COMBINATIONS AND COUNTING STRATEGIES FOR AMC 8: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (1990 AJHSME, Problem #14) A bag contains only blue balls and green balls. There are 6 blue balls. If the probability of drawing a blue ball at random from this bag is $\frac{1}{4}$, then the number of green balls in the bag is

(A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 36

Strategy. Every ball is equally likely to be drawn, so the probability of a blue ball is the number of blue balls divided by the number of balls in the bag. Set this fraction equal to $\frac{1}{4}$ to find the number of balls.

Solution. A ball drawn at random is equally likely to be any ball in the bag, so the outcomes are the balls and the favorable outcomes are the 6 blue balls. If the bag holds n balls in all, then

$$P(\text{blue}) = \frac{6}{n} = \frac{1}{4},$$

so $n = 4 \times 6 = 24$. Every ball in the bag is blue or green, so $24 - 6 = 18$ of the balls are green. As a check, $\frac{6}{6+18} = \frac{6}{24} = \frac{1}{4}$.

Answer: (B) 18. □

2. (2011 AMC 8, Problem #12) Angie, Bridget, Carlos, and Diego are seated at random around a square table, one person to a side. What is the probability that Angie and Carlos are seated opposite each other?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

Strategy. A seating is an arrangement of the four people at the four sides, and the seatings are equally likely. Count all of them, then count those with Carlos opposite Angie.

Solution. Each side of the table gets one person, so a seating is an arrangement of the 4 people at the 4 sides. There are $4! = 4 \times 3 \times 2 \times 1 = 24$ seatings, and seating the people at random makes them equally likely. In a favorable seating, Angie sits at any of the 4 sides, Carlos sits at the one side opposite her, and Bridget and Diego sit at the other two sides in $2! = 2$ orders. By the Multiplication Principle there are $4 \times 1 \times 2 = 8$ favorable seatings, so the probability is

$$\frac{8}{24} = \frac{1}{3}.$$

As a check, for each side Angie may take, Carlos sits at each of the other three sides in exactly 2 seatings. So he is equally likely to be at any of these three sides, and exactly one of them is opposite Angie.

Answer: (B) $\frac{1}{3}$. □

3. (2016 AMC 10B, Problem #12) Two different numbers are selected at random from $\{1, 2, 3, 4, 5\}$ and multiplied together. What is the probability that the product is even?

(A) 0.2 **(B)** 0.4 **(C)** 0.5 **(D)** 0.7 **(E)** 0.8

Strategy. The product is even when at least one of the two numbers is even. The complement, two odd numbers, is a single case that is easy to count.

Solution. Exchanging the two numbers does not change their product, so take the outcomes to be the unordered pairs of different numbers from the set, which are equally likely. There

are

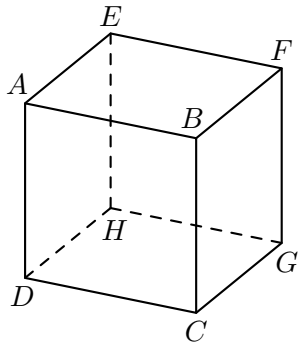
$$C(5,2) = \frac{5 \times 4}{2 \times 1} = 10$$

such pairs. A product of two whole numbers is odd exactly when both numbers are odd, so the complement of the event "the product is even" is made of the pairs chosen from the odd numbers 1, 3 and 5. There are $C(3,2) = 3$ such pairs: $\{1,3\}$, $\{1,5\}$ and $\{3,5\}$. By complementary counting, $10 - 3 = 7$ pairs give an even product, so the probability is

$$\frac{7}{10} = 0.7.$$

Answer: (D) 0.7. □

4. (2015 AMC 8, Problem #12) How many pairs of parallel edges, such as $\overline{AB}$ and $\overline{GH}$ or $\overline{EH}$ and $\overline{FG}$, does a cube have?



(A) 6 **(B)** 12 **(C)** 18 **(D)** 24 **(E)** 36

Strategy. Sort the 12 edges by direction. Two edges are parallel exactly when they run in the same direction, so a pair of parallel edges is a combination of 2 of the edges in one direction.

