TRIANGLES AND THE PYTHAGOREAN THEOREM FOR AMC 8

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1. Sides of a Triangle

This section covers which three lengths can be the sides of a triangle, and what equal sides say about the angles.

1.1. The Triangle Inequality.

In triangle ABC, the names AB, BC and CA stand for the sides and also for their lengths. The side BC is **opposite** $\angle A$, the side CA is opposite $\angle B$, and the side AB is opposite $\angle C$. Two triangles are **congruent** if they have the same shape and size, that is, if one can be moved onto the other so that they coincide.

The straight segment between two points is the shortest path between them. In triangle ABC, the path from A to C and then to B is not straight, so it is longer than the segment AB. The same argument applies to each side.

Theorem 1.1.1 (Triangle Inequality)**.** *In every triangle, each side is shorter than the sum of the other two sides:*

$$AB < BC + CA, \qquad BC < CA + AB, \qquad CA < AB + BC.$$

Conversely, three positive lengths are the sides of a triangle whenever each of them is less than the sum of the other two. It is enough to check the longest length: if the lengths are $a \le b \le c$ and $c < a+b$, then the other two inequalities hold automatically, since $a \le c < b+c$ and $b \le c < a+c$. If the longest length equals the sum of the other two, the three points lie on one line and there is no triangle.

Corollary 1.1.2 (Third Side)**.** *If two sides of a triangle have lengths a and b with $a \ge b$, then the third side c satisfies*

$$a - b < c < a + b.$$

Proof. The inequality $c < a + b$ is the triangle inequality for the side c. The triangle inequality for the side a gives $a < b + c$, that is, $c > a - b$. □

For example, if two sides of a triangle are 7 and 3, the third side is strictly between 4 and 10.

To list the triangles with integer sides and a given perimeter p, write the sides in order as $a \le b \le c$. The triangle inequality for the longest side, $c < a + b = p - c$, says that $c < \frac{p}{2}$. Also $c \ge \frac{p}{3}$, because c is the largest of three numbers with sum p. So only a few values of c need to be tried, and for each of them the pairs $a \le b \le c$ with $a + b = p - c$ are listed.

Examples.

Problem 1.1.3 (1992 AJHSME, Problem #17)**.** The sides of a triangle have lengths 6.5, 10, and s, where s is a whole number. What is the smallest possible value of s?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Strategy. Use the range for the third side.

Solution. By the corollary on the third side, with $a = 10$ and $b = 6.5$,

$$10 - 6.5 < s < 10 + 6.5, \quad \text{that is,} \quad 3.5 < s < 16.5.$$

The smallest whole number in this range is 4. Indeed, $4 + 6.5 = 10.5 > 10$, so 4, 6.5 and 10 are the sides of a triangle, while for $s = 3$ the sum $3 + 6.5 = 9.5$ is less than 10.

Answer: (B) 4. □

Problem 1.1.4 (2003 AMC 10A, Problem #7)**.** How many non-congruent triangles with perimeter 7 have integer side lengths?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. Write the sides as $a \le b \le c$. The triangle inequality bounds the longest side c.

Solution. Here $a+b+c=7$. The longest side satisfies $c < \frac{7}{2} = 3.5$, so $c \le 3$, and $c \ge \frac{7}{3}$, so $c \ge 3$. Hence $c = 3$ and $a + b = 4$ with $a \le b \le 3$, which gives $(a, b) = (1, 3)$ or $(a, b) = (2, 2)$. The triangles with sides $1, 3, 3$ and $2, 2, 3$ are not congruent, since their sides are different.

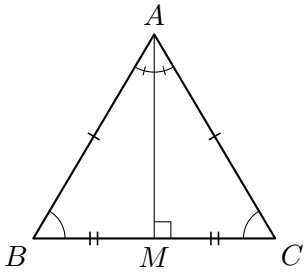
Answer: (B) 2. □

1.2. **Isosceles and Equilateral Triangles.** The previous handout stated without proof that the angles opposite the equal sides of an isosceles triangle are equal. The theorem below proves this fact and its converse.

Definition 1.2.1. A triangle is isosceles if at least two of its sides are equal, and **equilateral** if all three of its sides are equal. In an isosceles triangle ABC with $AB = AC$, the equal sides AB and AC are the **legs**, BC is the **base**, $\angle A$ is the **vertex angle**, and $\angle B$ and $\angle C$ are the **base angles**. Every equilateral triangle is isosceles. In figures, equal sides carry the same number of tick marks, and equal angles the same number of arcs or of ticks on their arcs.

Definition 1.2.2. An **altitude** of a triangle is the segment from a vertex perpendicular to the line through the opposite side. Its length is the **height** of the triangle for that side as the base, and the area of the triangle is $\frac{1}{2} \cdot \text{base} \cdot \text{height}$. The **distance** between two parallel lines is the length of a segment perpendicular to both of them.

In the figure below, $AB = AC$, the segment AM bisects $\angle A$, and the other marks show what the next theorem proves. The notation $AM \perp BC$ means that AM is perpendicular to BC.



Theorem 1.2.3 (Isosceles Triangle Theorem)**.** *Let ABC be a triangle with $AB = AC$, and let the bisector of $\angle A$ meet BC at M. Then*

(a) $\angle B = \angle C$;
(b) $BM = MC$ and $AM \perp BC$.

