COMBINATIONS AND COUNTING STRATEGIES FOR AMC 8: SOLUTIONS

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1. Solutions

Beginner.

1. (2013 AMC 8, Problem #8) A fair coin is tossed 3 times. What is the probability of at least two consecutive heads?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{3}{4}$

Strategy. There are only $2^3 = 8$ equally likely sequences of tosses, so list them all.

Solution. Each toss has 2 equally likely results, so by the Multiplication Principle there are $2 \times 2 \times 2 = 8$ equally likely sequences:

HHH, HHT, HTH, THH, HTT, THT, TTH, TTT.

Two heads appear next to each other only in HHH, HHT and THH. In HTH the two heads are separated by a tail. So 3 of the 8 outcomes are favorable, and the probability is $\frac{3}{8}$.

Answer: (C) $\frac{3}{8}$. □

2. (1991 AJHSME, Problem #9) How many whole numbers from 1 through 46 are divisible by either 3 or 5 or both?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 25 **(E)** 27

Strategy. Count the multiples of 3 and the multiples of 5, and use the Principle of Inclusion-Exclusion, since some numbers are multiples of both.

Solution. Among the whole numbers from 1 through 46, let A be the set of multiples of 3 and B the set of multiples of 5. Since $3 \times 15 = 45 \leq 46 < 48 = 3 \times 16$, the multiples of 3 are $3 \times 1, 3 \times 2, \ldots, 3 \times 15$, so $|A| = 15$. Since $5 \times 9 = 45 \leq 46 < 50 = 5 \times 10$, the multiples of 5 are $5 \times 1, 5 \times 2, \ldots, 5 \times 9$, so $|B| = 9$. The numbers in both sets are the multiples of 5 that are also divisible by 3. Among $5, 10, \ldots, 45$, these are 15, 30 and 45, so $|A \cap B| = 3$. By the Principle of Inclusion-Exclusion,

$$|A \cup B| = |A| + |B| - |A \cap B| = 15 + 9 - 3 = 21.$$

Answer: (B) 21. □

3. (2014 AMC 8, Problem #12) A magazine printed photos of three celebrities along with three photos of the celebrities as babies. The baby pictures did not identify the celebrities. Readers were asked to match each celebrity with the correct baby picture. What is the probability that a reader guessing at random will match all three correctly?

(A) $\frac{1}{9}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

Strategy. A matching places the three baby pictures next to the three celebrities in some order, so the matchings are the arrangements of 3 objects.

Solution. Line up the three celebrities. A matching places the three baby pictures next to them in some order, so there are $3! = 3 \times 2 \times 1 = 6$ possible matchings. A reader guessing at random is equally likely to choose any of them, and exactly one matching is completely correct. The probability is therefore $\frac{1}{6}$.

Answer: (B) $\frac{1}{6}$. □

4. (2001 AMC 8, Problem #14) Tyler has entered a buffet line in which he chooses one kind of meat, two different vegetables and one dessert. If the order of food items is not important, how many different meals might he choose?
Meat: beef, chicken, pork
Vegetables: baked beans, corn, potatoes, tomatoes
Dessert: brownies, chocolate cake, chocolate pudding, ice cream
(A) 4 **(B)** 24 **(C)** 72 **(D)** 80 **(E)** 144

Strategy. Choose the meat, then the vegetables, then the dessert, and multiply. The two vegetables are an unordered selection, so count them with a combination.

Solution. There are 3 ways to choose the meat. The two vegetables are two different vegetables chosen from four, in no particular order, so they can be chosen in

$$C(4,2) = \frac{4 \times 3}{2 \times 1} = 6$$

ways. There are 4 ways to choose the dessert. By the Multiplication Principle, the number of meals is

$$3 \times 6 \times 4 = 72.$$

Answer: (C) 72. □

Intermediate.

5. (2019 AMC 8, Problem #11) The eighth grade class at Lincoln Middle School has 93 students. Each student takes a math class or a foreign language class or both. There are 70 eighth graders taking a math class, and there are 54 eighth graders taking a foreign language class. How many eighth graders take *only* a math class and *not* a foreign language class?
(A) 16 **(B)** 23 **(C)** 31 **(D)** 39 **(E)** 70

Strategy. Use the Principle of Inclusion-Exclusion to find how many students take both classes, then subtract them from the math students.

Solution. Let M be the set of students taking a math class and F the set of students taking a foreign language class. Every student takes at least one of the two, so $|M \cup F| = 93$. By the Principle of Inclusion-Exclusion,

$$|M \cup F| = |M| + |F| - |M \cap F|, \quad \text{so} \quad 93 = 70 + 54 - |M \cap F|.$$

This gives $|M \cap F| = 124 - 93 = 31$, so 31 students take both classes. The students who take only math are the math students who are not among these 31:

$$70 - 31 = 39.$$

Answer: (D) 39. □

6. (2012 AMC 8, Problem #14) In the BIG N, a middle school football conference, each team plays every other team exactly once. If a total of 21 conference games were played during the 2012 season, how many teams were members of the BIG N conference?
(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Strategy. Each game is played by a pair of teams, and the order of the two teams does not matter. So the number of games is the number of ways to choose 2 of the teams.

