TRIANGLES AND THE PYTHAGOREAN THEOREM FOR AMC 8: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (1996 AJHSME, Problem #8) Points A and B are 10 units apart. Points B and C are 4 units apart. Points C and D are 3 units apart. If A and D are as close as possible, then the number of units between them is

(A) 0 **(B)** 3 **(C)** 9 **(D)** 11 **(E)** 17

Strategy. Compare the broken path from A to D, then to C, then to B with the straight segment AB.

Solution. The path that runs from A to D, from D to C and from C to B has length $AD + 3 + 4$. A straight segment is the shortest path between its endpoints, so this path is at least as long as the segment AB:

$$AD + 7 \geq 10, \quad \text{that is,} \quad AD \geq 3.$$

The distance 3 can occur. Put the four points on one line in the order A, D, C, B, with $AD = 3$, $DC = 3$ and $CB = 4$. Then $AB = 3 + 3 + 4 = 10$, $BC = 4$ and $CD = 3$, as the problem requires, and $AD = 3$. So the smallest possible distance is 3.

Answer: (B) 3. □

2. (2017 AMC 10A, Problem #7) Jerry and Silvia wanted to go from the southwest corner of a square field to the northeast corner. Jerry walked due east and then due north to reach the goal, but Silvia headed northeast and reached the goal walking in a straight line. Which of the following is closest to how much shorter Silvia's trip was, compared to Jerry's trip?

(A) 30% **(B)** 40% **(C)** 50% **(D)** 60% **(E)** 70%

Strategy. Jerry walks along two sides of the square and Silvia along its diagonal. Compare the two lengths for a square of side s.

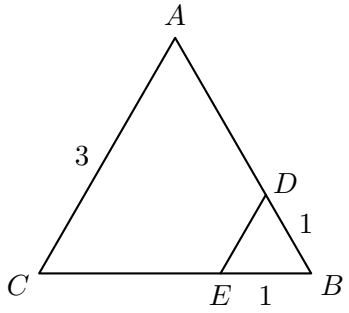
Solution. Let the field be a square of side s. Jerry walks the length of two sides, $2s$ in all. Silvia walks along a diagonal, which cuts the square into two 45-45-90 triangles, so her trip has length $s\sqrt{2}$. Her trip is shorter by $2s - s\sqrt{2}$, and as a fraction of Jerry's trip this is

$$\frac{2s - s\sqrt{2}}{2s} = 1 - \frac{\sqrt{2}}{2}.$$

Since $1.41^2 = 1.9881$ and $1.42^2 = 2.0164$, the number $\sqrt{2}$ lies between 1.41 and 1.42, so $\frac{\sqrt{2}}{2}$ lies between 0.705 and 0.71. The fraction is therefore between 0.29 and 0.295: Silvia's trip is between 29% and 29.5% shorter. Of the choices, 30% is the closest.

Answer: (A) 30%. □

3. (1987 AHSME, Problem #2) As shown in the figure, a triangular corner with side lengths $DB = EB = 1$ is cut from equilateral triangle ABC of side length 3. The perimeter of the remaining quadrilateral $ADEC$ is



(A) 6 **(B)** $6\frac{1}{2}$ **(C)** 7 **(D)** $7\frac{1}{2}$ **(E)** 8

Strategy. The corner triangle DBE has two equal sides and a 60° angle between them.

Solution. Each angle of the equilateral triangle ABC measures 60°, so $\angle DBE = 60°$. Triangle DBE is isosceles, since $DB = EB$, and by the corollary on equilateral triangles an isosceles triangle with a 60° angle is equilateral. Hence $DE = DB = 1$. The other three sides of $ADEC$ are

$$AD = AB - DB = 3 - 1 = 2, \qquad EC = BC - BE = 3 - 1 = 2, \qquad CA = 3,$$

so the perimeter of $ADEC$ is $2 + 1 + 2 + 3 = 8$.