Conversely,

(c) if $\angle B = \angle C$ in a triangle ABC, then $AB = AC$.

Proof. Fold the triangle along the line AM. Since $\angle BAM = \angle CAM$, the ray AB falls on the ray AC, and since $AB = AC$, the point B falls on C. The point M lies on the fold and stays where it is. So the two halves of the triangle cover each other exactly, which gives $\angle B = \angle C$, $BM = CM$ and $\angle AMB = \angle AMC$. The last two angles form a linear pair, so each of them measures $90°$.

For (c), fold along the line through the midpoint of BC perpendicular to BC. This fold exchanges B and C. Since $\angle B = \angle C$, the ray BA falls on the ray CA and the ray CA falls on the ray BA, so their common point A falls on itself. Then the segment AB falls on the segment AC, and $AB = AC$. □

There is only one perpendicular from A to the line BC, so the altitude from A is the segment AM. Hence in an isosceles triangle the altitude to the base bisects the base and the vertex angle, and it cuts the triangle into two right triangles with the same three sides.

Corollary 1.2.4 (Equilateral Triangles)**.** *Each angle of an equilateral triangle measures* $60°$*. Conversely, a triangle whose three angles measure* $60°$ *is equilateral, and so is an isosceles triangle with one angle of* $60°$*.*

Proof. In an equilateral triangle, part (a) applied to two pairs of equal sides shows that all three angles are equal. They add to $180°$, so each measures $60°$. If all three angles measure $60°$, part (c) applied to two pairs of angles shows that all three sides are equal. If an isosceles triangle has a vertex angle of $60°$, its two equal base angles share the remaining $120°$, so each measures $60°$; if a base angle measures $60°$, so does the other one, and the vertex angle measures $180° - 120° = 60°$. In both cases all three angles measure $60°$, so the triangle is equilateral. □

Examples.

Problem 1.2.5 (2009 AMC 8, Problem #19)**.** Two angles of an isosceles triangle measure $70°$ and $x°$. What is the sum of the three possible values of x?

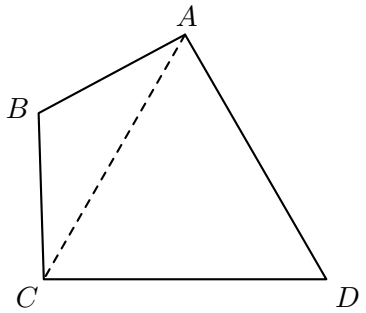
(A) 95 **(B)** 125 **(C)** 140 **(D)** 165 **(E)** 180

Strategy. The $70°$ angle is either the vertex angle or a base angle. Treat the two cases separately.

Solution. If the $70°$ angle is the vertex angle, the two equal base angles share the remaining $110°$, so each measures $55°$ and $x = 55$. If the $70°$ angle is a base angle, the other base angle also measures $70°$, and the vertex angle measures $180° - 140° = 40°$, so $x = 70$ or $x = 40$. The three possible values are 40, 55 and 70, and their sum is 165.

Answer: (D) 165. □

Problem 1.2.6 (2005 AMC 8, Problem #9)**.** In quadrilateral $ABCD$, sides $\overline{AB}$ and $\overline{BC}$ both have length 10, sides $\overline{CD}$ and $\overline{DA}$ both have length 17, and the measure of angle ADC is $60°$. What is the length of diagonal $\overline{AC}$?



(A) 13.5 **(B)** 14 **(C)** 15.5 **(D)** 17 **(E)** 18.5

Strategy. Look at triangle ACD, which contains the $60°$ angle.

Solution. Triangle ACD is isosceles, since $DA = DC = 17$, and its vertex angle is $\angle ADC = 60°$. By the corollary on equilateral triangles, an isosceles triangle with a $60°$ angle is equilateral, so $AC = DA = 17$. The sides of length 10 are not needed.

Answer: (D) 17. □

2. Right Triangles

A triangle with a right angle is a **right triangle**. This section proves the Pythagorean theorem, which relates the three sides of a right triangle, and its converse.

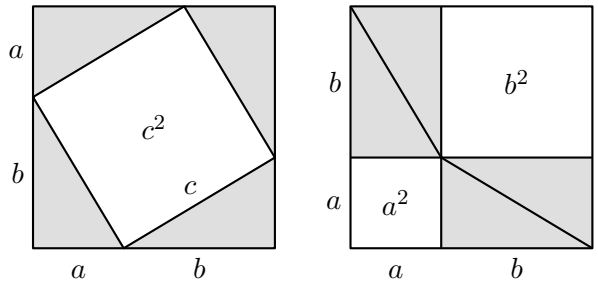
2.1. **The Pythagorean Theorem.**

Definition 2.1.1. In a right triangle, the two sides that form the right angle are the **legs**, and the side opposite the right angle is the **hypotenuse**.

Theorem 2.1.2 (Pythagorean Theorem)**.** *In a right triangle with legs a and b and hypotenuse c,*

$$a^2 + b^2 = c^2.$$

In other words, the square on the hypotenuse has the same area as the squares on the two legs together.



