad with $CD = 3$, and A lies on Main Street with $AB = 3$, where B is the point at which Main Street meets the railroad. Take CD as the base of triangle ACD. Its height is the distance from A to the line through C and D, which is the railroad. Main Street runs east and west and the railroad runs north and south, so they are perpendicular. Hence this distance is $AB = 3$, and the foot of the altitude is B, outside the segment CD. By the area of a triangle,

$$[ACD] = \tfrac{1}{2} \cdot CD \cdot AB = \tfrac{1}{2} \cdot 3 \cdot 3 = 4.5$$

square miles.

Answer: (C) 4.5. □

5. (2015 AMC 8, Problem #2) Point O is the center of the regular octagon $ABCDEFGH$, and X is the midpoint of the side $\overline{AB}$. What fraction of the area of the octagon is shaded?



(A) $\frac{11}{32}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{13}{32}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{15}{32}$

Strategy. The segments from O to the vertices cut the octagon into eight congruent triangles. The shaded region is three of them and half of a fourth, because OX joins O to the midpoint of a side.

Solution. The octagon is regular, so its sides are equal and its center O is equally far from all eight vertices. So the segments from O to the vertices cut the octagon into eight triangles

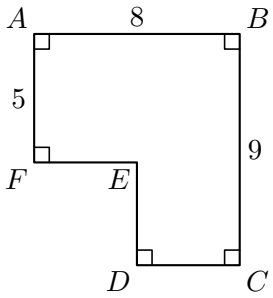
OAB, OBC, ..., OHA with the same three side lengths. These triangles are congruent, so each has $\frac{1}{8}$ of the area of the octagon. In the figure, the shaded region consists of the triangles OBC, OCD, ODE and OXB. Since X is the midpoint of $\overline{AB}$, the triangles OAX and OXB have equal bases $AX = XB$ and the same height from O, so each is half of triangle OAB. Hence the shaded fraction is

$$3 \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{8} = \frac{7}{16}.$$

Answer: (D) $\frac{7}{16}$. □

Intermediate.

6. (2005 AMC 8, Problem #13) The area of polygon $ABCDEF$ is 52 with $AB = 8$, $BC = 9$ and $FA = 5$. What is $DE + EF$?



(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Strategy. Put back the missing corner to complete an 8×9 rectangle. The missing corner is a rectangle whose area is the difference of the two areas and whose height is known.

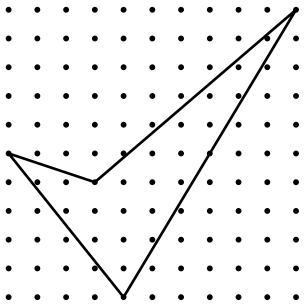
Solution. In the figure, every angle of the hexagon is a right angle except the one at E. Extend $\overline{AF}$ downward and $\overline{CD}$ to the left until they meet at G. Then $ABCG$ is a rectangle with sides $AB = 8$ and $BC = 9$, and the hexagon is this rectangle with the rectangle $FEDG$ removed. So

$$[FEDG] = 8 \cdot 9 - 52 = 72 - 52 = 20.$$

Opposite sides of a rectangle are equal, so $AG = BC = 9$ and $FG = AG - AF = 9 - 5 = 4$. In the rectangle $FEDG$, $DE = FG = 4$, and its area is $DE \cdot EF = 20$, so $EF = 5$. Hence $DE + EF = 4 + 5 = 9$.

Answer: (C) 9. □

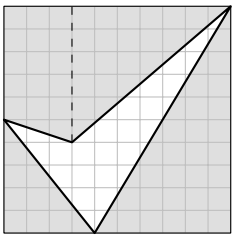
7. (2004 AMC 8, Problem #14) What is the area enclosed by the geoboard quadrilateral below?



(A) 15 **(B)** $18\frac{1}{2}$ **(C)** $22\frac{1}{2}$ **(D)** 27 **(E)** 41

Strategy. Enclose the quadrilateral in the square formed by the outer pegs and subtract the parts of the square outside it. One vertex lies inside the square, so the part next to it is cut along a grid line into two simpler pieces.