Solution. The 12 edges of a cube run in three directions, 4 edges in each. In the figure, $\overline{AB}$, $\overline{DC}$, $\overline{EF}$ and $\overline{HG}$ run from left to right, $\overline{AE}$, $\overline{BF}$, $\overline{CG}$ and $\overline{DH}$ run from front to back, and $\overline{AD}$, $\overline{BC}$, $\overline{EH}$ and $\overline{FG}$ run up and down. Two edges of the cube are parallel exactly when they run in the same direction. A pair of edges is an unordered selection of 2 edges, so each direction gives

$$C(4,2) = \frac{4 \times 3}{2 \times 1} = 6$$

pairs of parallel edges. Every pair of parallel edges runs in exactly one of the three directions, so by the Addition Principle the cube has $6 + 6 + 6 = 18$ pairs of parallel edges.

Answer: (C) 18. □

5. (2004 AMC 8, Problem #17) Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen?

(A) 1 **(B)** 3 **(C)** 6 **(D)** 10 **(E)** 12

Strategy. Split into cases by the number of pencils the first friend has. In each case, count the ways the other two friends can share the remaining pencils.

Solution. The pencils are identical, so a way is determined by how many pencils each friend has: three whole numbers, each at least 1, that add to 6. Call the friends the first, the second and the third. The second and third friends have at least 1 pencil each, so the first

friend has k pencils, where k is 1, 2, 3 or 4. These four values of k give mutually exclusive cases that together cover every way.

If the first friend has k pencils, the other two share the remaining $6-k$ pencils, at least 1 each. The second friend can have any number from 1 to $5-k$, which leaves at least 1 for the third friend, and then the third friend's number is fixed. So the case gives $5-k$ ways, and the cases $k = 1, 2, 3, 4$ give 4, 3, 2 and 1 ways. By the Addition Principle, the number of ways is

$$4 + 3 + 2 + 1 = 10.$$

Answer: (D) 10. □

Intermediate.

6. (1987 AJHSME, Problem #25) Ten balls numbered 1 to 10 are in a jar. Jack reaches into the jar and randomly removes one of the balls. Then Jill reaches into the jar and randomly removes a different ball. What is the probability that the sum of the two numbers on the balls removed is even?

(A) $\frac{4}{9}$ **(B)** $\frac{9}{19}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{10}{19}$ **(E)** $\frac{5}{9}$

Strategy. Exchanging the two balls does not change the sum, so the unordered pairs of balls can serve as the outcomes. The sum is even for two odd numbers or two even numbers, and odd otherwise.

Solution. Jack removes one of the 10 balls and then Jill removes one of the other 9, so by the Multiplication Principle there are

$$P(10, 2) = 10 \times 9 = 90$$

ordered draws, all equally likely. Exchanging Jack's ball with Jill's ball does not change the sum, and each unordered pair of balls comes from exactly 2 ordered draws. So the

$$C(10, 2) = \frac{10 \times 9}{2 \times 1} = 45$$

unordered pairs are also equally likely, and they can be taken as the outcomes. An odd number plus an even number is odd, while two odd numbers or two even numbers have an even sum. Five of the balls are odd (1, 3, 5, 7, 9) and five are even, so $C(5, 2) = 10$ pairs have two odd numbers and $C(5, 2) = 10$ pairs have two even numbers. These cases do not overlap, so by the Addition Principle $10 + 10 = 20$ pairs give an even sum, and the probability is

$$\frac{20}{45} = \frac{4}{9}.$$

Counting ordered draws gives the same answer: each of the 20 pairs comes from 2 draws, so 40 of the 90 draws are favorable, and $\frac{40}{90} = \frac{4}{9}$.

Answer: (A) $\frac{4}{9}$. □

7. (2020 AMC 10B, Problem #11) Ms. Carr asks her students to read any 5 of the 10 books on a reading list. Harold randomly selects 5 books from this list, and Betty does the same. What is the probability that there are exactly 2 books that they both select?

(A) $\frac{1}{8}$ **(B)** $\frac{5}{36}$ **(C)** $\frac{14}{45}$ **(D)** $\frac{25}{63}$ **(E)** $\frac{1}{2}$

Strategy. Fix Harold's selection and count Betty's. Exactly 2 books in common means 2 of Harold's 5 books and 3 of the 5 books he did not select, chosen separately.