Proof. Each square in the figure has side $a + b$, so the two squares have the same area. Each of them contains four copies of the right triangle, shaded. In the left square, the four triangles leave a region with four sides of length c. At each corner of this region, the two acute angles of the triangle and the corner angle together make a straight angle. The acute angles of a right triangle add to 90°, so the corner angle measures 90°, and the region is a square of area c^2. In the right square, the four triangles form two rectangles of size a by b, and they leave a square of side a and a square of side b. Taking the four triangles away from the two equal areas leaves equal areas, so $c^2 = a^2 + b^2$. □

Since $c^2 = a^2 + b^2$ is larger than a^2 and larger than b^2, the hypotenuse is the longest side of a right triangle. Two sides of a right triangle give the third:

$$c = \sqrt{a^2 + b^2}, \qquad a = \sqrt{c^2 - b^2}.$$

For example, a right triangle with legs 6 and 8 has hypotenuse $\sqrt{36 + 64} = \sqrt{100} = 10$.

Remark 2.1.3 (Square Roots)**.** For positive numbers x and y, $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$. A square root is in **simplest form** when the number under the root sign has no square factor other than 1. For example,

$$\sqrt{50} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}.$$

A square root in a denominator is removed by multiplying the top and the bottom by it:

$$\frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}.$$

Examples.

Problem 2.1.4. Find the missing side of each right triangle.

(a) The legs have lengths 4 and 6.
(b) The hypotenuse has length 7 and one leg has length 3.

Strategy. Use the Pythagorean theorem, then write each square root in simplest form.

Solution. (a) The hypotenuse c satisfies $c^2 = 4^2 + 6^2 = 16 + 36 = 52$, so

$$c = \sqrt{52} = \sqrt{4} \cdot \sqrt{13} = 2\sqrt{13}.$$

(b) The other leg b satisfies $3^2 + b^2 = 7^2$, so $b^2 = 49 - 9 = 40$ and

$$b = \sqrt{40} = \sqrt{4} \cdot \sqrt{10} = 2\sqrt{10}.$$

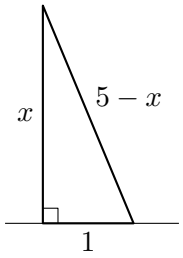
Answer: (a) $2\sqrt{13}$; (b) $2\sqrt{10}$. □

Problem 2.1.5 (2009 AMC 10B, Problem #10)**.** A flagpole is originally 5 meters tall. A hurricane snaps the flagpole at a point x meters above the ground so that the upper part, still attached to the stump, touches the ground 1 meter away from the base. What is x?

(A) 2.0 **(B)** 2.1 **(C)** 2.2 **(D)** 2.3 **(E)** 2.4

Strategy. The stump, the ground and the broken part form a right triangle. Write all three sides in terms of x.

Solution. The break leaves an upright stump of length x, and the broken part, the top $5-x$ meters of the pole, reaches from the top of the stump down to the ground 1 meter from the base. The stump, the broken part and this 1-meter stretch of ground form a triangle. Its angle at the base of the pole is a right angle, because the pole stood vertically, so the broken part, which is opposite this angle, is the hypotenuse.



For this triangle the Pythagorean theorem says

$$(5 - x)^2 = x^2 + 1, \quad \text{that is,} \quad 25 - 10x + x^2 = x^2 + 1.$$

Subtracting x^2 from both sides leaves $25 - 10x = 1$, so $x = \frac{24}{10} = 2.4$. As a check, $2.4^2 + 1 = 6.76 = 2.6^2$, and $2.6 = 5 - 2.4$.

Answer: (E) 2.4. □

2.2. **The Converse of the Pythagorean Theorem.**

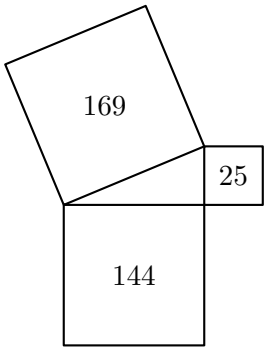
Theorem 2.2.1 (Converse of the Pythagorean Theorem)**.** *If the sides of a triangle satisfy the equation $a^2 + b^2 = c^2$, then the angle opposite the side c is a right angle.*

Proof. Draw a right triangle with legs a and b. By the Pythagorean theorem, its hypotenuse is $\sqrt{a^2 + b^2} = c$, so it has the same three side lengths as the given triangle. Two triangles with the same three side lengths are congruent (a triangle built from three rods cannot be bent into another shape), so their angles are equal. Hence the angle of the given triangle opposite c is a right angle. □

To test whether a triangle is a right triangle, square its three sides. Only the longest side can be the hypotenuse, so the triangle is a right triangle exactly when the two smaller squares add up to the largest one.

Examples.

Problem 2.2.2 (2003 AMC 8, Problem #6)**.** Given the areas of the three squares in the figure, what is the area of the interior triangle?



(A) 13 **(B)** 30 **(C)** 60 **(D)** 300 **(E)** 1800

Strategy. The side of each square is a side of the triangle. Check whether the triangle is a right triangle.