Answer: (E) 8. □

4. (2004 AMC 10B, Problem #8) Minneapolis-St. Paul International Airport is 8 miles southwest of downtown St. Paul and 10 miles southeast of downtown Minneapolis. Which of the following is closest to the number of miles between downtown St. Paul and downtown Minneapolis?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

Strategy. Describe both downtowns as seen from the airport. Northeast and northwest are perpendicular directions, so the two distances are the legs of a right triangle.

Solution. Let P be the airport, S downtown St. Paul and M downtown Minneapolis. The airport is 8 miles southwest of S, so S is 8 miles northeast of P; in the same way, M is 10 miles northwest of P. Northeast is halfway between north and east, 45° from north, and northwest is 45° from north on the other side, so

$$\angle SPM = 45° + 45° = 90°.$$

Triangle SPM has legs 8 and 10, and by the Pythagorean theorem

$$SM = \sqrt{8^2 + 10^2} = \sqrt{164}.$$

Since $12.8^2 = 163.84$ and $12.9^2 = 166.41$, the distance $\sqrt{164}$ is between 12.8 and 12.9 miles. Of the choices, 13 is the closest.

Answer: (A) 13. □

5. (2015 AMC 8, Problem #6) In $\triangle ABC$, $AB = BC = 29$, and $AC = 42$. What is the area of $\triangle ABC$?

(A) 100 **(B)** 420 **(C)** 500 **(D)** 609 **(E)** 701

Strategy. The triangle is isosceles with base AC. The altitude to the base cuts it into two right triangles, and their common leg is the height.

Solution. Since $AB = BC$, triangle ABC is isosceles with base AC. The altitude from B meets AC at its midpoint M, so $AM = 21$, and triangle AMB has a right angle at M and hypotenuse $AB = 29$. By the Pythagorean theorem,

$$BM = \sqrt{29^2 - 21^2} = \sqrt{841 - 441} = \sqrt{400} = 20.$$

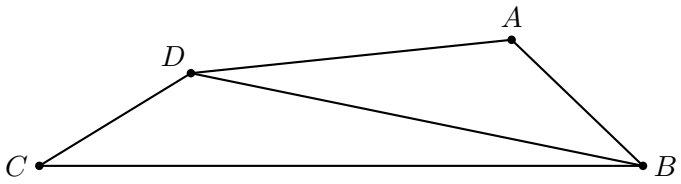
The height for the base AC is $BM = 20$, so the area is

$$\tfrac{1}{2} \cdot AC \cdot BM = \tfrac{1}{2} \cdot 42 \cdot 20 = 420.$$

Answer: (B) 420. □

Intermediate.

6. (2009 AMC 10A, Problem #12) In quadrilateral $ABCD$, $AB = 5$, $BC = 17$, $CD = 5$, $DA = 9$, and BD is an integer. What is BD?



(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Strategy. The diagonal BD is a side of two triangles, ABD and CBD. Each triangle gives a range for BD.

Solution. By the corollary on the third side, each side of a triangle is longer than the difference of the other two sides and shorter than their sum. In triangle ABD the other two sides are $DA = 9$ and $AB = 5$, so

$$9 - 5 < BD < 9 + 5, \quad \text{that is,} \quad 4 < BD < 14.$$

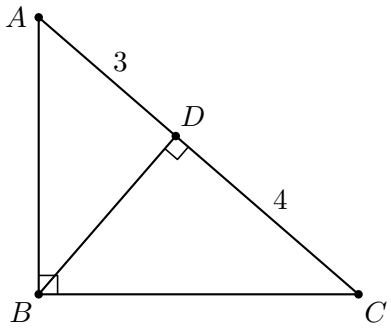
In triangle CBD the other two sides are $BC = 17$ and $CD = 5$, so

$$17 - 5 < BD < 17 + 5, \quad \text{that is,} \quad 12 < BD < 22.$$

Both ranges hold only when $12 < BD < 14$, and the only integer strictly between 12 and 14 is 13.

