PERMUTATIONS AND ARRANGEMENTS FOR AMC 8

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1. Factorials

Counting arrangements leads to products of consecutive whole numbers, such as $5\times4\times3\times2\times1$. These products have a short notation.

Definition 1.1. For a positive integer n, the **factorial** $n!$ (read "n factorial") is the product of the integers from 1 to n:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

The table lists the first ten values.

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40,320	362,880	3,628,800

Factorials grow quickly, so it rarely helps to multiply them out. It helps more to notice that each factorial contains every smaller factorial as a factor.

Proposition 1.2. For every integer $n \geq 2$, $n! = n \times (n-1)!$. More generally, if m and n are positive integers with $m < n$, then

$$\frac{n!}{m!} = n \times (n-1) \times \cdots \times (m+1),$$

the product of the integers from $m+1$ to n.

Proof. The product $n! = 1 \times 2 \times \cdots \times m \times (m+1) \times \cdots \times n$ begins with $1 \times 2 \times \cdots \times m = m!$. Dividing by $m!$ leaves the product of the integers from $m+1$ to n. The case $m = n-1$ gives $n! = n \times (n-1)!$. □

For example,

$$\frac{8!}{5!} = 8 \times 7 \times 6 = 336.$$

To add or subtract factorials, write each one as a multiple of the smallest one, and factor the smallest one out:

$$7! - 6! = 7 \times 6! - 6! = (7-1) \times 6! = 6 \times 720 = 4320.$$

Examples.

Problem 1.3 (1996 AHSME, Problem #3)**.** $\dfrac{(3!)!}{3!} =$

(A) 1 **(B)** 2 **(C)** 6 **(D)** 40 **(E)** 120

Strategy. The numerator is the factorial of the number 3!. Replace 3! by its value, and then simplify the quotient with Proposition 1.2.

Solution. The expression (3!)! is the factorial of the number $3! = 6$, so $(3!)! = 6!$. By Proposition 1.2,

$$\frac{6!}{3!} = 6 \times 5 \times 4 = 120.$$

Directly, $6! = 720$ and $720 \div 6 = 120$.

Answer: (E) 120. □

Problem 1.4 (2016 AMC 10A, Problem #1)**.** What is the value of $\dfrac{11! - 10!}{9!}$?

(A) 99 **(B)** 100 **(C)** 110 **(D)** 121 **(E)** 132

Strategy. Write 11! and 10! as multiples of 9!, and factor out 9!.

Solution. By Proposition 1.2, $11! = 11 \times 10 \times 9!$ and $10! = 10 \times 9!$. So

$$11! - 10! = 110 \times 9! - 10 \times 9! = 100 \times 9!,$$

and dividing by 9! gives 100.

Answer: (B) 100. □

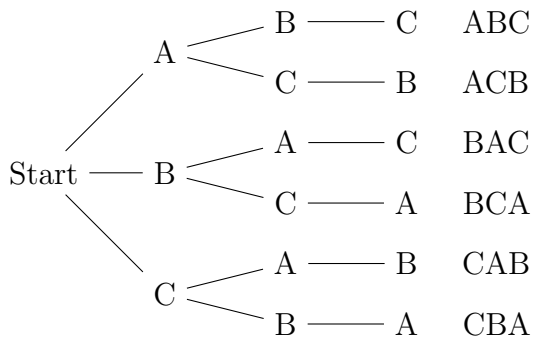
2. Permutations

This section counts the ways to put distinct objects in a row, using all of them or only some of them. Both counts come from the Multiplication Principle of the previous handout.

2.1. **Arranging All of the Objects.**

Definition 2.1.1. A **permutation** of a collection of distinct objects is an arrangement of all of the objects in a row. Two permutations are different when some position holds different objects.

The letters A, B and C have six permutations. Sorted by the first letter and then by the second, they are ABC, ACB, BAC, BCA, CAB and CBA. The tree diagram builds them one position at a time: three choices for the first letter, then two for the second, then one for the third.