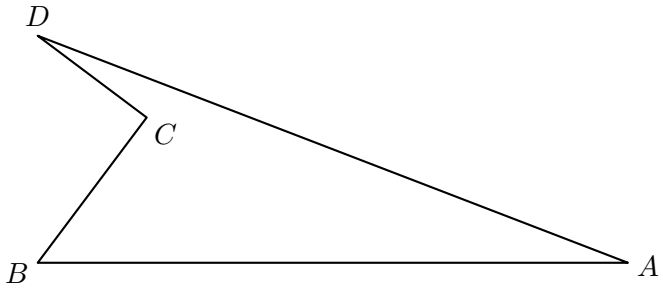
Solution. The squares have sides $\sqrt{25} = 5$, $\sqrt{144} = 12$ and $\sqrt{169} = 13$, and these are the sides of the triangle. Since

$$5^2 + 12^2 = 25 + 144 = 169 = 13^2,$$

the converse of the Pythagorean theorem shows that the angle between the sides 5 and 12 is a right angle. These two sides are the legs, so the area of the triangle is $\frac{1}{2} \cdot 5 \cdot 12 = 30$.

Answer: (B) 30. □

Problem 2.2.3 (2017 AMC 8, Problem #18)**.** In the non-convex quadrilateral $ABCD$ shown below, $\angle BCD$ is a right angle, $AB = 12$, $BC = 4$, $CD = 3$, and $AD = 13$. What is the area of quadrilateral $ABCD$?



(A) 12 **(B)** 24 **(C)** 26 **(D)** 30 **(E)** 36

Strategy. Draw the diagonal BD. The quadrilateral is triangle ABD with triangle BCD cut out.

Solution. In the right triangle BCD, $BD^2 = BC^2 + CD^2 = 16 + 9 = 25$, so $BD = 5$. In triangle ABD,

$$BD^2 + AB^2 = 25 + 144 = 169 = AD^2,$$

so by the converse of the Pythagorean theorem $\angle ABD = 90°$, with legs $BD = 5$ and $AB = 12$. As the figure shows, C lies inside triangle ABD (on the other side of BD the quadrilateral would be convex), so the quadrilateral is triangle ABD with triangle BCD removed. Its area is

$$\tfrac{1}{2} \cdot 5 \cdot 12 - \tfrac{1}{2} \cdot 4 \cdot 3 = 30 - 6 = 24.$$

Answer: (B) 24. □

2.3. **Pythagorean Triples.**

Definition 2.3.1. A **Pythagorean triple** is a list of positive integers (a, b, c) with $a^2 + b^2 = c^2$.

A right triangle with legs a and b has hypotenuse $\sqrt{a^2 + b^2} = c$, so the numbers of a Pythagorean triple are the sides of a right triangle. These are the triples whose numbers have no common factor greater than 1 and whose largest number is at most 25:

a	b	c	$a^2 + b^2 = c^2$
3	4	5	$9 + 16 = 25$
5	12	13	$25 + 144 = 169$
8	15	17	$64 + 225 = 289$
7	24	25	$49 + 576 = 625$

Proposition 2.3.2 (Multiples of a Triple)**.** If (a, b, c) is a Pythagorean triple and k is any positive number, then the triangle with sides ka, kb and kc is a right triangle.

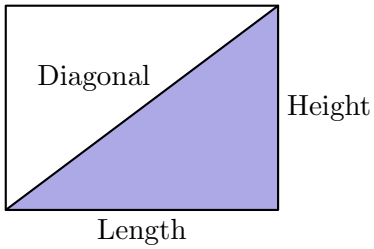
Proof. Since $(ka)^2 + (kb)^2 = k^2(a^2 + b^2) = k^2c^2 = (kc)^2$, the right triangle with legs ka and kb has hypotenuse kc. By the converse of the Pythagorean theorem, any triangle with sides ka, kb and kc is a right triangle. □

For example, the triangles with sides 6, 8 and 10; 30, 40 and 50; 10, 24 and 26; and 1.5, 2 and 2.5 are right triangles. When two sides of a right triangle have a common factor, divide it out first and look for a triple in the table. When the sides are in the ratio $3 : 4 : 5$, they can be written as $3k$, $4k$ and $5k$ for some number k, as with any ratio.

A diagonal of a rectangle is the hypotenuse of a right triangle whose legs are two sides of the rectangle.

Example.

Problem 2.3.3 (2003 AMC 10B, Problem #6)**.** Many television screens are rectangles that are measured by the length of their diagonals. The ratio of the horizontal length to the height in a standard television screen is $4 : 3$. The horizontal length of a "27-inch" television screen is closest, in inches, to which of the following?



(A) 20 **(B)** 20.5 **(C)** 21 **(D)** 21.5 **(E)** 22

Strategy. Write the horizontal length and the height as $4k$ and $3k$. The diagonal is the hypotenuse of a right triangle with these legs.

Solution. Let the horizontal length be $4k$ inches and the height $3k$ inches. The diagonal is the hypotenuse of the shaded right triangle, whose legs $3k$ and $4k$ are k times the triple $(3, 4, 5)$, so the diagonal is $5k$ inches. A "27-inch" screen has a diagonal of 27 inches, so $5k = 27$ and $k = 5.4$. The horizontal length is $4k = 21.6$ inches, which is closest to 21.5.