Answer: (C) 13. □

7. (2009 AMC 10A, Problem #10) Triangle ABC has a right angle at B. Point D is the foot of the altitude from B, $AD = 3$, and $DC = 4$. What is the area of $\triangle ABC$?



(A) $4\sqrt{3}$ **(B)** $7\sqrt{3}$ **(C)** 21 **(D)** $14\sqrt{3}$ **(E)** 42

Strategy. The figure has three right triangles: ADB, CDB and ABC. Call the altitude h and apply the Pythagorean theorem to all three.

Solution. Let $h = BD$. Triangles ADB and CDB have right angles at D, so

$$AB^2 = 3^2 + h^2 = 9 + h^2 \qquad \text{and} \qquad BC^2 = 4^2 + h^2 = 16 + h^2.$$

As the figure shows, D lies between A and C, so $AC = 3 + 4 = 7$. Triangle ABC has its right angle at B, so AC is its hypotenuse and $AB^2 + BC^2 = AC^2 = 49$. Substituting the two expressions above gives

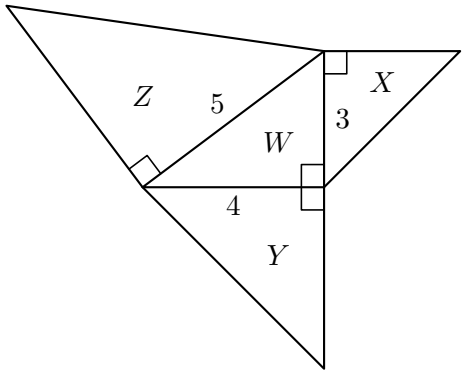
$$(9 + h^2) + (16 + h^2) = 49, \quad \text{that is,} \quad 25 + 2h^2 = 49,$$

so $h^2 = 12$ and $h = \sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$. The segment BD is the altitude to the side AC, so the area of triangle ABC is

$$\tfrac{1}{2} \cdot AC \cdot BD = \tfrac{1}{2} \cdot 7 \cdot 2\sqrt{3} = 7\sqrt{3}.$$

Answer: (B) $7\sqrt{3}$. □

8. (2002 AMC 8, Problem #16) Right isosceles triangles are constructed on the sides of a $3 - 4 - 5$ right triangle, as shown. A capital letter represents the area of each triangle. Which one of the following is true?



(A) $X + Z = W + Y$ **(B)** $W + X = Z$ **(C)** $3X + 4Y = 5Z$
(D) $X + W = \dfrac{1}{2}(Y + Z)$ **(E)** $X + Y = Z$

Strategy. A right isosceles triangle with legs of length s has area $\frac{1}{2}s^2$, half the area of the square with side s. The Pythagorean theorem, read as a statement about the squares on the sides, then compares X, Y and Z.

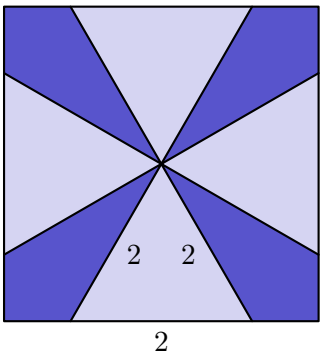
Solution. The right-angle marks show that each side of the right triangle with sides 3, 4 and 5 is a leg of the right isosceles triangle built on it: the side 3 for the triangle with area X, the side 4 for Y, and the side 5 for Z. A right isosceles triangle with legs s has area $\frac{1}{2}s^2$, so

$$X = \tfrac{1}{2} \cdot 3^2 = \tfrac{9}{2}, \qquad Y = \tfrac{1}{2} \cdot 4^2 = 8, \qquad Z = \tfrac{1}{2} \cdot 5^2 = \tfrac{25}{2}.$$

Since $3^2 + 4^2 = 5^2$, halving each term gives $X + Y = Z$. The other four statements are false. The triangle with sides 3, 4, 5 has area $W = \frac{1}{2} \cdot 3 \cdot 4 = 6$, and the two sides of the equation in (A) are 17 and 14, in (B) $\frac{21}{2}$ and $\frac{25}{2}$, in (C) $\frac{91}{2}$ and $\frac{1125}{2}$, and in (D) $\frac{21}{2}$ and $\frac{41}{4}$.