Solution. Suppose the conference has n teams. Each pair of teams plays exactly one game, so the number of games equals the number of pairs of teams:

$$C(n,2) = \frac{n(n-1)}{2} = 21, \quad \text{so} \quad n(n-1) = 42.$$

The product $n(n-1)$ increases as n increases, and $7 \times 6 = 42$, so $n = 7$ is the only solution.

Answer: (B) 7. □

7. (1985 AJHSME, Problem #15) How many whole numbers between 100 and 400 contain the digit 2?

(A) 100 **(B)** 120 **(C)** 138 **(D)** 140 **(E)** 148

Strategy. Count the numbers with no digit 2 and subtract them from the total.

Solution. The whole numbers between 100 and 400 are 101 through 399. The number 100 has no digit 2, so adding it does not change the answer, and it is enough to count among the 300 numbers from 100 to 399.

A number in this range with no digit 2 has hundreds digit 1 or 3 (2 choices), and its tens and units digits can each be any digit except 2 (9 choices each). By the Multiplication Principle, $2 \times 9 \times 9 = 162$ of the numbers have no digit 2. The rest contain the digit 2:

$$300 - 162 = 138.$$

Answer: (C) 138. □

8. (2015 AMC 8, Problem #15) At Euler Middle School, 198 students voted on two issues in a school referendum with the following results. 149 voted in favor of the first issue and 119 voted in favor of the second issue. If there were exactly 29 students who voted against both issues, how many students voted in favor of both issues?

(A) 49 **(B)** 70 **(C)** 79 **(D)** 99 **(E)** 149

Strategy. The students who voted in favor of at least one issue are all the students except the 29 who voted against both. Count them, and then use the Principle of Inclusion-Exclusion.

Solution. Let F be the set of students who voted in favor of the first issue and S the set of students who voted in favor of the second issue, so $|F| = 149$ and $|S| = 119$. Each student voted on both issues, so a student who is in neither set voted against both issues. The students in $F \cup S$ are therefore the complement of these 29 students, and

$$|F \cup S| = 198 - 29 = 169.$$

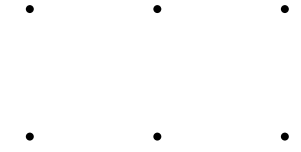
By the Principle of Inclusion-Exclusion,

$$|F \cup S| = |F| + |S| - |F \cap S|, \quad \text{so} \quad 169 = 149 + 119 - |F \cap S|.$$

This gives $|F \cap S| = 268 - 169 = 99$, so 99 students voted in favor of both issues.

Answer: (D) 99. □

9. (2005 AMC 8, Problem #21) How many distinct triangles can be drawn using three of the dots below as vertices?



(A) 9 **(B)** 12 **(C)** 18 **(D)** 20 **(E)** 24

Strategy. A triangle is determined by its set of three vertices. Count all sets of three dots, and subtract the sets of three dots that lie on one line, since those give no triangle.

Solution. The figure has 6 dots, in two rows of three. A set of three dots is an unordered selection, so there are

$$C(6,3) = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$

sets of three dots. Three dots are the vertices of a triangle unless they lie on one line. The three dots of a row lie on one line, which gives 2 such sets. Any other set of three dots has two dots in one row and one dot in the other row. The only line through the two dots in the same row is that row, which does not contain the third dot, so these three dots form a triangle. Different sets of vertices give different triangles, so the number of triangles is

$$20 - 2 = 18.$$

Answer: (C) 18. □

Advanced.

10. (2011 AMC 8, Problem #18) A fair six-sided die is rolled twice. What is the probability that the first number that comes up is greater than or equal to the second number?

(A) $\frac{1}{6}$ **(B)** $\frac{5}{12}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{7}{12}$ **(E)** $\frac{5}{6}$

Strategy. Split the outcomes into cases by the second number.

Solution. Each roll has 6 equally likely results, so by the Multiplication Principle the two rolls have $6 \times 6 = 36$ equally likely outcomes. If the second number is k, the first number can be any of $k, k+1, \ldots, 6$, which is $7-k$ choices. The cases $k = 1, 2, \ldots, 6$ do not overlap, so by the Addition Principle the number of favorable outcomes is

$$6 + 5 + 4 + 3 + 2 + 1 = 21.$$

The probability is therefore $\frac{21}{36} = \frac{7}{12}$.

Answer: (D) $\frac{7}{12}$. □

11. (2007 AMC 8, Problem #21) Two cards are dealt from a deck of four red cards labeled A, B, C, D and four green cards labeled A, B, C, D. A winning pair is two of the same color or two of the same letter. What is the probability of drawing a winning pair?

(A) $\frac{2}{7}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{4}{7}$ **(E)** $\frac{5}{8}$

Strategy. Count all the pairs of cards. Then count the winning pairs in two cases, the same color and the same letter, and check that the cases do not overlap.

