TRIANGLES AND THE PYTHAGOREAN THEOREM FOR AMC 8: SOLUTIONS

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Contents

1. Solutions 2
2. Answer Key 8

1. Solutions

Beginner.

1. (2015 AMC 8, Problem #8) What is the smallest whole number larger than the perimeter of any triangle with a side of length 5 and a side of length 19?

(A) 24 **(B)** 29 **(C)** 43 **(D)** 48 **(E)** 57

Strategy. Find the range of the third side, and with it the range of the perimeter.

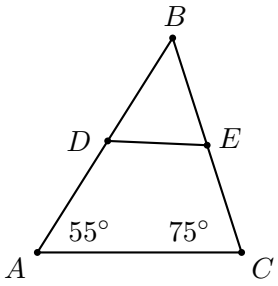
Solution. Let the third side be c. By the corollary on the third side,

$$19 - 5 < c < 19 + 5, \quad \text{that is,} \quad 14 < c < 24.$$

The perimeter $5 + 19 + c = 24 + c$ is therefore less than $24 + 24 = 48$, so 48 is larger than the perimeter of every such triangle. No smaller whole number works. The third side can be any number between 14 and 24, for example $c = 23.5$, which gives the perimeter 47.5, and every whole number up to 47 is less than 47.5. So the smallest whole number larger than every possible perimeter is 48.

Answer: (D) 48. □

2. (1993 AHSME, Problem #2) In $\triangle ABC$, $\angle A = 55°$, $\angle C = 75°$, D is on side $\overline{AB}$ and E is on side $\overline{BC}$. If $DB = BE$, then $\angle BED =$



(A) 50° **(B)** 55° **(C)** 60° **(D)** 65° **(E)** 70°

Strategy. Find $\angle B$, then use the isosceles triangle DBE.

Solution. By the angle sum in triangle ABC,

$$\angle B = 180° - 55° - 75° = 50°.$$

Since $DB = BE$, triangle DBE is isosceles with vertex angle $\angle DBE = \angle B = 50°$. Its base angles are equal, so

$$\angle BED = \angle BDE = \frac{180° - 50°}{2} = 65°.$$

Answer: (D) 65°. □

3. (1972 AHSME, Problem #1) The lengths in inches of the three sides of each of four triangles I, II, III, and IV are as follows:

I	$3, 4, 5$
II	$4, 7\frac{1}{2}, 8\frac{1}{2}$
III	$7, 24, 25$
IV	$3\frac{1}{2}, 4\frac{1}{2}, 5\frac{1}{2}$

Of these four given triangles, the only right triangles are:

(A) I and II **(B)** I and III **(C)** I and IV **(D)** I, II, and III **(E)** I, II, and IV

Strategy. Write the mixed numbers as fractions. For each triangle, square the three lengths and check whether the two smaller squares add up to the largest one.

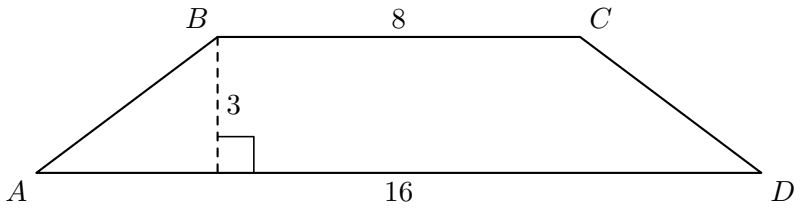
Solution. In each triangle the last length is the longest.

- Triangle I: $3^2 + 4^2 = 9 + 16 = 25 = 5^2$.
- Triangle II: $4^2 + \left(\frac{15}{2}\right)^2 = 16 + \frac{225}{4} = \frac{289}{4} = \left(\frac{17}{2}\right)^2$. These sides are the triple $(8, 15, 17)$ halved.
- Triangle III: $7^2 + 24^2 = 49 + 576 = 625 = 25^2$.
- Triangle IV: $\left(\frac{7}{2}\right)^2 + \left(\frac{9}{2}\right)^2 = \frac{49}{4} + \frac{81}{4} = \frac{130}{4}$, while $\left(\frac{11}{2}\right)^2 = \frac{121}{4}$.