Theorem 2.1.2. *There are $n!$ permutations of n distinct objects.*

Proof. Fill the positions from left to right. The first position can hold any of the n objects, the second any of the $n - 1$ objects not yet used, the third any of the $n - 2$ objects left, and so on, until one object is left for the last position. Which objects are left depends on the earlier choices, but how many are left does not, and different choices give different arrangements. By the Multiplication Principle there are $n \times (n - 1) \times \cdots \times 1 = n!$ permutations. □

For example, five students can stand in a line in $5! = 120$ ways.

When the $n!$ permutations are listed in alphabetical or numerical order, the ones with the same first object come together. For each first object, the other $n - 1$ objects can follow in $(n - 1)!$ orders, so the list is made of n groups of $(n - 1)!$ permutations each. Inside each group the same holds for the second object, with smaller groups of $(n-2)!$ permutations, and so on. The previous handout found the 17th of the 24 numbers made from the digits 2, 4, 5 and 7 in this way, with groups of $3! = 6$ numbers.

Example.

Problem 2.1.3. The six-digit numbers that use each of the digits 1, 2, 3, 4, 5 and 6 exactly once are listed in order from smallest to largest. What is the 500th number in the list?

Strategy. There are $6! = 720$ such numbers. In increasing order they are sorted by the first digit, then by the second, and so on. Count the numbers in each group to find the group that holds the 500th number.

Solution. For each first digit, the other five digits can be arranged in $5! = 120$ ways. So the numbers with each first digit fill the positions shown in the table.

First digit	1	2	3	4	5	6
Positions	1–120	121–240	241–360	361–480	481–600	601–720

The 500th number begins with 5. Since $500 - 480 = 20$, it is the 20th of the numbers that begin with 5. Among these, each second digit gives $4! = 24$ numbers, and the smallest second digit is 1. So the first 24 numbers that begin with 5, in positions 481 to 504, begin with 51, and the 500th number is the 20th of them.

The remaining digits are 2, 3, 4 and 6, and each third digit gives $3! = 6$ numbers. In this group, third digit 2 fills places 1 to 6, third digit 3 fills places 7 to 12, third digit 4 fills places 13 to 18, and third digit 6 fills places 19 to 24. So the number begins with 516, and it is the 2nd of the numbers that begin with 516. Its last three digits are 2, 3 and 4, and in increasing order the first two such numbers are 516234, the 499th number, and 516243, the 500th.

Answer: 516243. □

2.2. **Arranging Some of the Objects.** The previous handout counted the ways to award gold, silver and bronze medals among 7 runners as $7 \times 6 \times 5 = 210$. Only three positions are filled, so the multiplication stops after three factors.

Theorem 2.2.1. *Let n and k be positive integers with $k \leq n$. The number of ways to choose k of n distinct objects and arrange them in a row, written $P(n, k)$, is*

$$P(n, k) = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1).$$

Proof. Fill the k positions from left to right. The first position has n options, the second $n-1$, and in general the jth has $n - j + 1$, since $j - 1$ objects are already used. The kth position has $n - k + 1$ options. The Multiplication Principle gives the product. □

The product has k factors, starting at n and going down by 1. For example, a code made of 3 different letters from A to H can be written in $P(8, 3) = 8 \times 7 \times 6 = 336$ ways. When $k = n$ every object is used, and $P(n, n) = n!$. When $k < n$, Proposition 1.2 gives the factorial form

$$P(n, k) = \frac{n!}{(n-k)!}.$$

Example.

Problem 2.2.2. A teacher has 9 different posters and room on a wall for 4 of them, side by side in a row. In how many ways can she choose and arrange the posters on the wall?

Strategy. Order matters, since the same four posters in a different order look different on the wall. Fill the four places from left to right.

Solution. The leftmost place can hold any of the 9 posters, the next any of the other 8, then 7, then 6. By Theorem 2.2.1 there are

$$P(9, 4) = 9 \times 8 \times 7 \times 6 = 3024$$

ways. In factorial form this is $9! \div 5!$.