Solution. Take the distance between neighboring pegs as 1, so the pegs form a 10×10 square with area 100. Counting pegs in the figure, the vertices of the quadrilateral are the top right corner of the square, the peg on the bottom edge 4 units from the bottom left corner, the peg on the left edge 5 units above that corner, and the peg that is 3 units from the left edge and 4 units above the bottom edge. The parts of the square outside the quadrilateral are shaded below. The dashed segment runs along a grid line from the inner vertex up to the top edge.



- At the bottom left corner, a right triangle with legs 4 and 5 has area 10.
- At the bottom right, a right triangle with legs $10 - 4 = 6$ and 10 has area 30.
- At the top, to the right of the dashed segment, a right triangle with legs $10 - 3 = 7$ along the top edge and $10 - 4 = 6$ along the dashed segment has area 21.
- At the top, to the left of the dashed segment, a trapezoid has vertical parallel sides $10 - 5 = 5$ along the left edge and 6 along the dashed segment, and height 3; its area is $\frac{1}{2}(5 + 6) \cdot 3 = \frac{33}{2}$.

Hence the area of the quadrilateral is

$$100 - \left(10 + 30 + 21 + \tfrac{33}{2}\right) = 100 - 77\tfrac{1}{2} = 22\tfrac{1}{2}.$$

Answer: (C) $22\frac{1}{2}$. □

8. (1951 AHSME, Problem #3) If the length of a diagonal of a square is $a + b$, then the area of the square is:

(A) $(a + b)^2$ **(B)** $\frac{1}{2}(a + b)^2$ **(C)** $a^2 + b^2$ **(D)** $\frac{1}{2}(a^2 + b^2)$ **(E)** None of these

Strategy. A square is a rhombus, so its area is half the product of its diagonals, and the two diagonals of a square have the same length.

Solution. A square has four equal sides, so it is a rhombus, and the area of a rhombus is half the product of its diagonals. Each diagonal of a square with side s cuts the square into two 45-45-90 triangles with legs s, so both diagonals have length $s\sqrt{2}$; here both have length $a+b$. Hence the area of the square is

$$\tfrac{1}{2}(a+b)(a+b) = \tfrac{1}{2}(a+b)^2.$$

As a check, the side of the square is $s = \frac{a+b}{\sqrt{2}}$, and $s^2 = \frac{(a+b)^2}{2}$.

Answer: (B) $\dfrac{1}{2}(a+b)^2$. □

9. (1959 AHSME, Problem #10) In triangle ABC, with $\overline{AB} = \overline{AC} = 3.6$, a point D is taken on $\overline{AB}$ at a distance 1.2 from A. Point D is joined to point E in the prolongation of $\overline{AC}$ so that triangle AED is equal in area to triangle ABC. Then $\overline{AE}$ equals:

(A) 4.8 **(B)** 5.4 **(C)** 7.2 **(D)** 10.8 **(E)** 12.6

Strategy. Draw $\overline{DC}$. Compare triangle ADC with triangle ABC, and then triangle ADE with triangle ADC; in each pair the two triangles have the same height.

Solution. The point E lies on the line AC, beyond C. Draw the segment DC.

Triangles ADC and ABC have their bases AD and AB on the line AB and the same height, the distance from C to that line. Triangles with the same height have areas in the ratio of their bases, so

$$[ADC] = \frac{AD}{AB} \cdot [ABC] = \frac{1.2}{3.6} \cdot [ABC] = \frac{1}{3}[ABC].$$

Triangles ADE and ADC have their bases AE and AC on the line AC and the same height, the distance from D to that line. So

$$[ADE] = \frac{AE}{AC} \cdot [ADC] = \frac{AE}{3.6} \cdot \frac{1}{3}[ABC] = \frac{AE}{10.8} \cdot [ABC].$$

Since $[ADE] = [ABC]$, this gives $\frac{AE}{10.8} = 1$, so $AE = 10.8$.

