ANGLES, PARALLEL LINES, AND POLYGONS FOR AMC 8: PRACTICE SET SOLUTIONS

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Contents

1. Solutions 2
2. Answer Key 9

1. Solutions

Beginner.

1. (1950 AHSME, Problem #12) As the number of sides of a polygon increases from 3 to n, the sum of the exterior angles formed by extending each side in succession:

(A) Increases **(B)** Decreases **(C)** Remains constant **(D)** Cannot be predicted **(E)** Becomes $(n-3)$ straight angles

Strategy. Extending the sides in succession makes one exterior angle at each vertex. The Exterior Angle Sum gives their total for every number of sides.

Solution. Extending each side in succession, past the vertex where it ends, makes one exterior angle at every vertex of the polygon. The polygon is taken to be convex, as the Exterior Angle Sum requires. At each vertex of a convex n-gon, the interior angle and the exterior angle form a linear pair, so they add to $180°$. The n interior angles and the n exterior angles therefore add to $n \cdot 180°$. By the Interior Angle Sum, the interior angles alone add to $(n-2) \cdot 180°$, so the exterior angles add to

$$n \cdot 180° - (n-2) \cdot 180° = 360°.$$

This is the Exterior Angle Sum. The total is $360°$, two straight angles, whether the polygon has 3 sides or more, so it remains constant as the number of sides increases.

Answer: (C) Remains constant. □

2. (1991 AHSME, Problem #4) Which of the following triangles cannot exist?

(A) An acute isosceles triangle **(B)** An isosceles right triangle
(C) An obtuse right triangle **(D)** A scalene right triangle
(E) A scalene obtuse triangle

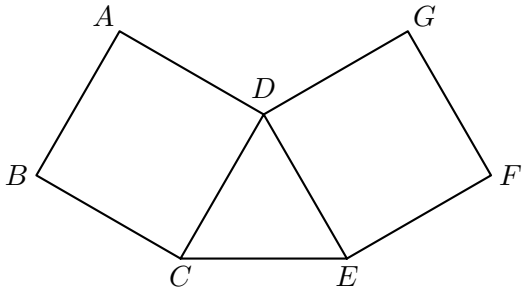
Strategy. A right angle and an obtuse angle together already add to more than $180°$.

Solution. An obtuse right triangle would have a right angle, of $90°$, and an obtuse angle, of more than $90°$. These two angles alone add to more than $180°$, but by the Triangle Angle Sum the three angles of a triangle add to exactly $180°$. So an obtuse right triangle cannot exist.

The other four triangles do exist. An equilateral triangle is isosceles, and its three angles are equal, since the angles opposite equal sides are equal; so each angle measures $60°$ and the triangle is acute. Cutting a square along a diagonal gives an isosceles right triangle: two of its sides are sides of the square, and the angle between them is a right angle. A triangle with angles $30°$, $60°$ and $90°$ is right, and one with angles $20°$, $40°$ and $120°$ is obtuse. Each of these two has three different angles, so no two of its sides are equal, because equal sides would have equal angles opposite them. Both are scalene.

Answer: (C) An obtuse right triangle. □

3. (1980 AHSME, Problem #4) In the adjoining figure, CDE is an equilateral triangle and $ABCD$ and $DEFG$ are squares. The measure of $\angle GDA$ is



(A) 90° **(B)** 105° **(C)** 120° **(D)** 135° **(E)** 150°

Strategy. Four angles fill the full turn around D: an angle of each square, an angle of the equilateral triangle, and $\angle GDA$.

Solution. Each angle of a square measures 90°, so $\angle ADC = \angle EDG = 90^\circ$. An equilateral triangle is a regular 3-gon, so each of its angles measures

$$180^\circ - \frac{360^\circ}{3} = 60^\circ,$$

and $\angle CDE = 60^\circ$. The four angles $\angle ADC$, $\angle CDE$, $\angle EDG$ and $\angle GDA$ fill the full turn around D without overlapping, so they add to 360°. Therefore

$$\angle GDA = 360^\circ - 90^\circ - 60^\circ - 90^\circ = 120^\circ.$$

Answer: (C) 120°. □

4. Two lines intersect, forming four angles. One of these four angles is 4 times as large as another one of them. What is the degree measure of the smallest of the four angles?

(A) 18 **(B)** 30 **(C)** 36 **(D)** 45 **(E)** 72

Strategy. Vertical angles are equal, so the four angles are x°, $(180-x)^\circ$, x° and $(180-x)^\circ$. An angle that is 4 times another cannot equal it, so the two angles are next to each other and add to 180°.