Solution. Each student's selection is an unordered choice of 5 of the 10 books, so each student has

$$C(10,5) = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 252$$

possible selections, all equally likely. The outcomes are the 252×252 pairs of selections, one for Harold and one for Betty. Fix one of Harold's selections. Betty's selection shares exactly 2 books with it when she takes exactly 2 of Harold's 5 books and her other 3 books from the 5 books that Harold did not select. She can choose the 2 books in $C(5,2) = 10$ ways and, whichever they are, the other 3 books in

$$C(5,3) = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$$

ways, so by the Multiplication Principle $10 \times 10 = 100$ of her selections work. This holds for each of Harold's 252 selections, so there are 252×100 favorable outcomes, and the probability is

$$\frac{252 \times 100}{252 \times 252} = \frac{100}{252} = \frac{25}{63}.$$

Answer: (D) $\frac{25}{63}$. □

8. (1991 AJHSME, Problem #23) The Pythagoras High School band has 100 female and 80 male members. The Pythagoras High School orchestra has 80 female and 100 male members. There are 60 females who are members in both band and orchestra. Altogether, there are 230 students who are in either band or orchestra or both. The number of males in the band who are NOT in the orchestra is

(A) 10 **(B)** 20 **(C)** 30 **(D)** 50 **(E)** 70

Strategy. Apply the Principle of Inclusion-Exclusion to all the members to find how many students are in both the band and the orchestra. The males among them are those students minus the 60 females.

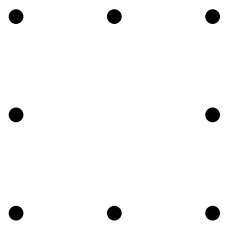
Solution. Let B be the set of band members and O the set of orchestra members. Then $|B| = 100 + 80 = 180$, $|O| = 80 + 100 = 180$ and $|B \cup O| = 230$. By the Principle of Inclusion-Exclusion,

$$|B \cup O| = |B| + |O| - |B \cap O|, \quad \text{so} \quad 230 = 180 + 180 - |B \cap O|.$$

This gives $|B \cap O| = 360 - 230 = 130$, so 130 students are in both the band and the orchestra. Of these, 60 are female, so $130 - 60 = 70$ are male. The band has 80 males, and 70 of them are also in the orchestra, so $80 - 70 = 10$ males in the band are not in the orchestra.

Answer: (A) 10. □

9. (2008 AMC 8, Problem #19) Eight points are spaced at intervals of one unit around a 2×2 square, as shown. Two of the 8 points are chosen at random. What is the probability that the points are one unit apart?



(A) $\frac{1}{4}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{4}{11}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{4}{7}$

Strategy. The outcomes are the combinations of 2 of the 8 points. Count the pairs one unit apart by going around the square, and check that no other pair is one unit apart.

Solution. The order in which the two points are chosen does not matter, so the outcomes are the

$$C(8,2) = \frac{8 \times 7}{2 \times 1} = 28$$

unordered pairs of points, all equally likely. The points are the four corners of the square and the midpoints of its four sides, which have length 2. Going around the square, each point is one unit from the next, and the last point is one unit from the first. This gives 8 pairs of points one unit apart, two on each side of the square.

No other pair of points is one unit apart. Two corners are at least 2 units apart. The midpoints of two opposite sides are 2 units apart, and the midpoints of two neighboring sides are opposite corners of a 1×1 square, so they are farther apart than its side, which is 1 unit. A corner is at least 2 units from every point of the two sides that do not pass through it, since each of those sides is 2 units from the corner.

So 8 of the 28 pairs are favorable, and the probability is

$$\frac{8}{28} = \frac{2}{7}.$$

Answer: (B) $\frac{2}{7}$. □

10. (2006 AMC 8, Problem #11) How many two-digit numbers have digits whose sum is a perfect square?

(A) 13 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

Strategy. The digit sum of a two-digit number is between 1 and 18, so it can only be one of the perfect squares 1, 4, 9 and 16. Count the numbers in each case and add.