Answer: (D) 10.8. □

10. (2022 AMC 8, Problem #18) The midpoints of the four sides of a rectangle are $(-3, 0)$, $(2, 0)$, $(5, 4)$, and $(0, 4)$. What is the area of the rectangle?

(A) 20 **(B)** 25 **(C)** 40 **(D)** 50 **(E)** 80

Strategy. The midpoints of the sides of a rectangle are the vertices of a rhombus whose diagonals are as long as the sides of the rectangle, so the rectangle has twice the area of the rhombus. Find the area of the rhombus as a parallelogram with a horizontal base.

Solution. Let the rectangle have sides w and h. Each side of the quadrilateral formed by the four midpoints cuts off a corner of the rectangle, a right triangle with legs $\frac{w}{2}$ and $\frac{h}{2}$. By the Pythagorean theorem, the four sides of the quadrilateral are equal, so it is a rhombus. Its diagonals join the midpoints of opposite sides of the rectangle, so they are parallel to the sides of the rectangle and as long as they are: their lengths are w and h. The area of a rhombus is half the product of its diagonals, so the rhombus has area $\frac{1}{2}wh$, half the area of the rectangle. As a check, the four corner triangles have total area $4 \cdot \frac{1}{2} \cdot \frac{w}{2} \cdot \frac{h}{2} = \frac{1}{2}wh$.

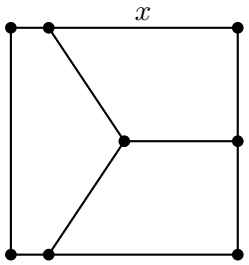
Taken in the order listed, the given points are the vertices of this rhombus. Its side from $(-3, 0)$ to $(2, 0)$ is horizontal with length 5, and the opposite side, from $(0, 4)$ to $(5, 4)$, lies

4 units higher. A rhombus is a parallelogram, so with base 5 and height 4 the rhombus has area $5 \cdot 4 = 20$. Hence the rectangle has area $2 \cdot 20 = 40$.

Answer: (C) 40. □

Advanced.

11. (1998 AHSME, Problem #8) A square with sides of length 1 is divided into two congruent trapezoids and a pentagon, which have equal areas, by joining the center of the square with points on three of the sides, as shown. Find x, the length of the longer parallel side of each trapezoid.



(A) $\frac{3}{5}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{5}{6}$ **(E)** $\frac{7}{8}$

Strategy. Each of the three pieces has area $\frac{1}{3}$. Write the area of one trapezoid with the trapezoid formula and solve for x.

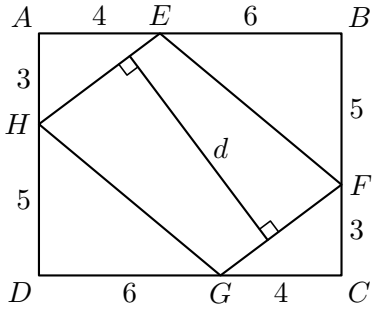
Solution. The three pieces have equal areas and together fill the square, whose area is 1, so each has area $\frac{1}{3}$. Let O be the center of the square. In the figure, a horizontal segment OM joins O to the right side, so M is the midpoint of that side and $OM = \frac{1}{2}$. The segments from O to the top and bottom sides end at points P and Q. The upper trapezoid has vertices P, the top right corner, M and O. Its parallel sides are OM and the part of the top side from P to the corner; the latter is the longer one, x. Its height is the distance between these two sides, which is the distance from O to the top side, $\frac{1}{2}$. By the trapezoid formula,

$$\tfrac{1}{2}\left(x + \tfrac{1}{2}\right) \cdot \tfrac{1}{2} = \tfrac{1}{3},$$

so $x + \frac{1}{2} = \frac{4}{3}$ and $x = \frac{5}{6}$.