By the converse of the Pythagorean theorem, I, II and III are right triangles. Triangle IV is not: if it were a right triangle, its longest side would be the hypotenuse, and the Pythagorean theorem would give $\frac{130}{4} = \frac{121}{4}$.

Answer: (D) I, II, and III. □

4. (1999 AMC 8, Problem #14) In trapezoid $ABCD$, the sides AB and CD are equal. The perimeter of $ABCD$ is



(A) 27 **(B)** 30 **(C)** 32 **(D)** 34 **(E)** 48

Strategy. Drop the perpendiculars from B and C to AD. Each of the equal sides AB and CD is the hypotenuse of a right triangle.

Solution. In the figure, $BC = 8$ and $AD = 16$ are the parallel sides, and the altitude is 3. Let the perpendiculars from B and C meet AD at P and Q. Then $BCQP$ is a rectangle, so $PQ = BC = 8$. As the figure shows, P and Q lie between A and D, so the two end pieces AP and QD have total length $16 - 8 = 8$. The right triangles APB and DQC have the same height $BP = CQ = 3$ and equal hypotenuses $AB = CD$, so their legs along AD are equal:

$$AP = \sqrt{AB^2 - 3^2} = \sqrt{CD^2 - 3^2} = QD.$$

Hence $AP = QD = 4$. Each of AB and CD is then the hypotenuse of a right triangle with legs 3 and 4, so $AB = CD = 5$ by the triple $(3, 4, 5)$. The perimeter is

$$8 + 16 + 5 + 5 = 34.$$

Answer: (D) 34. □

Intermediate.

5. (2005 AMC 8, Problem #15) How many different isosceles triangles have integer side lengths and perimeter 23?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 9 **(E)** 11

Strategy. Call the equal sides a and the third side b. The perimeter shows that b is odd, and the triangle inequality bounds b.

Solution. Let the two equal sides have length a and the third side length b, so $2a + b = 23$. Then $b = 23 - 2a$ is odd. By the triangle inequality, $b < a + a = 23 - b$, so $b < 11.5$; the

other inequality, $a < a + b$, always holds. Hence b is one of 1, 3, 5, 7, 9, 11, and each value gives one triangle:

$$(11, 11, 1), \quad (10, 10, 3), \quad (9, 9, 5), \quad (8, 8, 7), \quad (7, 7, 9), \quad (6, 6, 11).$$

None of these is equilateral, since 23 is not a multiple of 3, so each of them has only one pair of equal sides and is counted once. There are 6 triangles.

Answer: (C) 6. □

6. (2011 AMC 8, Problem #16) Let A be the area of the triangle with sides of length 25, 25, and 30. Let B be the area of the triangle with sides of length 25, 25, and 40. What is the relationship between A and B?

(A) $A = \frac{9}{16}B$ **(B)** $A = \frac{3}{4}B$ **(C)** $A = B$ **(D)** $A = \frac{4}{3}B$ **(E)** $A = \frac{16}{9}B$

Strategy. Both triangles are isosceles. Find each altitude to the base from a right triangle.

Solution. In the triangle with sides 25, 25, 30, the altitude to the base of length 30 meets the base at its midpoint, so it is a leg of a right triangle with hypotenuse 25 and other leg 15. The altitude is $\sqrt{625 - 225} = \sqrt{400} = 20$, and

$$A = \tfrac{1}{2} \cdot 30 \cdot 20 = 300.$$

In the triangle with sides 25, 25, 40, half of the base is 20, and the altitude is

$$\sqrt{625 - 400} = \sqrt{225} = 15, \qquad \text{so} \qquad B = \tfrac{1}{2} \cdot 40 \cdot 15 = 300.$$

Hence $A = B$. Each triangle is made of two right triangles with sides 15, 20, 25, joined along different legs.

Answer: (C) $A = B$. □

7. (2002 AMC 10A, Problem #13) The sides of a triangle have lengths of 15, 20, and 25. Find the length of the shortest altitude.

(A) 6 **(B)** 12 **(C)** 12.5 **(D)** 13 **(E)** 15

Strategy. Recognize a multiple of a Pythagorean triple, then compute the area in two ways.