Solution. A two-digit number has a tens digit from 1 to 9 and a units digit from 0 to 9, so its digit sum is at least $1+0=1$ and at most $9+9=18$. The perfect squares from 1 to 18 are 1, 4, 9 and 16. Each number has only one digit sum, so these four cases are mutually exclusive, and every number to be counted is in one of them. In each case, the tens digit determines the units digit.

Case 1: The digit sum is 1. The only such number is 10.

Case 2: The digit sum is 4. The numbers are 13, 22, 31 and 40, so there are 4.

Case 3: The digit sum is 9. Every tens digit from 1 to 9 works, with the units digit equal to 9 minus the tens digit, which gives the 9 numbers $18, 27, 36, \ldots, 90$.

Case 4: The digit sum is 16. The units digit is at most 9, so the tens digit is at least 7, which gives 79, 88 and 97, so there are 3.

By the Addition Principle, the number of two-digit numbers whose digit sum is a perfect square is

$$1 + 4 + 9 + 3 = 17.$$

Answer: (C) 17. □

Advanced.

11. (2007 AMC 10A, Problem #12) Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

(A) 56 **(B)** 58 **(C)** 60 **(D)** 62 **(E)** 64

Strategy. The two guides are different people, so the tourists are split into two groups that can be told apart. Count all the ways the tourists can choose, then subtract the ways that leave a guide with no tourists.

Solution. Each of the 6 tourists chooses one of the 2 guides, so by the Multiplication Principle there are $2^6 = 64$ ways to make the choices. A grouping is not allowed when one guide has no tourists, which happens only when all six tourists choose the same guide. There are 2 such choices, one for each guide. The number of allowed groupings is $64 - 2 = 62$.

Answer: (D) 62. □

12. (2000 AMC 8, Problem #21) Keiko tosses one penny and Ephraim tosses two pennies. The probability that Ephraim gets the same number of heads that Keiko gets is

(A) $\frac{1}{4}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

Strategy. Record the results of the three pennies in order, which gives 8 equally likely outcomes. Split the favorable outcomes into cases by the result of Keiko's penny.

Solution. Each penny lands heads (H) or tails (T) with equal chance. Record Keiko's penny first, then Ephraim's first penny, then his second. By the Multiplication Principle there are $2 \times 2 \times 2 = 8$ equally likely outcomes. Ephraim's pennies are recorded in order, so HT and TH are different results for him; his numbers of heads alone, 0, 1 and 2, are not equally likely. Keiko's penny shows tails or heads, which gives two mutually exclusive cases.

Case 1: Keiko gets tails, so she has 0 heads. Ephraim also needs 0 heads, which happens only with TT. This is 1 outcome.

Case 2: Keiko gets heads, so she has 1 head. Ephraim needs exactly 1 head, which happens with HT or TH. These are 2 outcomes.

By the Addition Principle, $1 + 2 = 3$ of the 8 outcomes are favorable, so the probability is $\frac{3}{8}$.

Answer: (B) $\frac{3}{8}$. □

13. (2016 AMC 8, Problem #21) A box contains 3 red chips and 2 green chips. Chips are drawn randomly, one at a time without replacement, until all 3 of the reds are drawn or until both green chips are drawn. What is the probability that the 3 reds are drawn?

(A) $\frac{3}{10}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{2}{3}$

Strategy. Imagine that the drawing continues until all five chips are out. This does not change which event happens first, and the 3 reds are used up first exactly when the last chip is green. The sequences of colors are equally likely, so count them with a combination.

Solution. Suppose the drawing goes on until the box is empty. The chips drawn after the drawing would have stopped do not change which event happened first, so it is enough to find the probability that, in the full order of the five chips, all 3 red chips come before the second green chip. This happens exactly when the last chip is green. If the last chip is green, the 3 red chips are among the first four, so they all come before that green chip. If the last chip is red, both green chips come before it, so the second green chip comes before the third red chip.