Answer: (D) $\frac{5}{6}$. □

12. (2004 AMC 8, Problem #24) In the figure, $ABCD$ is a rectangle and $EFGH$ is a parallelogram. Using the measurements given in the figure, what is the length d of the segment that is perpendicular to $\overline{HE}$ and $\overline{FG}$?



(A) 6.8 **(B)** 7.1 **(C)** 7.6 **(D)** 7.8 **(E)** 8.1

Strategy. Find the area of the parallelogram by subtracting the four corner triangles from the rectangle. The segment of length d is the height of the parallelogram for the base HE, so d is the area divided by HE.

Solution. From the figure, $AB = 4+6 = 10$ and $AD = 3+5 = 8$, so $[ABCD] = 10 \cdot 8 = 80$. The parts of the rectangle outside $EFGH$ are four right triangles at its corners, with areas

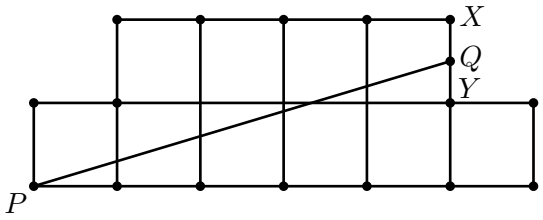
$$[AEH] = \tfrac{1}{2} \cdot 4 \cdot 3 = 6, \qquad [EBF] = \tfrac{1}{2} \cdot 6 \cdot 5 = 15,$$
$$[FCG] = \tfrac{1}{2} \cdot 3 \cdot 4 = 6, \qquad [GDH] = \tfrac{1}{2} \cdot 6 \cdot 5 = 15.$$

So $[EFGH] = 80 - (6 + 15 + 6 + 15) = 38$. In the right triangle AEH, the Pythagorean theorem gives $HE = \sqrt{3^2 + 4^2} = 5$. The segment of length d is perpendicular to the opposite sides HE and FG, so d is the distance between them, which is the height of the parallelogram for the base HE. A parallelogram with base b and area K has height $\frac{K}{b}$, so

$$d = \frac{38}{5} = 7.6.$$

Answer: (C) 7.6. □

13. (2010 AMC 8, Problem #17) The diagram shows an octagon consisting of 10 unit squares. The portion below $\overline{PQ}$ is a unit square and a triangle with base 5. If $\overline{PQ}$ bisects the area of the octagon, what is the ratio $\dfrac{XQ}{QY}$?



(A) $\frac{2}{5}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{3}{5}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

Strategy. The part below $\overline{PQ}$ has area 5. Without the unit square it is a right triangle with base 5 and area 4, and the height of this triangle locates Q.

Solution. The octagon has area 10, and $\overline{PQ}$ bisects it, so the part below $\overline{PQ}$ has area 5. That part is a unit square and a triangle with base 5, so the triangle has area $5 - 1 = 4$. Let R be the dot on the bottom edge directly below Y. In the figure, the triangle is PRQ: its base $PR = 5$ lies along the bottom edge, and Q lies directly above R, so its height for this base is RQ. A triangle with base b and area K has height $\frac{2K}{b}$, so

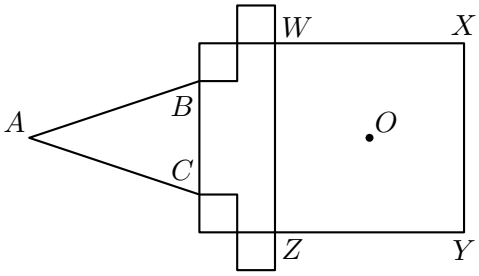
$$RQ = \frac{2 \cdot 4}{5} = \frac{8}{5}.$$

The points Y and X lie on the line RQ, 1 and 2 units above R. Hence $QY = \frac{8}{5} - 1 = \frac{3}{5}$ and $XQ = 2 - \frac{8}{5} = \frac{2}{5}$, and

$$\frac{XQ}{QY} = \frac{2/5}{3/5} = \frac{2}{3}.$$

Answer: (D) $\dfrac{2}{3}$. □

14. (2003 AMC 8, Problem #25) In the figure, the area of square $WXYZ$ is 25 cm^2. The four smaller squares have sides 1 cm long, either parallel to or coinciding with the sides of the large square. In $\triangle ABC$, $AB = AC$, and when $\triangle ABC$ is folded over side $\overline{BC}$, point A coincides with O, the center of square $WXYZ$. What is the area of $\triangle ABC$, in square centimeters?