Answer: 3024. □

3. Arrangements with Restrictions

Many problems put conditions on an arrangement. This section gives a method for each of three common conditions: some objects must go in, or avoid, certain positions; some objects must stay together; and some objects must stay apart.

3.1. **Fixed Positions.** When some objects must go in certain positions, or must avoid certain positions, make these **restricted choices** first. Then arrange the remaining objects in the remaining positions in every possible way. For example, if six people stand in a line and Ann must stand at the left end, Ann's place is decided in 1 way and the other five people fill the other five places in $5! = 120$ ways, so there are 120 lines.

Restricted choices go first because the Multiplication Principle needs the number of options at each step not to depend on the earlier choices. If a restricted choice were made last, its number of options could depend on which positions the earlier choices had used. An object that must avoid some positions is placed the same way: count the positions it may take, and place it before the objects that have no restrictions.

Example.

Problem 3.1.1. Seven people stand in a line for a photo. Ann must stand in the middle of the line, and Ben must not stand at either end. In how many orders can the seven people stand?

Strategy. Place the restricted people first: Ann, then Ben. After them, the other five people fill the remaining places in any order.

Solution. Number the places 1 to 7. Ann takes place 4, in 1 way. Ben cannot take place 1 or place 7, and place 4 is taken, so he has 4 places: 2, 3, 5 or 6. The other five people fill the remaining five places in $5! = 120$ ways. By the Multiplication Principle there are

$$1 \times 4 \times 120 = 480$$

orders. If the other five people were placed before Ben, the number of places left for him would depend on whether they had taken the end places.

Answer: 480. □

3.2. **Objects Kept Together.** To count arrangements in which certain objects must be next to each other, treat them as a single unit, called a **block**. Arrange the block together with the other objects, and then multiply by the number of ways to order the objects inside the block.

Proposition 3.2.1. If k particular objects among n distinct objects must be next to each other, the number of arrangements in a row is $(n-k+1)! \times k!$.

Proof. The block and the other $n-k$ objects are $n-k+1$ units, which can be arranged in $(n-k+1)!$ ways, and the k objects inside the block can be ordered in $k!$ ways. Each arrangement with the k objects together comes from exactly one order of the units and one order inside the block. □

If the problem also fixes the order inside the block, the factor $k!$ becomes 1. For example, if Ann must stand immediately to the left of Ben in a line of five people, Ann and Ben form a block with a fixed order. The block and the other three people are 4 units, so there are $4! \times 1 = 24$ lines. Several blocks are handled the same way: each block is one unit and contributes its own factor for the order inside it.

Examples.

Problem 3.2.2. Seven friends, three of whom are sisters, stand in a line for a photo. In how many orders can they stand if the three sisters must stand next to each other?

Strategy. Glue the three sisters into one block.

Solution. The block and the other 4 friends are 5 units, which can stand in $5! = 120$ orders. Inside the block the sisters can stand in $3! = 6$ orders. So there are $120 \times 6 = 720$ orders. Proposition 3.2.1 with $n = 7$ and $k = 3$ gives the same count, $5! \times 3!$.

Answer: 720. □

Problem 3.2.3. Three boys and four girls stand in a row. In how many ways can they stand if the three boys stand next to each other and the four girls stand next to each other?

Strategy. Make two blocks, one of boys and one of girls.

Solution. The two blocks can be in $2! = 2$ orders: boys first or girls first. Inside the blocks, the boys can be ordered in $3! = 6$ ways and the girls in $4! = 24$ ways. By the Multiplication Principle there are

$$2 \times 6 \times 24 = 288$$

ways.

Answer: 288. □

3.3. **Objects Kept Apart.** To count arrangements in which no two of certain objects are next to each other, first arrange the other objects. A row of m objects has $m + 1$ **gaps**: one before the first object, one between each two neighbors, and one after the last object. These are the $m - 1$ gaps between neighbors counted in the previous handout, together with the two ends. For example, five novels in a row, each shown as N, make six gaps:

$$__ \text{ N } __ \text{ N } __ \text{ N } __ \text{ N } __ \text{ N } __$$

Then put the restricted objects into the gaps, at most one in each gap.