Solution. The lengths 15, 20, 25 are 5 times 3, 4, 5, so by the proposition on multiples of a triple the triangle is a right triangle with legs 15 and 20. Its area is $\frac{1}{2} \cdot 15 \cdot 20 = 150$. For each side taken as the base, $\frac{1}{2} \cdot \text{base} \cdot \text{height} = 150$, so the altitude to that side is 300 divided by the side:

$$\frac{300}{15} = 20, \qquad \frac{300}{20} = 15, \qquad \frac{300}{25} = 12.$$

The altitudes to the legs are the other legs. The shortest altitude is 12, the one to the longest side.

Answer: (B) 12. □

8. (1962 AHSME, Problem #6) A square and an equilateral triangle have equal perimeters. The area of the triangle is $9\sqrt{3}$ square inches. Expressed in inches the diagonal of the square is:

(A) $\frac{9}{2}$ **(B)** $2\sqrt{5}$ **(C)** $4\sqrt{2}$ **(D)** $\frac{9\sqrt{2}}{2}$ **(E)** none of these

Strategy. Write the height of the triangle in terms of its side with a 30-60-90 triangle, and use $\frac{1}{2} \cdot \text{base} \cdot \text{height}$ to find the side. Then find the side and the diagonal of the square.

Solution. Let a be the side of the triangle. The altitude from one vertex meets the opposite side at its midpoint (Isosceles Triangle Theorem), so it cuts the triangle into two 30-60-90 triangles with hypotenuse a and shortest side $\frac{a}{2}$. The altitude is the side opposite the $60°$ angle, so its length is $\frac{\sqrt{3}}{2}a$, and the area of the triangle is

$$\frac{1}{2}\cdot a\cdot\frac{\sqrt{3}}{2}a=\frac{\sqrt{3}}{4}a^2.$$

Setting this equal to $9\sqrt{3}$ gives $a^2=36$, so $a=6$. The perimeter of the triangle is 18, so the square has side $\frac{18}{4}=\frac{9}{2}$. A diagonal of a square with side s has length $s\sqrt{2}$, so the diagonal of the square is

$$\frac{9}{2}\sqrt{2}=\frac{9\sqrt{2}}{2}.$$

Answer: (D) $\frac{9\sqrt{2}}{2}$. □

Advanced.

9. (2006 AMC 10B, Problem #10) In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?

(A) 43 **(B)** 44 **(C)** 45 **(D)** 46 **(E)** 47

Strategy. Call the sides x, $3x$ and 15, and apply the triangle inequality to each side.

Solution. Let the sides be x, $3x$ and 15, where x is a positive integer. The three triangle inequalities are

$$3x<x+15,\qquad 15<x+3x,\qquad x<3x+15.$$

The first gives $x<7.5$, the second gives $x>3.75$, and the third always holds. So x is 4, 5, 6 or 7, and the perimeter $x+3x+15=4x+15$ is greatest for $x=7$. The triangle with sides 7, 21, 15 exists, since $7+15=22>21$, and its perimeter is 43.

Answer: (A) 43. □

10. (2010 AMC 10A, Problem #7) Crystal has a running course marked out for her daily run. She starts this run by heading due north for one mile. She then runs northeast for one mile, then southeast for one mile. The last portion of her run takes her on a straight line back to where she started. How far, in miles, is this last portion of her run?

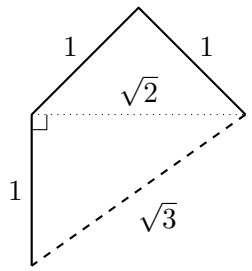
(A) 1 **(B)** $\sqrt{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $2\sqrt{2}$

Strategy. Split each diagonal mile into a north-south part and an east-west part, then find how far north and how far east of her starting point Crystal ends.

Solution. A mile to the northeast is the hypotenuse of a 45-45-90 triangle, so it goes $\frac{\sqrt{2}}{2}$ mile north and $\frac{\sqrt{2}}{2}$ mile east. In the same way, a mile to the southeast goes $\frac{\sqrt{2}}{2}$ mile south and $\frac{\sqrt{2}}{2}$ mile east. In all, Crystal ends

$$1+\tfrac{\sqrt{2}}{2}-\tfrac{\sqrt{2}}{2}=1\text{ mile north}\qquad\text{and}\qquad\tfrac{\sqrt{2}}{2}+\tfrac{\sqrt{2}}{2}=\sqrt{2}\text{ miles east}$$

of her starting point.