Answer: (E) $X + Y = Z$. □

9. (2019 AMC 10B, Problem #8) The figure below shows a square and four equilateral triangles, with each triangle having a side lying on a side of the square, such that each triangle has side length 2 and the third vertices of the triangles meet at the center of the square. The region inside the square but outside the triangles is shaded. What is the area of the shaded region?



(A) 4 **(B)** $12-4\sqrt{3}$ **(C)** $3\sqrt{3}$ **(D)** $4\sqrt{3}$ **(E)** $16-4\sqrt{3}$

Strategy. The altitude of each triangle from the center of the square is perpendicular to a side of the square, so it is half the side of the square. The height of an equilateral triangle comes from a 30-60-90 triangle.

Solution. Let O be the center of the square, and consider the triangle whose base lies on the bottom side. Its altitude from O is perpendicular to the bottom side, so its length is the distance from O to that side, which is half the side of the square. The altitude of an equilateral triangle with side 2 is the longer leg of a 30-60-90 triangle with hypotenuse 2, so it has length $\frac{\sqrt{3}}{2}\cdot 2=\sqrt{3}$. Hence the square has side $2\sqrt{3}$ and area $(2\sqrt{3})^2=12$.

Each triangle has area $\frac{1}{2}\cdot 2\cdot\sqrt{3}=\sqrt{3}$. The four triangles do not overlap. In each triangle the altitude from O bisects the 60° angle at O, so the triangle stays within 30° of its altitude. The altitudes of two different triangles point to different sides of the square, so they are at least 90° apart, and $30^\circ+30^\circ<90^\circ$. The shaded region is the square with the four triangles removed, so its area is

$$12-4\sqrt{3}.$$

Answer: (B) $12-4\sqrt{3}$. □

10. (2011 AMC 10B, Problem #14) A rectangular parking lot has a diagonal of 25 meters and an area of 168 square meters. In meters, what is the perimeter of the parking lot?

(A) 52 **(B)** 58 **(C)** 62 **(D)** 68 **(E)** 70

Strategy. The diagonal is the hypotenuse of a right triangle whose legs are two sides of the lot. A triple with hypotenuse 25 fits the area; the square of the sum of the sides shows that the perimeter can take no other value.

Solution. Let the sides of the lot be a and b meters. The diagonal is the hypotenuse of a right triangle with legs a and b, so $a^2+b^2=25^2=625$, and the area gives $ab=168$. The Pythagorean triple $(7,24,25)$ has hypotenuse 25, and $7\cdot 24=168$, so a lot of 7 by 24 meters satisfies both conditions.

No other perimeter is possible. For any such lot,

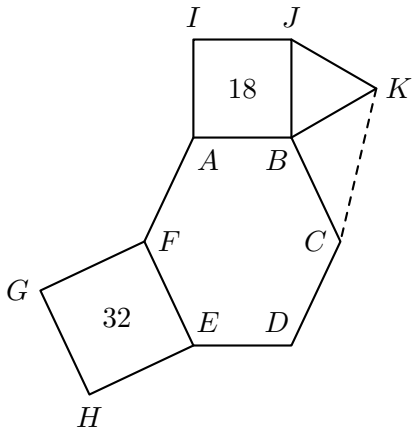
$$(a+b)^2=a^2+2ab+b^2=625+2\cdot 168=961=31^2,$$

so $a+b=31$, and the perimeter is $2(a+b)=62$.

Answer: (C) 62. □

Advanced.

11. (2015 AMC 8, Problem #21) In the given figure hexagon $ABCDEF$ is equiangular, $ABJI$ and $FEHG$ are squares with areas 18 and 32 respectively, $\triangle JBK$ is equilateral and $FE=BC$. What is the area of $\triangle KBC$?