(A) $\frac{15}{4}$ **(B)** $\frac{21}{4}$ **(C)** $\frac{27}{4}$ **(D)** $\frac{21}{2}$ **(E)** $\frac{27}{2}$

Strategy. Take BC as the base. Folding over BC does not change distances from the line BC, so the height from A equals the distance from O to that line.

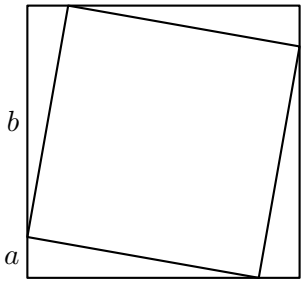
Solution. The square $WXYZ$ has area 25, so its side is 5, and its center O is $\frac{5}{2}$ from each side. In the figure, the small square with corner B has its left side on the line BC and its top side on the line WX. Its right side lies on the same line as the left side of the small square just above and to the right of it, whose right side lies on the line WZ. So the line BC is parallel to WZ, at distance $1 + 1 = 2$ from it on the side away from O, and B is 1 below the line WX. In the same way, C is 1 above the line ZY. Hence $BC = 5-1-1 = 3$, and the distance from O to the line BC is $\frac{5}{2} + 2 = \frac{9}{2}$.

When the triangle is folded over BC, the point A lands on O. Folding over a line does not change the distance of a point from that line, so A is also $\frac{9}{2}$ from the line BC. With BC as the base,

$$[ABC] = \tfrac{1}{2} \cdot 3 \cdot \tfrac{9}{2} = \tfrac{27}{4}.$$

Answer: (C) $\frac{27}{4}$. □

15. (2012 AMC 8, Problem #25) A square with area 4 is inscribed in a square with area 5, with one vertex of the smaller square on each side of the larger square. A vertex of the smaller square divides a side of the larger square into two segments, one of length a and the other of length b. What is the value of ab?



(A) $\frac{1}{5}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 4

Strategy. The large square is the small square together with four right triangles at its corners, each with legs a and b. Their total area is $5 - 4$.

Solution. Each vertex of the small square lies on a side of the large one, so the part of the large square outside the small square consists of four right triangles, one at each corner of the large square, and the hypotenuse of each is a side of the small square. At each vertex of the small square, its right angle and the acute angles of the two triangles there make a straight angle, so those two acute angles add to $90°$. Hence all four triangles have the same angles, and their hypotenuses are equal, so they are congruent, each with legs a and b. Their total area is the difference of the areas of the squares, $5 - 4 = 1$, so

$$4 \cdot \tfrac{1}{2}ab = 1 \qquad \text{and} \qquad ab = \tfrac{1}{2}.$$

As a check, the side of the large square is $a + b = \sqrt{5}$, and the Pythagorean theorem in one of the triangles gives $a^2 + b^2 = 2^2 = 4$, so $2ab = (a + b)^2 - (a^2 + b^2) = 5 - 4 = 1$.

Answer: (C) $\dfrac{1}{2}$. □

2. Answer Key

Problem	Letter	Value
1	E	150
2	D	64
3	B	6
4	C	4.5
5	D	$\frac{7}{16}$
6	C	9
7	C	$22\frac{1}{2}$
8	B	$\frac{1}{2}(a+b)^2$
9	D	10.8
10	C	40
11	D	$\frac{5}{6}$
12	C	7.6
13	D	$\frac{2}{3}$
14	C	$\frac{27}{4}$
15	C	$\frac{1}{2}$