COMBINATIONS AND COUNTING STRATEGIES FOR AMC 8

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1. Combinations

This section counts the ways to choose objects when the order of the chosen objects does not matter. The counts use factorials and the number $P(n,k) = n \times (n-1) \times \cdots \times (n-k+1)$ of ordered selections, or permutations, of k of n different objects.

1.1. Counting Combinations.

Definition 1.1.1. A **combination** of k objects from a set of n different objects is an unordered selection of k of the objects. The number of such combinations is written $C(n,k)$, ${}_nC_k$, or $\binom{n}{k}$.

The k objects of a combination can be put in order in $k!$ ways, so $P(n,k)$ counts each combination $k!$ times. Dividing by $k!$ gives

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{k!} \qquad (1 \le k \le n).$$

When $k < n$, writing $P(n,k)$ as $\frac{n!}{(n-k)!}$ gives the factorial form

$$C(n,k) = \frac{n!}{k!\,(n-k)!}.$$

For example, a 3-person committee chosen from 10 people is an unordered selection, so the number of committees is

$$C(10,3) = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120.$$

Choosing k of n different objects also decides the other $n-k$. So, when the two groups can be told apart, for example by their sizes or because they go to different people, $C(n,k)$ is also the number of ways to split n different objects into a group of k and a group of the other $n-k$.

Examples.

Problem 1.1.2 (1996 AHSME, Problem #10)**.** How many line segments have both their endpoints located at the vertices of a given cube?

(A) 12 **(B)** 15 **(C)** 24 **(D)** 28 **(E)** 56

Strategy. A segment is determined by its two endpoints, and the order of the endpoints does not matter. So the segments are the combinations of 2 of the vertices.

Solution. A cube has 8 vertices. Any two different vertices X and Y are the endpoints of exactly one segment, and the segment from X to Y is the same as the segment from Y to X. So each segment is an unordered selection of 2 of the 8 vertices, and the number of segments is

$$C(8,2) = \frac{8 \times 7}{2 \times 1} = 28.$$

As a check, the segments are the 12 edges, the 12 diagonals of the faces (2 on each of the 6 faces) and the 4 diagonals that pass through the center of the cube, and $12 + 12 + 4 = 28$.

Answer: (D) 28. □

Problem 1.1.3. A science club has 7 seventh graders and 5 eighth graders. A team of 3 seventh graders and 2 eighth graders is to be chosen for a competition. In how many ways can the team be chosen?

Strategy. Choose the seventh graders and the eighth graders separately. Each choice is an unordered selection, and the two choices together give the team.

Solution. The 3 seventh graders are an unordered selection from 7, so they can be chosen in

$$C(7,3) = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

ways. Whichever seventh graders are chosen, the 2 eighth graders can be chosen in

$$C(5,2) = \frac{5 \times 4}{2 \times 1} = 10$$

ways. Each team comes from exactly one choice of seventh graders and one choice of eighth graders, so by the Multiplication Principle the number of teams is $35 \times 10 = 350$.

Answer: 350. □

1.2. **Permutations Versus Combinations.** To decide between the two, check whether the order of the selected objects matters. If exchanging two of the selected objects gives a different outcome, count permutations. If it gives the same outcome, count combinations.

For example, take four students A, B, C and D. Choosing a president and a vice-president is a permutation, because A as president with B as vice-president differs from B as president with A as vice-president. There are $P(4,2) = 4 \times 3 = 12$ ways. Choosing a 2-person committee is a combination, because the committee $\{A, B\}$ is the same as the committee $\{B, A\}$. There are

$$C(4,2) = \frac{4 \times 3}{2 \times 1} = 6$$

ways, half of 12, since each committee gives $2! = 2$ choices of president and vice-president.

Example.

Problem 1.2.1 (2006 AMC 8, Problem #20)**.** A singles tournament had six players. Each player played every other player only once, with no ties. If Helen won 4 games, Ines won 3 games, Janet won 2 games, Kendra won 2 games and Lara won 2 games, how many games did Monica win?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Strategy. Count the games. A game is determined by the pair of players in it, and exchanging the two players gives the same game, so count combinations. Each game has exactly one winner.

Solution. The six players are Helen, Ines, Janet, Kendra, Lara and Monica. Each pair of players played exactly one game, and the game between Helen and Ines is the same as the game between Ines and Helen. So the games are the combinations of 2 of the 6 players, and there are

$$C(6,2) = \frac{6 \times 5}{2 \times 1} = 15$$

games. Counting ordered pairs of players would give $P(6,2) = 30$, which counts each game twice. No game ended in a tie, so each of the 15 games has exactly one winner, and the numbers of games won by the six players add up to 15. The first five players won $4 + 3 + 2 + 2 + 2 = 13$ games, so Monica won $15 - 13 = 2$ games.