Solution. Let one of the four angles measure x°. The angle across from it is its vertical angle, so it also measures x°. Each of the other two angles forms a linear pair with it, so each measures $(180-x)^\circ$. One angle is 4 times another, so these two angles are not equal. Hence one of them measures x° and the other $(180-x)^\circ$, and together they make 180°. If the smaller of the two measures s°, the larger measures $4s^\circ$, and

$$s + 4s = 180,$$

so $s = 36$. The four angles measure 36°, 144°, 36° and 144°, and the smallest is 36°.

Answer: (C) 36. □

5. (1988 AHSME, Problem #6) A figure is an equiangular parallelogram if and only if it is a

(A) rectangle **(B)** regular polygon **(C)** rhombus **(D)** square **(E)** trapezoid

Strategy. Two consecutive angles of a parallelogram are same-side interior angles for a pair of parallel sides, so they are supplementary.

Solution. Let $ABCD$ be a parallelogram, so $AB \parallel DC$ and $AD \parallel BC$. With side AD as a transversal of the parallel lines AB and DC, the angles $\angle A$ and $\angle D$ are same-side interior angles, so $\angle A + \angle D = 180°$. The same holds for any two consecutive angles of the parallelogram. If the parallelogram is equiangular, then $\angle A = \angle D$, so each angle measures $90°$ and the figure is a rectangle. Conversely, a rectangle is a parallelogram whose four angles all measure $90°$, so it is an equiangular parallelogram. The equiangular parallelograms are exactly the rectangles.

None of the other choices describes the same figures. A 1-by-2 rectangle is an equiangular parallelogram, but it is not a square, a rhombus or a regular polygon, because its sides are not all equal. A trapezoid with angles $60°$, $60°$, $120°$ and $120°$ is not equiangular.

Answer: (A) rectangle. □

Intermediate.

6. (1994 AHSME, Problem #9) If $\angle A$ is four times $\angle B$, and the complement of $\angle B$ is four times the complement of $\angle A$, then $\angle B =$

(A) $10°$ **(B)** $12°$ **(C)** $15°$ **(D)** $18°$ **(E)** $22.5°$

Strategy. Write $\angle B = x°$ and $\angle A = 4x°$. Each complement is 90 minus the angle, so the condition on the complements becomes an equation in x.

Solution. Let $\angle B = x°$, so $\angle A = 4x°$. The complement of $\angle B$ measures $(90 - x)°$, and the complement of $\angle A$ measures $(90 - 4x)°$. The complement of $\angle B$ is four times the complement of $\angle A$, so

$$90 - x = 4(90 - 4x) = 360 - 16x.$$

Then $15x = 270$, so $x = 18$. As a check, $\angle A = 72°$, the complements of $\angle B$ and $\angle A$ measure $72°$ and $18°$, and $72 = 4 \cdot 18$.

Answer: (D) $18°$. □

7. (1973 AHSME, Problem #16) If the sum of all the angles except one of a convex polygon is $2190°$, then the number of sides of the polygon must be

(A) 13 **(B)** 15 **(C)** 17 **(D)** 19 **(E)** 21

Strategy. The full angle sum, $(n - 2) \cdot 180°$, is more than $2190°$ by the one angle left out, and that angle is less than $180°$ because the polygon is convex.

Solution. Let the angle left out measure $a°$. By the Interior Angle Sum, the angles of an n-gon add to $(n - 2) \cdot 180°$, so

$$(n - 2) \cdot 180 = 2190 + a.$$

The polygon is convex, so $0 < a < 180$, and dividing by 180 gives

$$12\tfrac{1}{6} = \frac{2190}{180} < n - 2 < \frac{2190 + 180}{180} = 13\tfrac{1}{6}.$$

The only whole number in this range is 13, so $n - 2 = 13$ and $n = 15$. The angle left out measures $13 \cdot 180° - 2190° = 150°$.

Answer: (B) 15. □

8. (1976 AHSME, Problem #14) The measures of the interior angles of a convex polygon are in arithmetic progression. If the smallest angle is $100°$ and the largest angle is $140°$, then the number of sides the polygon has is

(A) 6 **(B)** 8 **(C)** 10 **(D)** 11 **(E)** 12

Strategy. In an arithmetic progression, terms the same distance from the two ends have a constant sum, so the average angle is the average of the smallest and the largest. Compare the total with the Interior Angle Sum.