Treat the five chips as different, so that each of the $5! = 120$ orders is equally likely. The sequence of colors of an order is determined by which 2 of the 5 places hold the green chips, so there are $C(5,2) = 10$ sequences of colors. Each of them comes from $3! \times 2! = 12$ orders, since the red chips can fill the red places in 3! ways and the green chips can fill the green places in 2! ways. So the 10 sequences of colors are equally likely. A sequence ends in green when one green place is the fifth and the other is one of the first four, which gives 4 sequences. The probability is

$$\frac{4}{10} = \frac{2}{5}.$$

Answer: (B) $\frac{2}{5}$. □

14. (2007 AMC 8, Problem #24) A bag contains four pieces of paper, each labeled with one of the digits 1, 2, 3 or 4, with no repeats. Three of these pieces are drawn, one at a time without replacement, to construct a three-digit number. What is the probability that the three-digit number is a multiple of 3?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

Strategy. The order of the digits matters for the number but not for whether it is a multiple of 3, which depends only on the sum of the digits. So count the combinations of 3 digits instead of the ordered draws.

Solution. The draws are the $P(4,3) = 4 \times 3 \times 2 = 24$ ordered selections of 3 of the 4 digits, all equally likely. A number is a multiple of 3 exactly when the sum of its digits is a multiple of 3, and exchanging two digits does not change the sum. So whether a draw gives a multiple of 3 depends only on the set of 3 digits drawn. Each set of 3 digits can be put in order in $3! = 6$ ways, so it comes from exactly 6 of the 24 draws, and the

$$C(4,3) = \frac{P(4,3)}{3!} = \frac{24}{6} = 4$$

sets are equally likely. Each set leaves out one of the four digits, so the sets are $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$ and $\{2,3,4\}$, with digit sums 6, 7, 8 and 9. Only the sums 6 and 9 are multiples of 3, so 2 of the 4 sets give a multiple of 3, and the probability is $\frac{2}{4} = \frac{1}{2}$.

Answer: (C) $\frac{1}{2}$. □

15. (2010 AMC 8, Problem #20) In a room, $\frac{2}{5}$ of the people are wearing gloves, and $\frac{3}{4}$ of the people are wearing hats. What is the minimum number of people in the room wearing both a hat and gloves?

(A) 3 **(B)** 5 **(C)** 8 **(D)** 15 **(E)** 20

Strategy. Since $\frac{2}{5}$ and $\frac{3}{4}$ of the people are whole numbers of people, the room holds a multiple of 20 people. Everyone who wears gloves or a hat is in the room, so the Principle of Inclusion-Exclusion gives a lower bound on the number who wear both.

Solution. Let n be the number of people in the room, G the set of people wearing gloves and H the set of people wearing hats, so $|G| = \frac{2}{5}n$ and $|H| = \frac{3}{4}n$. These are whole numbers. The number $\frac{2}{5}n$ is a whole number exactly when $2n$ is a multiple of 5, that is, when n is a multiple of 5. In the same way, $\frac{3}{4}n$ is a whole number exactly when $3n$ is a multiple of 4, that is, when n is a multiple of 4. So n is a multiple of 20, and $n \geq 20$.

Everyone in $G \cup H$ is in the room, so $|G \cup H| \leq n$. By the Principle of Inclusion-Exclusion,

$$|G \cap H| = |G| + |H| - |G \cup H| \geq \frac{2}{5}n + \frac{3}{4}n - n = \frac{3}{20}n \geq \frac{3}{20} \times 20 = 3.$$

So at least 3 people wear both a hat and gloves. Exactly 3 is possible: in a room of 20 people, 8 wear gloves and 15 wear hats, and if 3 of them wear both, then $8 - 3 = 5$ wear only gloves and $15 - 3 = 12$ wear only hats, and $5 + 12 + 3 = 20$ is everyone in the room. The minimum is 3.

Answer: (A) 3. □

2. Answer Key

Problem	Letter	Value
1	B	18
2	B	$\frac{1}{3}$
3	D	0.7
4	C	18
5	D	10
6	A	$\frac{4}{9}$
7	D	$\frac{25}{63}$
8	A	10
9	B	$\frac{2}{7}$
10	C	17
11	D	62
12	B	$\frac{3}{8}$
13	B	$\frac{2}{5}$
14	C	$\frac{1}{2}$
15	A	3