Answer: (C) 2. □

2. Probability

Many counting problems ask for a probability. An **experiment**, such as tossing a coin or rolling a die, has a set of possible results called **outcomes**. An **event** is a set of outcomes, such as "the die shows an even number."

Definition 2.1. If an experiment has equally likely outcomes, the **probability** of an event is

$$P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}},$$

where the **favorable outcomes** are the outcomes in the event.

The P in $P(\text{event})$ stands for probability, not for the number $P(n, k)$ of ordered selections. A probability is a number from 0 to 1. Finding one is a counting problem: count the favorable outcomes and the total outcomes, often with the Multiplication Principle or with combinations. The outcomes counted must be equally likely. When a fair coin is tossed several times, for example, the equally likely outcomes are the ordered sequences of heads and tails, not the numbers of heads.

Examples.

Problem 2.2. Two fair coins are tossed. What is the probability that exactly one of them lands heads?

Strategy. List the equally likely outcomes as ordered pairs: the result of the first coin, then the result of the second.

Solution. Each coin lands heads (H) or tails (T) with equal chance, so by the Multiplication Principle there are $2 \times 2 = 4$ equally likely outcomes: HH, HT, TH and TT, where the first letter is the result of the first coin. Exactly one head appears in two of them, HT and TH, so

$$P(\text{exactly one head}) = \frac{2}{4} = \frac{1}{2}.$$

The three results "no heads," "one head" and "two heads" are not equally likely, because "one head" happens in two ways and each of the others in one way. So the probability is not $\frac{1}{3}$.

Answer: $\frac{1}{2}$. □

Problem 2.3 (2016 AMC 8, Problem #13)**.** Two different numbers are randomly selected from the set

$$\{-2, -1, 0, 3, 4, 5\}$$

and multiplied together. What is the probability that the product is 0?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

Strategy. The product does not depend on the order of the two numbers, so take the outcomes to be the unordered pairs, which are equally likely. Count all pairs, then the pairs whose product is 0.

Solution. The outcomes are the combinations of 2 of the 6 numbers, and each is equally likely to be selected. There are

$$C(6, 2) = \frac{6 \times 5}{2 \times 1} = 15$$

of them. A product of two numbers is 0 exactly when one of the numbers is 0, because the product of two nonzero numbers is not 0. A pair that contains 0 is made of 0 and one of the other 5 numbers, so 5 pairs have product 0. The probability is

$$\frac{5}{15} = \frac{1}{3}.$$

Answer: (D) $\frac{1}{3}$. □

3. Counting Strategies

This section covers three ways to organize a count: casework, complementary counting, and the Principle of Inclusion-Exclusion.

3.1. **Casework.** The Addition Principle counts outcomes that split into groups, no two of which have an outcome in common. Such groups are called **mutually exclusive**.

Theorem 3.1.1 (Addition Principle)**.** *If the ways to do a task split into two mutually exclusive groups, one of m ways and one of n ways, then the task can be done in $m + n$ ways. The same holds for more than two groups: add the sizes of all the groups.*

For example, a student who takes one mathematics course, either one of 3 geometry courses or one of 2 algebra courses, has $3 + 2 = 5$ choices.

Casework splits the outcomes into cases, counts each case, and adds the counts. The cases must be mutually exclusive, so that no outcome is counted twice, and **exhaustive**, meaning that every outcome belongs to some case. The Addition Principle then gives the total. Choose the cases so that each one is easy to count, often with the Multiplication Principle or with combinations. When an outcome is made of several numbers, the cases are often the possible values of one of them.

Example.

Problem 3.1.2 (2005 AMC 8, Problem #14)**.** The Little Twelve Basketball Conference has two divisions, with six teams in each division. Each team plays each of the other teams in its own division twice and every team in the other division once. How many conference games are scheduled?

(A) 80 **(B)** 96 **(C)** 100 **(D)** 108 **(E)** 192

Strategy. Split the games into two cases: games between two teams of the same division, and games between teams of different divisions. In each case, count the pairs of teams that meet.

Solution. Every game is played by two teams, which are either in the same division or in different divisions. So the two cases below are mutually exclusive and exhaustive.

Case 1: The two teams are in the same division. In one division, a pair of teams can be chosen in $C(6, 2) = \frac{6\times5}{2\times1} = 15$ ways, so the two divisions have $2 \times 15 = 30$ such pairs. Each pair plays twice, which gives $30 \times 2 = 60$ games.