Solution. Let the polygon have n sides. Write its angles in increasing order, and below them the same angles in decreasing order. Since the angles are in arithmetic progression, each column then adds to $100^\circ + 140^\circ = 240^\circ$. The two rows together add to $240n$ degrees, so the angles add to $120n$ degrees. By the Interior Angle Sum,

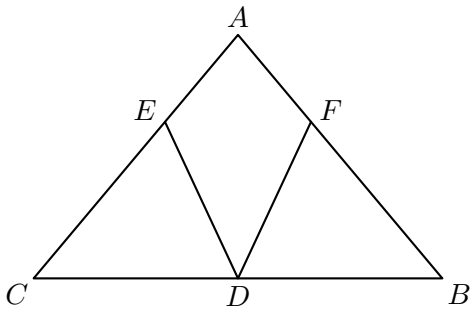
$$120n = (n-2) \cdot 180 = 180n - 360,$$

so $60n = 360$ and $n = 6$. For example, the six angles 100°, 108°, 116°, 124°, 132° and 140° add to $720^\circ = (6-2) \cdot 180^\circ$.

The exterior angles give the same answer: they also form an arithmetic progression, from 80° to 40°, so their average is 60°, and by the Exterior Angle Sum $60n = 360$.

Answer: (A) 6. □

9. (1977 AHSME, Problem #4) In triangle ABC, $AB = AC$ and $\angle A = 80^\circ$. If points D, E and F lie on sides BC, AC and AB, respectively, and $CE = CD$ and $BF = BD$, then $\angle EDF$ equals



(A) 30° **(B)** 40° **(C)** 50° **(D)** 65° **(E)** none of these

Strategy. Find the base angles of triangle ABC first, then those of the isosceles triangles CDE and BDF. The three angles at D above line BC make a straight angle.

Solution. Since $AB = AC$, the angles opposite these sides are equal, so $\angle B = \angle C$. By the Triangle Angle Sum they add to $180^\circ - 80^\circ = 100^\circ$, so $\angle B = \angle C = 50^\circ$.

In triangle CDE, the sides CE and CD are equal, so the angles opposite them, $\angle CDE$ and $\angle CED$, are equal. They add to $180^\circ - \angle C = 130^\circ$, so $\angle CDE = 65^\circ$. In the same way, $BF = BD$ in triangle BDF gives $\angle BDF = \angle BFD = 65^\circ$.

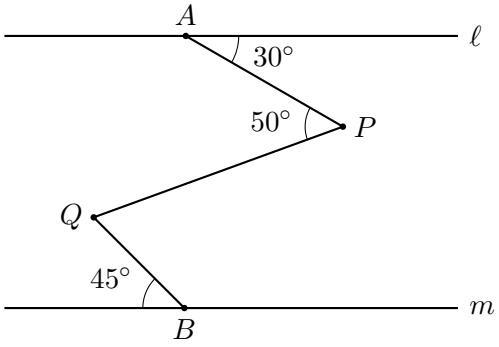
The angles $\angle BDF$, $\angle FDE$ and $\angle EDC$ together make up the straight angle BDC, so

$$\angle EDF = 180^\circ - 65^\circ - 65^\circ = 50^\circ.$$

The answer does not depend on where D lies on BC.

Answer: (C) 50°. □

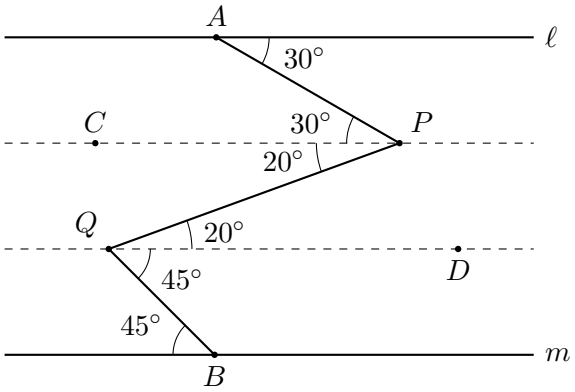
10. In the figure, lines ℓ and m are parallel, point A lies on ℓ, point B lies on m, and points P and Q lie between the two lines. What is the degree measure of $\angle PQB$?



(A) 55 **(B)** 65 **(C)** 75 **(D)** 80 **(E)** 95

Strategy. Draw the lines through P and Q parallel to ℓ. They split $\angle APQ$ and $\angle PQB$ into parts that are alternate interior angles to the angles at A and B or to each other.

Solution. Draw the line through P parallel to ℓ and the line through Q parallel to ℓ. Both are also parallel to m. Choose a point C on the first line to the left of P and a point D on the second line to the right of Q, as shown.