In the figure, north is up, and the dashed segment is the last portion. It is the hypotenuse of a right triangle with legs 1 and $\sqrt{2}$, so its length is $\sqrt{1+2} = \sqrt{3}$.

Answer: (C) $\sqrt{3}$. □

11. (2005 AMC 10B, Problem #10) In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is BD?

(A) 3 **(B)** $2\sqrt{3}$ **(C)** 4 **(D)** 5 **(E)** $4\sqrt{2}$

Strategy. Draw the altitude from C to the line AB. It is a leg of two right triangles, one with hypotenuse CB and one with hypotenuse CD.

Solution. Let M be the midpoint of AB. Triangle ABC is isosceles with $CA = CB$, so by the Isosceles Triangle Theorem CM is perpendicular to AB, and $AM = MB = 1$. In the right triangle CMB,

$$CM^2 = CB^2 - MB^2 = 49 - 1 = 48.$$

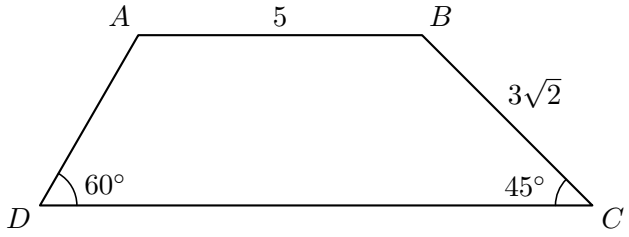
The point D lies on the line AB beyond B, so triangle CMD has a right angle at M, and

$$MD^2 = CD^2 - CM^2 = 64 - 48 = 16, \qquad MD = 4.$$

Then $BD = MD - MB = 4 - 1 = 3$.

Answer: (A) 3. □

12. (1984 AHSME, Problem #8) Figure $ABCD$ is a trapezoid with $AB \parallel DC$, $AB = 5$, $BC = 3\sqrt{2}$, $\angle BCD = 45°$ and $\angle CDA = 60°$. The length of DC is



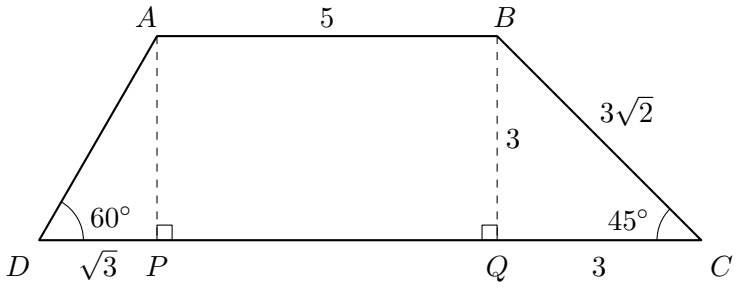
(A) $7 + \frac{2}{3}\sqrt{3}$ **(B)** 8 **(C)** $9\frac{1}{2}$ **(D)** $8 + \sqrt{3}$ **(E)** $8 + 3\sqrt{3}$

Strategy. Drop perpendiculars from A and B to DC. They cut off a 30-60-90 triangle and a 45-45-90 triangle.

Solution. Let the perpendiculars from A and B meet DC at P and Q. Then $ABQP$ is a rectangle, so $PQ = AB = 5$ and $AP = BQ$. Triangle BQC has a right angle at Q and a 45° angle at C, so it is a 45-45-90 triangle with hypotenuse $BC = 3\sqrt{2}$, and its legs are $BQ = QC = \frac{3\sqrt{2}}{\sqrt{2}} = 3$. So $AP = 3$. Triangle APD has a right angle at P and a 60° angle at D, so AP is the side opposite the 60° angle and DP is the side opposite the 30° angle.

Hence $AP = DP \cdot \sqrt{3}$, and

$$DP = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}.$$



The angles at D and C are acute, so P and Q lie between D and C, and

$$DC = DP + PQ + QC = \sqrt{3} + 5 + 3 = 8 + \sqrt{3}.$$

Answer: (D) $8 + \sqrt{3}$. □

2. Answer Key

Problem	Letter	Value
1	D	48
2	D	65°
3	D	I, II, and III
4	D	34
5	C	6
6	C	$A = B$
7	B	12
8	D	$\frac{9\sqrt{2}}{2}$
9	A	43
10	C	$\sqrt{3}$
11	A	3
12	D	$8 + \sqrt{3}$