(A) $6\sqrt{2}$ **(B)** 9 **(C)** 12 **(D)** $9\sqrt{2}$ **(E)** 32

Strategy. The areas of the squares give BK and BC. The four angles around B add to 360°, and three of them are known.

Solution. The square $ABJI$ has area 18, so its side is $AB=BJ=\sqrt{18}=\sqrt{9}\cdot\sqrt{2}=3\sqrt{2}$. Triangle JBK is equilateral, so $BK=BJ=3\sqrt{2}$ and $\angle JBK=60°$. The square $FEHG$ has area 32, so $BC=FE=\sqrt{32}=\sqrt{16}\cdot\sqrt{2}=4\sqrt{2}$.

The angles of a hexagon add to $(6-2)\cdot 180°=720°$, and the six angles of an equiangular hexagon are equal, so each measures 120°. Around B the figure shows four angles that together make a full turn: $\angle ABJ=90°$ in the square, $\angle JBK=60°$ in the triangle, $\angle KBC$, and $\angle CBA=120°$ in the hexagon. Hence

$$\angle KBC=360°-90°-60°-120°=90°.$$

Triangle KBC is a right triangle with legs $BK=3\sqrt{2}$ and $BC=4\sqrt{2}$, so its area is

$$\tfrac{1}{2}\cdot 3\sqrt{2}\cdot 4\sqrt{2}=\tfrac{1}{2}\cdot 12\cdot 2=12.$$

Answer: (C) 12. □

12. (2015 AMC 12B, Problem #10) How many noncongruent integer-sided triangles with positive area and perimeter less than 15 are neither equilateral, isosceles, nor right triangles?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Strategy. List the triangles with three different integer sides and perimeter at most 14, sorted by the longest side. Then square the sides of each to find the right triangles.

Solution. The triangle is neither equilateral nor isosceles, so its sides are three different integers $a<b<c$. Two triangles with the same three sides are congruent, so each list of sides counts once. Three lengths $a<b<c$ are the sides of a triangle with positive area exactly when $c<a+b$, and the perimeter $p=a+b+c$ is at most 14. From $c<a+b=p-c$ it follows that $c<\frac{p}{2}\le 7$, so $c\le 6$; also $c\ge 3$, since a, b and c are different positive integers. For each c, the pairs $a<b<c$ with $c<a+b\le 14-c$ are:

- $c = 6$: $a + b$ is 7 or 8, which gives $(2, 5)$, $(3, 4)$ and $(3, 5)$;
- $c = 5$: $a + b$ is between 6 and 9 and $b \le 4$, which gives $(2, 4)$ and $(3, 4)$;
- $c = 4$: $a + b \ge 5$ and $b \le 3$, which gives only $(2, 3)$;
- $c = 3$: the only pair is $(1, 2)$, and $1 + 2 = 3$ is not greater than 3.

This gives six triangles. Only the longest side can be the hypotenuse, so by the Pythagorean theorem and its converse a triangle is a right triangle exactly when $a^2 + b^2 = c^2$:

Sides	$a^2 + b^2$	c^2
$2, 3, 4$	13	16
$2, 4, 5$	20	25
$2, 5, 6$	29	36
$3, 4, 5$	25	25
$3, 4, 6$	25	36
$3, 5, 6$	34	36

Only the triangle with sides 3, 4, 5 is a right triangle. Removing it leaves 5 triangles.

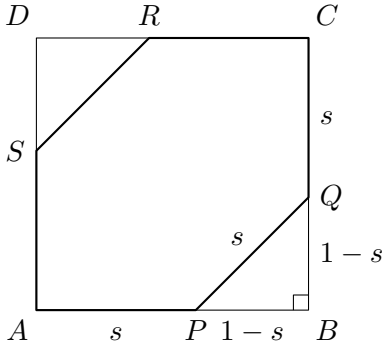
Answer: (C) 5. □

13. (2022 AMC 10A, Problem #5) Square $ABCD$ has side length 1. Points P, Q, R, and S each lie on a side of $ABCD$ such that $APQCRS$ is an equilateral convex hexagon with side length s. What is s?