With AP as the transversal of the parallel lines ℓ and PC, the 30° angle at A and $\angle APC$ are alternate interior angles, so $\angle APC = 30°$. Ray PC lies inside $\angle APQ$, so

$$\angle CPQ = \angle APQ - \angle APC = 50° - 30° = 20°.$$

With PQ as the transversal of the parallel lines PC and QD, the angles $\angle CPQ$ and $\angle PQD$ are alternate interior angles, so $\angle PQD = 20°$. With QB as the transversal of the parallel lines QD and m, the 45° angle at B and $\angle DQB$ are alternate interior angles, so $\angle DQB = 45°$. Ray QD lies inside $\angle PQB$, so

$$\angle PQB = \angle PQD + \angle DQB = 20° + 45° = 65°.$$

Answer: (B) 65. □

Advanced.

11. (2003 AMC 8, Problem #20) What is the measure of the acute angle formed by the hands of a clock at 4:20 a.m.?

(A) 0° **(B)** 5° **(C)** 8° **(D)** 10° **(E)** 12°

Strategy. The twelve hour marks are 30° apart. In 20 minutes the hour hand moves a third of the way from the 4 to the 5.

Solution. The twelve hour marks divide the full turn of 360° into twelve equal angles of 30°. Measure the position of each hand clockwise from the 12.

In 20 minutes the minute hand makes a third of a full turn, so at 4:20 it is $\frac{1}{3}\cdot 360^\circ = 120^\circ$ from the 12, pointing at the 4. At 4:00 the hour hand points at the 4, which is $4\cdot 30^\circ = 120^\circ$ from the 12, and in the next hour it moves 30°, to the 5. In 20 minutes it moves a third of that, 10°, so at 4:20 it is 130° from the 12. The two hands are therefore

$$130^\circ - 120^\circ = 10^\circ$$

apart, and this is the acute angle they form.

Answer: (D) 10°. □

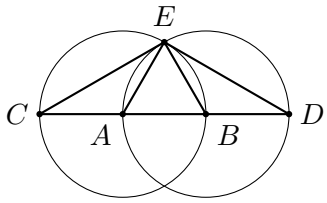
12. (2016 AMC 8, Problem #23) Two congruent circles centered at points A and B each pass through the other circle's center. The line containing both A and B is extended to intersect the circles at points C and D.
The circles intersect at two points, one of which is E. What is the degree measure of $\angle CED$?

(A) 90 **(B)** 105 **(C)** 120 **(D)** 135 **(E)** 150

Strategy. The segments AB, AE, BE, AC and BD are all radii, so they are equal. This makes triangle ABE equilateral and triangles ACE and BDE isosceles, each with an exterior angle of 60°.

Solution. Let r be the radius of the two circles. Each circle passes through the center of the other, so $AB = r$. Point E lies on both circles, so $AE = BE = r$. Triangle ABE is therefore equilateral. An equilateral triangle is a regular 3-gon, so each of its angles measures $180^\circ - \frac{360^\circ}{3} = 60^\circ$.

The line through A and B meets the circle centered at A again at a point on the far side of A from B, and the circle centered at B again at a point on the far side of B from A. These are C and D, so the four points lie on the line in the order C, A, B, D, with $AC = BD = r$.



In triangle ACE, $AC = AE$, so $\angle ACE = \angle AEC$. Since C, A and B lie on a line, $\angle EAB$ is the exterior angle of triangle ACE at A. By the Exterior Angle Theorem,

$$\angle ACE + \angle AEC = \angle EAB = 60^\circ,$$

so $\angle AEC = 30^\circ$. In the same way, $\angle BED = 30^\circ$ in triangle BDE. Rays EA and EB lie inside $\angle CED$, so by angle addition

$$\angle CED = \angle CEA + \angle AEB + \angle BED = 30^\circ + 60^\circ + 30^\circ = 120^\circ.$$

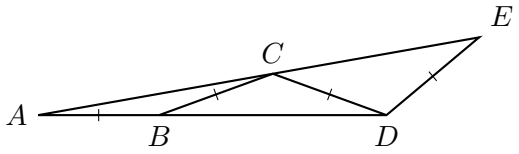
Answer: (C) 120. □

13. (1978 AHSME, Problem #12) In $\triangle ADE$, $\angle ADE = 140^\circ$, points B and C lie on sides AD and AE, respectively, and points A, B, C, D, E are distinct. If lengths AB, BC, CD, and DE are all equal, then the measure of $\angle EAD$ is

(A) 5° **(B)** 6° **(C)** 7.5° **(D)** 8° **(E)** 10°

Strategy. The four equal segments form a chain of isosceles triangles ABC, BCD and CDE. Write $\angle EAD = x^\circ$ and follow the chain with the Exterior Angle Theorem until the angle at E is known.