Case 2: The two teams are in different divisions. A team from the first division and a team from the second division can be chosen in $6 \times 6 = 36$ ways, and each such pair plays once, which gives 36 games.

By the Addition Principle, the number of conference games is

$$60 + 36 = 96.$$

As a check, each team plays $5 \times 2 + 6 = 16$ games. Adding these numbers for all 12 teams gives $12 \times 16 = 192$, which counts every game twice, once for each of its two teams, and $192 \div 2 = 96$.

Answer: (B) 96. □

3.2. **Complementary Counting. Complementary counting** counts the outcomes that are not wanted and subtracts them from the total:

$$\text{wanted outcomes} = \text{all outcomes} - \text{unwanted outcomes},$$

where each term is a number of outcomes. The unwanted outcomes are called the **complement** of the wanted outcomes. This is casework with two cases, the wanted outcomes and their complement.

It helps when the complement is easier to count, for example when the wanted outcomes have "at least one" of something and the complement has none.

Examples.

Problem 3.2.1 (2016 AMC 8, Problem #17)**.** An ATM password at Fred's Bank is composed of four digits from 0 to 9, with repeated digits allowable. If no password may begin with the sequence 9, 1, 1, then how many passwords are possible?

(A) 30 **(B)** 7290 **(C)** 9000 **(D)** 9990 **(E)** 9999

Strategy. Count the passwords that begin with 9, 1, 1 and subtract them from the total.

Solution. Each of the four digits has 10 choices, so by the Multiplication Principle there are

$$10 \times 10 \times 10 \times 10 = 10{,}000$$

passwords in all. A password that begins with 9, 1, 1 has its first three digits fixed and any of the 10 digits last, so there are $1 \times 1 \times 1 \times 10 = 10$ such passwords. The number of allowed passwords is

$$10{,}000 - 10 = 9990.$$

Answer: (D) 9990. □

Problem 3.2.2 (2015 AMC 8, Problem #7)**.** Each of two boxes contains three chips numbered 1, 2, 3. A chip is drawn randomly from each box and the numbers on the two chips are multiplied. What is the probability that their product is even?

(A) $\frac{1}{9}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{4}{9}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{5}{9}$

Strategy. An odd product needs two odd chips, which is a single case. Count the odd products and subtract them from the total.

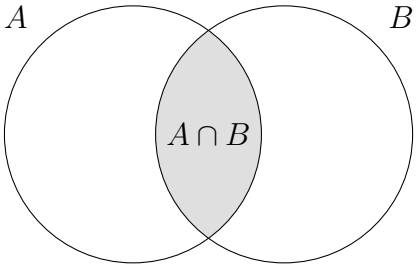
Solution. A product of two whole numbers is even when at least one of them is even, which splits into three cases: even and even, even and odd, odd and even. The complement is the single case odd and odd, which gives an odd product, so count it instead.

A chip is drawn from each box, so by the Multiplication Principle there are $3 \times 3 = 9$ equally likely outcomes. Two of the three chips in each box, 1 and 3, are odd, so $2 \times 2 = 4$ outcomes give an odd product. The other $9 - 4 = 5$ outcomes give an even product, so the probability that the product is even is $\frac{5}{9}$.

Answer: (E) $\frac{5}{9}$. □

3.3. **Principle of Inclusion-Exclusion for Two Sets.** For a set A, $|A|$ is the number of elements of A. The **union** $A \cup B$ is the set of elements that are in A or B (or both), and the **intersection** $A \cap B$ is the set of elements that are in both A and B. The Principle of Inclusion-Exclusion counts the elements of $A \cup B$ when A and B may have elements in common.

A **Venn diagram** draws each set as a circle. In the Venn diagram below, the elements of $A \cap B$ lie in the shaded region where the two circles overlap.



The sum $|A|+|B|$ counts each element of $A\cap B$ twice, once in A and once in B, and every other element of $A\cup B$ once. Subtracting $|A\cap B|$ once counts every element of $A\cup B$ exactly once.

Theorem 3.3.1 (Principle of Inclusion-Exclusion)**.** *For any two finite sets A and B,*

$$|A\cup B| = |A| + |B| - |A\cap B|.$$

Example.

Problem 3.3.2. In a class of 30 students, 18 play soccer, 15 play basketball, and 8 play both sports. How many students in the class play neither sport?

Strategy. Use the Principle of Inclusion-Exclusion to count the students who play at least one sport, then subtract from the class size.