Answer: (D) 21.5. □

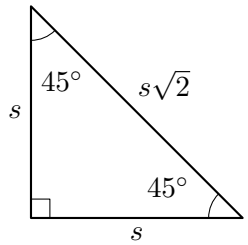
3. Special Right Triangles

This section finds the side ratios of two right triangles: half of a square and half of an equilateral triangle.

3.1. **The 45-45-90 Triangle.** Let a right triangle have equal legs. Its two acute angles are opposite the equal legs, so they are equal by the Isosceles Triangle Theorem; they add to 90°, so each measures 45°. Such a triangle is called a **45-45-90 triangle**. Conversely, a right triangle with a 45° angle has a second angle of $90° - 45° = 45°$, so by part (c) of the Isosceles Triangle Theorem its legs are equal.

Theorem 3.1.1. *In a 45-45-90 triangle with legs s, the hypotenuse is $s\sqrt{2}$.*

Proof. By the Pythagorean theorem, the hypotenuse c satisfies $c^2 = s^2+s^2 = 2s^2$, so $c = \sqrt{2}\cdot s$. □



A diagonal of a square with side s cuts the square into two 45-45-90 triangles, so the diagonal has length $s\sqrt{2}$. In the other direction, a 45-45-90 triangle with hypotenuse h has legs $\frac{h}{\sqrt{2}} = \frac{\sqrt{2}}{2}h$.

Example.

Problem 3.1.2 (2012 AMC 10B, Problem #12)**.** Point B is due east of point A. Point C is due north of point B. The distance between points A and C is $10\sqrt{2}$ meters, and $\angle BAC = 45°$. Point D is 20 meters due north of point C. The distance AD is between which two integers?

(A) 30 and 31 **(B)** 31 and 32 **(C)** 32 and 33 **(D)** 33 and 34 **(E)** 34 and 35

Strategy. Triangle ABC is a 45-45-90 triangle. Find its legs, then use the right triangle ABD.

Solution. East and north are perpendicular directions, so $\angle ABC = 90°$. With $\angle BAC = 45°$, triangle ABC is a 45-45-90 triangle with hypotenuse $AC = 10\sqrt{2}$, so its legs are $AB = BC = \frac{10\sqrt{2}}{\sqrt{2}} = 10$. Point D is 20 meters north of C, so B, C and D lie on one north-south line, $BD = 10 + 20 = 30$, and $\angle ABD = 90°$. By the Pythagorean theorem,

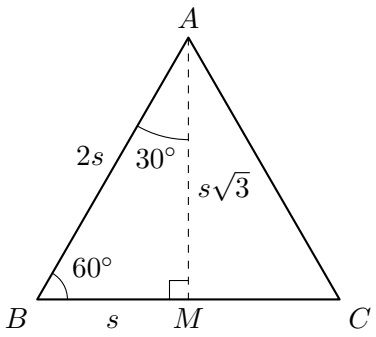
$$AD = \sqrt{AB^2 + BD^2} = \sqrt{100 + 900} = \sqrt{1000}.$$

Since $31^2 = 961 < 1000 < 1024 = 32^2$, the distance AD is between 31 and 32.

Answer: (B) 31 and 32. □

3.2. **The 30-60-90 Triangle.** Let ABC be an equilateral triangle with side $2s$, and let M be the midpoint of BC. By the Isosceles Triangle Theorem, AM is perpendicular to BC and bisects $\angle A$. So triangle ABM has a 60° angle at B, a 30° angle at A and a right angle at M. Such a triangle is called a **30-60-90 triangle**. Here its hypotenuse is $AB = 2s$, and the side opposite the 30° angle is $BM = s$.

Theorem 3.2.1. *In a 30-60-90 triangle, the side opposite the 30° angle is half the hypotenuse, and the side opposite the 60° angle is $\sqrt{3}$ times the side opposite the 30° angle. That is, the sides are s, $s\sqrt{3}$ and $2s$.*



Proof. Let the hypotenuse be h, and let x be the side opposite the 30° angle. Reflect the triangle across the leg that joins the 30° angle to the right angle. The two right angles at the end of that leg form a straight angle, so the side x and its mirror image make one segment of length $2x$. The triangle and its image together form a triangle with sides h, h and $2x$, whose angles measure 60°, 60° and $30° + 30° = 60°$. By the corollary on equilateral triangles it is equilateral, so $2x = h$, that is, $x = \frac{h}{2}$. Writing $x = s$, the hypotenuse is $2s$, and by the Pythagorean theorem the remaining side is

$$\sqrt{(2s)^2 - s^2} = \sqrt{3s^2} = s\sqrt{3}.$$ □

In an equilateral triangle with side a, an altitude is the longer leg of a 30-60-90 triangle with hypotenuse a and shortest side $\frac{a}{2}$, so the height of the triangle is $\frac{\sqrt{3}}{2}a$.

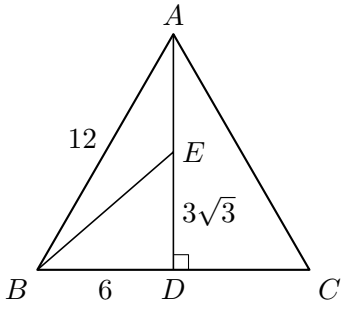
Examples.