Solution. Let $\angle EAD = x^\circ$. The figure below is drawn with the answer, $x = 10$; the tick marks show the four equal segments.



In triangle ABC, $AB = BC$, so $\angle BCA = \angle BAC = x^\circ$. Point B lies on side AD, so $\angle CBD$ is the exterior angle of triangle ABC at B, and by the Exterior Angle Theorem $\angle CBD = x^\circ + x^\circ = 2x^\circ$.

In triangle BCD, $BC = CD$, so $\angle CDB = \angle CBD = 2x^\circ$.

Point C lies on side AE, so $\angle DCE$ is the exterior angle of triangle ACD at C. Its remote interior angles are $\angle CAD = x^\circ$ and $\angle ADC$. Since B lies on side AD, $\angle ADC$ is the same angle as $\angle BDC$, so $\angle ADC = 2x^\circ$. Hence $\angle DCE = x^\circ + 2x^\circ = 3x^\circ$.

In triangle CDE, $CD = DE$, so $\angle DEC = \angle DCE = 3x^\circ$.

Finally, since C lies on side AE, $\angle AED$ is the same angle as $\angle CED = 3x^\circ$. The angles of triangle ADE are therefore x°, 140° and $3x^\circ$, and by the Triangle Angle Sum

$$x + 140 + 3x = 180.$$

Then $4x = 40$ and $x = 10$. As a check, $\angle CDE = 180^\circ - 30^\circ - 30^\circ = 120^\circ$, and $\angle ADE = \angle ADC + \angle CDE = 20^\circ + 120^\circ = 140^\circ$.

Answer: (E) 10°. □

14. (2012 AMC 10A, Problem #10) Mary divides a circle into 12 sectors. The central angles of these sectors, measured in degrees, are all integers and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?

(A) 5 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

Strategy. The twelve central angles fill the full turn around the center, so they add to 360°. Write them with the smallest angle and the common difference, and make the difference as large as possible.

Solution. The sectors fill the circle without overlapping, so their central angles fill the full turn around the center and add to 360°. List the angles in increasing order. With smallest angle a° and common difference d°, they are

$$a,\ a + d,\ a + 2d,\ \ldots,\ a + 11d,$$

where a and d are integers, $a \geq 1$ and $d \geq 0$. Their sum is

$$12a + (1 + 2 + \cdots + 11)d = 12a + 66d = 360,$$

so $2a + 11d = 60$. Since $2a$ and 60 are even, $11d$ is even, so d is even. Since $a \geq 1$, $11d < 60$, so d is 0, 2 or 4, and then $a = 30$, 19 or 8. The smallest possible sector angle is 8°, from the angles $8^\circ, 12^\circ, 16^\circ, \ldots, 52^\circ$.

Answer: (C) 8. □

15. (1962 AHSME, Problem #31) The ratio of the interior angles of two regular polygons with sides of unit length is 3 : 2. How many such pairs are there?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** infinitely many

Strategy. Each interior angle of a regular n-gon measures $180^\circ - \frac{360^\circ}{n}$. The ratio $3:2$ becomes an equation in the two numbers of sides, and one of them must be small.

Solution. The angles of a regular polygon depend only on its number of sides, so the unit side length plays no role. Let the polygon with the larger angles have a sides and the other one b sides. Each interior angle of a regular n-gon measures $180^\circ - \frac{360^\circ}{n}$, so the ratio $3:2$ says

$$2\left(180 - \frac{360}{a}\right) = 3\left(180 - \frac{360}{b}\right).$$

Dividing by 180 gives $2 - \frac{4}{a} = 3 - \frac{6}{b}$, that is,

$$\frac{6}{b} = 1 + \frac{4}{a}.$$

The right side is more than 1, so $\frac{6}{b} > 1$ and $b < 6$. Hence b is 3, 4 or 5, and solving $\frac{4}{a} = \frac{6}{b} - 1$ gives $a = 4$, 8 or 20, respectively. The three pairs are a square and an equilateral triangle (angles 90° and 60°), a regular octagon and a square (135° and 90°), and a regular 20-gon and a regular pentagon (162° and 108°). So there are 3 such pairs.

Answer: (C) 3. □

2. Answer Key

Problem	Letter	Value
1	C	Remains constant
2	C	An obtuse right triangle
3	C	120°
4	C	36
5	A	rectangle
6	D	18°
7	B	15
8	A	6
9	C	50°
10	B	65
11	D	10°
12	C	120
13	E	10°
14	C	8
15	C	3