(A) $\frac{\sqrt{2}}{3}$ **(B)** $\frac{1}{2}$ **(C)** $2 - \sqrt{2}$ **(D)** $1 - \frac{\sqrt{2}}{4}$ **(E)** $\frac{2}{3}$

Strategy. The hexagon cuts off the corner triangle PBQ, a 45-45-90 triangle whose hypotenuse PQ is a side of the hexagon. Write its legs in terms of s.

Solution. The hexagon $APQCRS$ runs along the sides of the square from A to P on AB, cuts across the corner B from P to Q on BC, runs along the square from Q to C and then to R on CD, and cuts across the corner D from R to S on DA.



Since $AP = s$ and $AB = 1$, the segment PB has length $1 - s$. Since $QC = s$ and $BC = 1$, the segment BQ also has length $1 - s$. Triangle PBQ has a right angle at B and equal legs, so it is a 45-45-90 triangle, and its hypotenuse is $PQ = (1 - s)\sqrt{2}$. This hypotenuse is a side of the hexagon, so

$$(1 - s)\sqrt{2} = s, \quad \text{that is,} \quad \sqrt{2} = s(1 + \sqrt{2}).$$

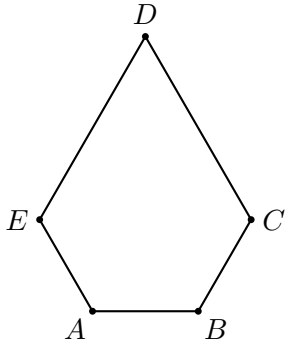
This equation has exactly one solution s, and $s = 2 - \sqrt{2}$ is a solution, since

$$(2 - \sqrt{2})(1 + \sqrt{2}) = 2 + 2\sqrt{2} - \sqrt{2} - 2 = \sqrt{2}.$$

So $s = 2 - \sqrt{2}$. The same computation at the corner D, where $RD = DS = 1 - s$, gives $RS = s$ as well, so all six sides of the hexagon equal s.

Answer: (C) $2 - \sqrt{2}$. □

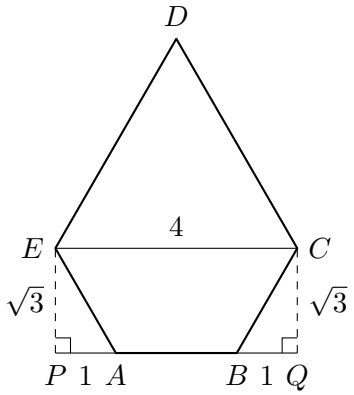
14. (1993 AHSME, Problem #14) The convex pentagon $ABCDE$ has $\angle A = \angle B = 120°$, $EA = AB = BC = 2$ and $CD = DE = 4$. What is the area of $ABCDE$?



(A) 10 **(B)** $7\sqrt{3}$ **(C)** 15 **(D)** $9\sqrt{3}$ **(E)** $12\sqrt{5}$

Strategy. Drop perpendiculars from E and C to the line AB. They cut off two 30-60-90 triangles and show that EC is parallel to AB with length 4. The diagonal EC then splits the pentagon into a trapezoid and an equilateral triangle.

Solution. Let the perpendiculars from E and C to the line AB meet it at P and Q. The angle $\angle EAB = 120°$ is obtuse, so P lies on the extension of BA beyond A, and $\angle EAP = 180° - 120° = 60°$. Triangle APE has a right angle at P, a 60° angle at A and hypotenuse $AE = 2$, so it is a 30-60-90 triangle with $AP = 1$, opposite the 30° angle, and $EP = \sqrt{3}$. In the same way, Q lies on the extension of AB beyond B, with $BQ = 1$ and $CQ = \sqrt{3}$.