Problem 3.2.2 (1961 AHSME, Problem #10)**.** Each side of triangle ABC is 12 units. D is the foot of the perpendicular dropped from A on BC, and E is the midpoint of AD. The length of BE, in the same unit, is:

(A) $\sqrt{18}$ **(B)** $\sqrt{28}$ **(C)** 6 **(D)** $\sqrt{63}$ **(E)** $\sqrt{98}$

Strategy. The segment AD is an altitude of the equilateral triangle. Then BE is the hypotenuse of the right triangle BDE.

Solution. Triangle ABC is equilateral, so its altitude AD bisects BC and $BD = 6$. Triangle ABD has a right angle at D and a 60° angle at B, so it is a 30-60-90 triangle with hypotenuse $AB = 12$. The side $BD = 6$ is opposite the 30° angle, so $AD = 6\sqrt{3}$ and $DE = \frac{1}{2}AD = 3\sqrt{3}$.



Triangle BDE has a right angle at D, so $BE = \sqrt{BD^2 + DE^2} = \sqrt{36 + 27} = \sqrt{63}$.

Answer: (D) $\sqrt{63}$. □

Problem 3.2.3 (1978 AHSME, Problem #7)**.** Opposite sides of a regular hexagon are 12 inches apart. The length of each side, in inches, is

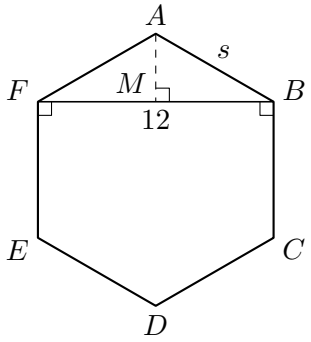
(A) 7.5 **(B)** $6\sqrt{2}$ **(C)** $5\sqrt{2}$ **(D)** $\frac{9}{2}\sqrt{3}$ **(E)** $4\sqrt{3}$

Strategy. In a regular hexagon $ABCDEF$, the diagonal FB is perpendicular to the opposite sides BC and EF. Find FB in terms of the side s from the isosceles triangle FAB.

Solution. Let the hexagon be $ABCDEF$ with side s. Each angle of a regular hexagon measures $120°$. Triangle FAB has $FA = AB = s$ and vertex angle $120°$, so its base angles are

$$\angle AFB = \angle ABF = \frac{180° - 120°}{2} = 30°.$$

Then $\angle FBC = 120° - 30° = 90°$, and in the same way $\angle BFE = 90°$. So FB is perpendicular to both BC and EF, and its length is the distance between these opposite sides: $FB = 12$.



The altitude from A to FB meets FB at its midpoint M and cuts triangle FAB into two 30-60-90 triangles. In triangle AMB, the hypotenuse is $AB = s$ and the angle at B measures $30°$, so $AM = \frac{s}{2}$ and $MB = \frac{s}{2}\sqrt{3}$. Hence

$$FB = 2 \cdot MB = s\sqrt{3} = 12, \qquad \text{so} \qquad s = \frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}.$$

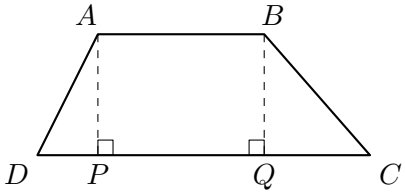
Answer: (E) $4\sqrt{3}$. □

4. Right Triangles in Figures

Many problems ask for a length that is not a side of any right triangle in the figure. One extra segment, usually a perpendicular, creates a right triangle that contains it.

4.1. **Triangles and Trapezoids.** In an isosceles triangle, the altitude to the base splits the triangle into two right triangles. Their legs are half the base and the altitude, and their hypotenuse is a leg of the isosceles triangle.

A **trapezoid** is a quadrilateral with a pair of parallel sides, called its **bases**. In a trapezoid $ABCD$ with $AB \parallel DC$ and AB the shorter base, draw the perpendiculars AP and BQ from A and B to the line DC. Then $ABQP$ is a rectangle, so $PQ = AB$. When the angles at D and C are acute, as in the figure, P and Q lie between D and C, so the perpendiculars split the trapezoid into the rectangle $ABQP$ and the right triangles APD and BQC, and $DC = DP + PQ + QC$. The common length $AP = BQ$ is the **altitude** of the trapezoid, the distance between its bases.



Examples.

Problem 4.1.1 (2007 AMC 8, Problem #14)**.** The base of isosceles $\triangle ABC$ is 24 and its area is 60. What is the length of one of the congruent sides?

(A) 5 **(B)** 8 **(C)** 13 **(D)** 14 **(E)** 18

Strategy. Find the altitude to the base from the area. It splits the triangle into two right triangles.

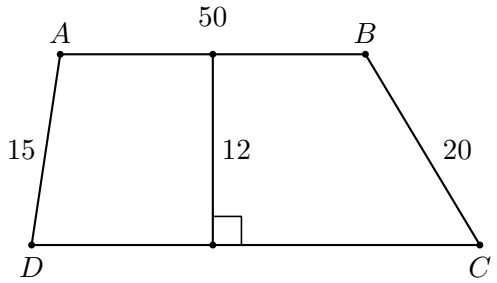
Solution. Let h be the altitude to the base. The area is $\frac{1}{2} \cdot 24 \cdot h = 60$, so $h = 5$. The altitude to the base meets the base at its midpoint, so each half of the triangle is a right triangle with legs 12 and 5. Its hypotenuse, one of the congruent sides, is

$$\sqrt{144 + 25} = \sqrt{169} = 13,$$

and 5, 12, 13 form a Pythagorean triple.