Solution. Whether a pair wins does not depend on the order in which the two cards are dealt, so take the outcomes to be the $C(8,2) = 28$ pairs of cards, which are equally likely. A pair of the same color is two of the four red cards or two of the four green cards, so there are $C(4,2) + C(4,2) = 6 + 6 = 12$ such pairs. A pair of the same letter is the red card and the green card with that letter, so there are 4 such pairs, one for each letter. No pair has both the same color and the same letter, because two cards with the same color and the same letter would be the same card. So the cases do not overlap, and by the Addition Principle there are $12 + 4 = 16$ winning pairs. The probability of drawing a winning pair is $\frac{16}{28} = \frac{4}{7}$.

Answer: (D) $\frac{4}{7}$. □

12. (2002 AMC 8, Problem #21) Harold tosses a nickel four times. The probability that he gets at least as many heads as tails is

(A) $\frac{5}{16}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{11}{16}$

Strategy. Count the sequences with 2, 3 or 4 heads, since these are the ones with at least as many heads as tails.

Solution. Each toss is heads or tails with equal chance, so by the Multiplication Principle there are $2 \times 2 \times 2 \times 2 = 16$ equally likely sequences of four tosses. The numbers of heads and tails add up to 4, so there are at least as many heads as tails exactly when there are at least 2 heads, that is, 2, 3 or 4 heads. A sequence with exactly k heads is determined by choosing which k of the four tosses are heads, so there are $C(4, k)$ such sequences:

$$C(4, 2) = 6, \quad C(4, 3) = 4, \quad C(4, 4) = 1.$$

So $6 + 4 + 1 = 11$ sequences are favorable, and the probability is $\frac{11}{16}$.

Answer: (E) $\frac{11}{16}$. □

13. (1990 AHSME, Problem #16) At one of George Washington's parties, each man shook hands with everyone except his spouse, and no handshakes took place between women. If 13 married couples attended, how many handshakes were there among these 26 people?

(A) 78 **(B)** 185 **(C)** 234 **(D)** 312 **(E)** 325

Strategy. A handshake is an unordered pair of people. Split the handshakes into cases by whether the two people are two men, two women, or a man and a woman.

Solution. The 13 married couples are 13 men and 13 women. Every handshake is between two men, between two women, or between a man and a woman, and these cases are mutually exclusive.

Case 1: Two men. Each man shook hands with every other man, so every pair of men shook hands once. There are $C(13, 2) = \frac{13 \times 12}{2 \times 1} = 78$ such handshakes.

Case 2: Two women. There were no such handshakes.

Case 3: A man and a woman. Each man shook hands with every woman except his wife, that is, with 12 women. Each of these handshakes includes exactly one man, so counting them man by man counts each one once, and there are $13 \times 12 = 156$ of them.

By the Addition Principle there were

$$78 + 0 + 156 = 234$$

handshakes. Complementary counting gives the same number: the 26 people form

$$C(26, 2) = \frac{26 \times 25}{2 \times 1} = 325$$

pairs, and the pairs that did not shake hands are the $C(13, 2) = 78$ pairs of women and the 13 married couples, so $325 - 78 - 13 = 234$.

Answer: (C) 234. □

14. (2020 AMC 8, Problem #23) Five different awards are to be given to three students. Each student will receive at least one award. In how many different ways can the awards be distributed?

(A) 120 **(B)** 150 **(C)** 180 **(D)** 210 **(E)** 240

Strategy. Split into cases by how many awards each student receives. In each case, choose who receives how many awards, then which awards, with combinations and the Multiplication Principle.

Solution. The numbers of awards the three students receive are whole numbers, each at least 1, that add to 5. No student receives 4 or more, since then the other two would share at most 1 award and one of them would receive none. If one student receives 3, the other two receive 1 each. Otherwise every student receives 1 or 2, and the only way to reach 5 is $2+2+1$. So the numbers of awards are $3,1,1$ or $2,2,1$ in some order.

Case 1: One student receives 3 awards and the others receive 1 each. The student who receives 3 can be chosen in 3 ways, and that student's awards in

$$C(5,3) = \frac{5!}{3!\,2!} = 10$$

ways. The other 2 awards go one to each of the other two students, in 2 ways. This gives $3 \times 10 \times 2 = 60$ ways.

Case 2: Two students receive 2 awards each and the third receives 1. The student who receives 1 award can be chosen in 3 ways, and that award in 5 ways. Call the other two students X and Y. The other 4 awards are split into a group of 2 for X and a group of the other 2 for Y, which can be done in $C(4,2) = 6$ ways. This gives $3 \times 5 \times 6 = 90$ ways.

The two cases do not overlap, because they give the students different numbers of awards. By the Addition Principle, the number of ways is

$$60 + 90 = 150.$$

Answer: (B) 150. □

2. Answer Key

Problem	Letter	Value
1	C	$\frac{3}{8}$
2	B	21
3	B	$\frac{1}{6}$
4	C	72
5	D	39
6	B	7
7	C	138
8	D	99
9	C	18
10	D	$\frac{7}{12}$
11	D	$\frac{4}{7}$
12	E	$\frac{11}{16}$
13	C	234
14	B	150