Solution. Let S be the set of students who play soccer and B the set of students who play basketball, so $|S| = 18$, $|B| = 15$ and $|S\cap B| = 8$. The students who play at least one sport form the union $S\cup B$. By the Principle of Inclusion-Exclusion,

$$|S\cup B| = |S| + |B| - |S\cap B| = 18 + 15 - 8 = 25.$$

The students who play neither sport are the complement of $S\cup B$ in the class, so there are $30-25=5$ of them. As a check, in a Venn diagram of S and B, the overlap holds the 8 students who play both sports, the rest of the soccer circle holds $18-8=10$ students, and the rest of the basketball circle holds $15-8=7$, so $8+10+7=25$ students play at least one sport.

Answer: 5. □

4. Problems

Beginner.

1. (2013 AMC 8, Problem #8) A fair coin is tossed 3 times. What is the probability of at least two consecutive heads?
(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{3}{4}$

2. (1991 AJHSME, Problem #9) How many whole numbers from 1 through 46 are divisible by either 3 or 5 or both?
(A) 18 **(B)** 21 **(C)** 24 **(D)** 25 **(E)** 27

3. (2014 AMC 8, Problem #12) A magazine printed photos of three celebrities along with three photos of the celebrities as babies. The baby pictures did not identify the celebrities. Readers were asked to match each celebrity with the correct baby picture. What is the probability that a reader guessing at random will match all three correctly?
(A) $\frac{1}{9}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

4. (2001 AMC 8, Problem #14) Tyler has entered a buffet line in which he chooses one kind of meat, two different vegetables and one dessert. If the order of food items is not important, how many different meals might he choose?
Meat: beef, chicken, pork
Vegetables: baked beans, corn, potatoes, tomatoes
Dessert: brownies, chocolate cake, chocolate pudding, ice cream
(A) 4 **(B)** 24 **(C)** 72 **(D)** 80 **(E)** 144

Intermediate.

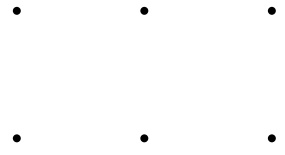
5. (2019 AMC 8, Problem #11) The eighth grade class at Lincoln Middle School has 93 students. Each student takes a math class or a foreign language class or both. There are 70 eighth graders taking a math class, and there are 54 eighth graders taking a foreign language class. How many eighth graders take *only* a math class and *not* a foreign language class?
(A) 16 **(B)** 23 **(C)** 31 **(D)** 39 **(E)** 70

6. (2012 AMC 8, Problem #14) In the BIG N, a middle school football conference, each team plays every other team exactly once. If a total of 21 conference games were played during the 2012 season, how many teams were members of the BIG N conference?
(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

7. (1985 AJHSME, Problem #15) How many whole numbers between 100 and 400 contain the digit 2?
(A) 100 **(B)** 120 **(C)** 138 **(D)** 140 **(E)** 148

8. (2015 AMC 8, Problem #15) At Euler Middle School, 198 students voted on two issues in a school referendum with the following results. 149 voted in favor of the first issue and 119 voted in favor of the second issue. If there were exactly 29 students who voted against both issues, how many students voted in favor of both issues?
(A) 49 **(B)** 70 **(C)** 79 **(D)** 99 **(E)** 149

9. (2005 AMC 8, Problem #21) How many distinct triangles can be drawn using three of the dots below as vertices?



(A) 9 **(B)** 12 **(C)** 18 **(D)** 20 **(E)** 24

Advanced.

10. (2011 AMC 8, Problem #18) A fair six-sided die is rolled twice. What is the probability that the first number that comes up is greater than or equal to the second number?

(A) $\frac{1}{6}$ **(B)** $\frac{5}{12}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{7}{12}$ **(E)** $\frac{5}{6}$

11. (2007 AMC 8, Problem #21) Two cards are dealt from a deck of four red cards labeled A, B, C, D and four green cards labeled A, B, C, D. A winning pair is two of the same color or two of the same letter. What is the probability of drawing a winning pair?

(A) $\frac{2}{7}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{4}{7}$ **(E)** $\frac{5}{8}$

12. (2002 AMC 8, Problem #21) Harold tosses a nickel four times. The probability that he gets at least as many heads as tails is

(A) $\frac{5}{16}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{11}{16}$

13. (1990 AHSME, Problem #16) At one of George Washington's parties, each man shook hands with everyone except his spouse, and no handshakes took place between women. If 13 married couples attended, how many handshakes were there among these 26 people?

(A) 78 **(B)** 185 **(C)** 234 **(D)** 312 **(E)** 325

14. (2020 AMC 8, Problem #23) Five different awards are to be given to three students. Each student will receive at least one award. In how many different ways can the awards be distributed?

(A) 120 **(B)** 150 **(C)** 180 **(D)** 210 **(E)** 240