Answer: (C) 13. □

Problem 4.1.2 (2011 AMC 8, Problem #20)**.** Quadrilateral $ABCD$ is a trapezoid, $AD = 15$, $AB = 50$, $BC = 20$, and the altitude is 12. What is the area of the trapezoid?



(A) 600 **(B)** 650 **(C)** 700 **(D)** 750 **(E)** 800

Strategy. Drop perpendiculars from A and B to the line DC. Find the pieces of the longer base from two right triangles.

Solution. In the figure, AB and DC are the parallel sides. Let the perpendiculars from A and B meet DC at P and Q. Each of them has length 12, and $ABQP$ is a rectangle, so $PQ = AB = 50$. In the right triangle APD,

$$DP = \sqrt{15^2 - 12^2} = \sqrt{81} = 9,$$

and in the right triangle BQC,

$$QC = \sqrt{20^2 - 12^2} = \sqrt{256} = 16.$$

These two triangles are 3 and 4 times the triangle with sides 3, 4, 5. As the figure shows, P and Q lie between D and C, so $DC = 9 + 50 + 16 = 75$. The trapezoid is made of the rectangle and the two right triangles, so its area is

$$50 \cdot 12 + \tfrac{1}{2} \cdot 9 \cdot 12 + \tfrac{1}{2} \cdot 16 \cdot 12 = 600 + 54 + 96 = 750.$$

Answer: (D) 750. □

4.2. **Grids and Compass Directions.** North and south are opposite directions, east and west are opposite directions, and the north-south direction is perpendicular to the east-west direction. For a trip along compass directions, find how far north or south of the start the trip ends by adding the moves north and subtracting the moves south, and find in the same way how far east or west it ends. The straight-line distance from the start to the end is then the hypotenuse of a right triangle whose legs are these two distances. Northeast is halfway between north and east,

so a move of length d to the northeast is the hypotenuse of a 45-45-90 triangle: it goes $\frac{\sqrt{2}}{2}d$ north and $\frac{\sqrt{2}}{2}d$ east. The same holds for northwest, southeast and southwest.

On a grid of unit squares, a segment joining two grid points is the hypotenuse of a right triangle whose legs run along the grid lines. If the segment goes a units across and b units up or down, its length is $\sqrt{a^2+b^2}$.

Examples.

Problem 4.2.1 (2005 AMC 8, Problem #7)**.** Bill walks $\frac{1}{2}$ mile south, then $\frac{3}{4}$ mile east, and finally $\frac{1}{2}$ mile south. How many miles is he, in a direct line, from his starting point?

(A) 1 **(B)** $1\frac{1}{4}$ **(C)** $1\frac{1}{2}$ **(D)** $1\frac{3}{4}$ **(E)** 2

Strategy. Add the moves in each direction. The two totals are the legs of a right triangle.

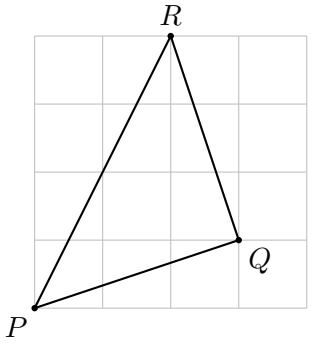
Solution. Bill walks $\frac{1}{2}+\frac{1}{2}=1$ mile south in all and $\frac{3}{4}$ mile east. His distance from the start is the hypotenuse of a right triangle with legs 1 and $\frac{3}{4}$:

$$\sqrt{1^2+\left(\tfrac{3}{4}\right)^2}=\sqrt{\tfrac{16}{16}+\tfrac{9}{16}}=\sqrt{\tfrac{25}{16}}=\tfrac{5}{4}=1\tfrac{1}{4}.$$

The sides $\frac{3}{4}$, 1, $\frac{5}{4}$ are the triple $(3,4,5)$ divided by 4.

Answer: (B) $1\frac{1}{4}$. □

Problem 4.2.2. Points P, Q and R lie at corners of a grid of unit squares, as shown. Find the side lengths of triangle PQR and the measure of $\angle QPR$.



Strategy. Each side is the hypotenuse of a right triangle whose legs run along the grid lines. Count the units across and up, then compare the squares of the sides.

Solution. The side PQ goes 3 units across and 1 unit up, so $PQ=\sqrt{9+1}=\sqrt{10}$. The side QR goes 1 unit across and 3 units up, so $QR=\sqrt{1+9}=\sqrt{10}$. The side PR goes 2 units across and 4 units up, so $PR=\sqrt{4+16}=\sqrt{20}=2\sqrt{5}$. Since

$$PQ^2+QR^2=10+10=20=PR^2,$$

the converse of the Pythagorean theorem shows that $\angle PQR$ is a right angle. The legs PQ and QR are equal, so PQR is a 45-45-90 triangle, and $\angle QPR=45^\circ$.