The pentagon is convex, so E and C lie on the same side of the line AB, both at distance $\sqrt{3}$ from it. Hence EC is parallel to AB, the quadrilateral $PQCE$ is a rectangle, and

$$EC = PQ = 1 + 2 + 1 = 4.$$

The quadrilateral $ABCE$ is the rectangle $PQCE$ with the right triangles APE and BQC removed, so its area is

$$4 \cdot \sqrt{3} - 2 \cdot \tfrac{1}{2} \cdot 1 \cdot \sqrt{3} = 4\sqrt{3} - \sqrt{3} = 3\sqrt{3}.$$

Triangle CDE has $CD = DE = EC = 4$, so it is equilateral, and its height is $\frac{\sqrt{3}}{2} \cdot 4 = 2\sqrt{3}$. Its area is $\frac{1}{2} \cdot 4 \cdot 2\sqrt{3} = 4\sqrt{3}$. Since the pentagon is convex, the diagonal EC splits it into

the quadrilateral $ABCE$ and the triangle CDE, and the area of the pentagon is

$$3\sqrt{3} + 4\sqrt{3} = 7\sqrt{3}.$$

Answer: (B) $7\sqrt{3}$. □

15. (1980 AHSME, Problem #9) A man walks x miles due west, turns 150° to his left and walks 3 miles in the new direction. If he finishes at a point $\sqrt{3}$ miles from his starting point, then x is

(A) $\sqrt{3}$ **(B)** $2\sqrt{3}$ **(C)** $\frac{3}{2}$ **(D)** 3 **(E)** not uniquely determined by the given information

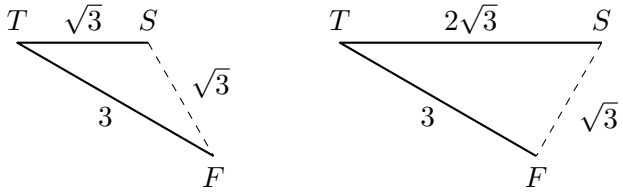
Strategy. Find the direction of the second leg, and split that leg into a part going east and a part going south with a 30-60-90 triangle. The Pythagorean theorem then gives an equation for x.

Solution. A man facing west has south on his left. A left turn of 90° points him south, and the remaining 60° of the turn bring him to a direction 30° south of east. The 3-mile part of the walk is the hypotenuse of a right triangle whose legs point east and south, and it makes a 30° angle with the eastward leg. So this is a 30-60-90 triangle: the southward part, opposite the 30° angle, is $\frac{3}{2}$ miles, and the eastward part is $\frac{3}{2}\sqrt{3}$ miles.

In all, the man ends $\frac{3}{2}$ miles south of his starting point. In the east-west direction he goes x miles west and then $\frac{3}{2}\sqrt{3}$ miles east, so the end is $|x - \frac{3}{2}\sqrt{3}|$ miles east or west of the start. The straight-line distance from the start to the end is $\sqrt{3}$, so by the Pythagorean theorem

$$\left(x - \tfrac{3}{2}\sqrt{3}\right)^2 + \left(\tfrac{3}{2}\right)^2 = 3, \quad \text{that is,} \quad \left(x - \tfrac{3}{2}\sqrt{3}\right)^2 = 3 - \tfrac{9}{4} = \tfrac{3}{4}.$$

Hence $x - \frac{3}{2}\sqrt{3}$ is $\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$, and x is $2\sqrt{3}$ or $\sqrt{3}$. Both values are positive, and both walks end $\sqrt{3}$ miles from the start. In the figure, north is up; the man starts at S, turns at T and ends at F.



The problem does not say which of the two walks the man took, so x is not uniquely determined.

Answer: (E) not uniquely determined by the given information. □

2. Answer Key

Problem	Letter	Value
1	B	3
2	A	30%
3	E	8
4	A	13
5	B	420
6	C	13
7	B	$7\sqrt{3}$
8	E	$X + Y = Z$
9	B	$12 - 4\sqrt{3}$
10	C	62
11	C	12
12	C	5
13	C	$2 - \sqrt{2}$
14	B	$7\sqrt{3}$
15	E	not uniquely determined by the given information