Answer: $PQ=QR=\sqrt{10}$, $PR=2\sqrt{5}$, and $\angle QPR=45^\circ$. □

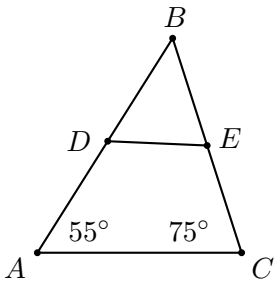
5. Problems

Beginner.

1. (2015 AMC 8, Problem #8) What is the smallest whole number larger than the perimeter of any triangle with a side of length 5 and a side of length 19?

(A) 24 **(B)** 29 **(C)** 43 **(D)** 48 **(E)** 57

2. (1993 AHSME, Problem #2) In $\triangle ABC$, $\angle A = 55°$, $\angle C = 75°$, D is on side $\overline{AB}$ and E is on side $\overline{BC}$. If $DB = BE$, then $\angle BED =$



(A) 50° **(B)** 55° **(C)** 60° **(D)** 65° **(E)** 70°

3. (1972 AHSME, Problem #1) The lengths in inches of the three sides of each of four triangles I, II, III, and IV are as follows:

I $3, 4, 5$

II $4, 7\frac{1}{2}, 8\frac{1}{2}$

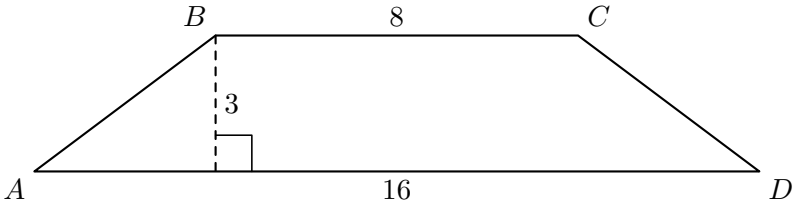
III $7, 24, 25$

IV $3\frac{1}{2}, 4\frac{1}{2}, 5\frac{1}{2}$

Of these four given triangles, the only right triangles are:

(A) I and II **(B)** I and III **(C)** I and IV **(D)** I, II, and III **(E)** I, II, and IV

4. (1999 AMC 8, Problem #14) In trapezoid $ABCD$, the sides AB and CD are equal. The perimeter of $ABCD$ is



(A) 27 **(B)** 30 **(C)** 32 **(D)** 34 **(E)** 48

Intermediate.

5. (2005 AMC 8, Problem #15) How many different isosceles triangles have integer side lengths and perimeter 23?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 9 **(E)** 11

6. (2011 AMC 8, Problem #16) Let A be the area of the triangle with sides of length 25, 25, and 30. Let B be the area of the triangle with sides of length 25, 25, and 40. What is the relationship between A and B?

(A) $A = \frac{9}{16}B$ **(B)** $A = \frac{3}{4}B$ **(C)** $A = B$ **(D)** $A = \frac{4}{3}B$ **(E)** $A = \frac{16}{9}B$

7. (2002 AMC 10A, Problem #13) The sides of a triangle have lengths of 15, 20, and 25. Find the length of the shortest altitude.

(A) 6 **(B)** 12 **(C)** 12.5 **(D)** 13 **(E)** 15

8. (1962 AHSME, Problem #6) A square and an equilateral triangle have equal perimeters. The area of the triangle is $9\sqrt{3}$ square inches. Expressed in inches the diagonal of the square is:

(A) $\frac{9}{2}$ **(B)** $2\sqrt{5}$ **(C)** $4\sqrt{2}$ **(D)** $\frac{9\sqrt{2}}{2}$ **(E)** none of these

Advanced.

9. (2006 AMC 10B, Problem #10) In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?

(A) 43 **(B)** 44 **(C)** 45 **(D)** 46 **(E)** 47

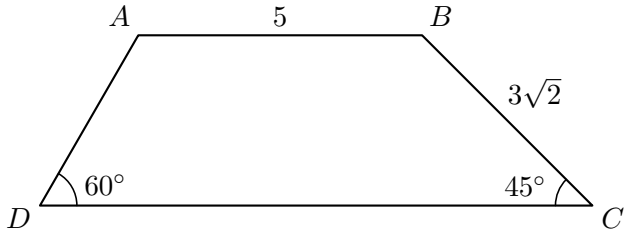
10. (2010 AMC 10A, Problem #7) Crystal has a running course marked out for her daily run. She starts this run by heading due north for one mile. She then runs northeast for one mile, then southeast for one mile. The last portion of her run takes her on a straight line back to where she started. How far, in miles, is this last portion of her run?

(A) 1 **(B)** $\sqrt{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $2\sqrt{2}$

11. (2005 AMC 10B, Problem #10) In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is BD?

(A) 3 **(B)** $2\sqrt{3}$ **(C)** 4 **(D)** 5 **(E)** $4\sqrt{2}$

12. (1984 AHSME, Problem #8) Figure $ABCD$ is a trapezoid with $AB \parallel DC$, $AB = 5$, $BC = 3\sqrt{2}$, $\angle BCD = 45^\circ$ and $\angle CDA = 60^\circ$. The length of DC is



(A) $7 + \frac{2}{3}\sqrt{3}$ **(B)** 8 **(C)** $9\frac{1}{2}$ **(D)** $8 + \sqrt{3}$ **(E)** $8 + 3\sqrt{3}$