Proposition 3.3.1. Let k and m be positive integers with $k \leq m + 1$. Suppose k distinct objects are arranged in a row together with m other distinct objects, and no two of the k objects may be next to each other. Then the number of arrangements is $m! \times P(m + 1, k)$.

Proof. Arrange the m other objects in $m!$ ways. Place the restricted objects one at a time into different gaps: the first has $m + 1$ gaps to choose from, the second m, and so on, which gives $P(m + 1, k)$ ways. Restricted objects in different gaps have at least one other object between them. Conversely, in any arrangement with no two restricted objects next to each other, removing the restricted objects leaves the other m objects in a row, and the restricted objects sit in different gaps of that row. So every allowed arrangement is counted exactly once. □

Example.

Problem 3.3.2. Five different novels and three different cookbooks are placed in a row on a shelf. In how many ways can this be done if no two cookbooks are next to each other?

Strategy. Arrange the novels first, then put the cookbooks into different gaps between and around them.

Solution. The novels can be arranged in $5! = 120$ ways, and they make 6 gaps. The first cookbook can go into any of the 6 gaps, the second into any of the other 5, and the third into any of the remaining 4, which gives $P(6,3) = 6 \times 5 \times 4 = 120$ ways. By Proposition 3.3.1 there are

$$5! \times P(6,3) = 120 \times 120 = 14{,}400$$

ways in total.

Answer: 14,400. □

4. Arrangements with Repeated Letters

A word with a repeated letter has fewer arrangements than a word with distinct letters, because exchanging two copies of the same letter does not change the word. Here an arrangement of the letters of a word is any order of its letters, whether or not it is a real word.

Consider the word BOOK. Give the two O's labels, O_1 and O_2, so that the four letters B, O_1, O_2, K are all different. They have $4! = 24$ arrangements. Erasing the labels turns each labeled arrangement into an arrangement of BOOK, and exactly two labeled arrangements give each one: for example, BO_1O_2K and BO_2O_1K both become BOOK. So BOOK has $24 \div 2 = 12$ arrangements. In alphabetical order they are

$$\text{BKOO, BOKO, BOOK, KBOO, KOBO, KOOB,}$$
$$\text{OBKO, OBOK, OKBO, OKOB, OOBK, OOKB.}$$

If a letter appears three times, its copies can be labeled 1, 2, 3 in $3! = 6$ ways in any arrangement, so each arrangement comes from 6 labeled arrangements. The same reasoning works for several repeated letters at once.

Theorem 4.1. *Suppose a word has n letters, one letter appears a times, a second letter appears b times, and so on, and every other letter appears once. Then the letters have*

$$\frac{n!}{a! \times b! \times \cdots}$$

different arrangements.

Proof. Label the copies of each repeated letter so that all n letters are different. The labeled letters have $n!$ arrangements. Take any arrangement of the word. The labels of the first repeated letter can be placed on its a positions in $a!$ ways, the labels of the second on its b positions in $b!$ ways, and so on, so by the Multiplication Principle exactly $a! \times b! \times \cdots$ labeled arrangements give this arrangement of the word. Every labeled arrangement gives exactly one arrangement of the word, so

$$(\text{number of arrangements of the word}) \times a! \times b! \times \cdots = n!,$$

which gives the formula. □

A letter that appears once contributes $1! = 1$, so only the repeated letters change the count. The same formula counts arrangements of any objects some of which are identical, such as the digits of a number or beads of the same color.

Examples.

Problem 4.2. How many different arrangements of the letters of the word BANANA are there?

Strategy. Count the copies of each letter and use Theorem 4.1.

Solution. BANANA has 6 letters: three A's, two N's and one B. With labels A_1, A_2, A_3 and N_1, N_2 there would be $6! = 720$ arrangements, and each arrangement of BANANA comes from $3! \times 2! = 12$ of them. So there are

$$\frac{6!}{3! \times 2!} = \frac{720}{12} = 60$$

arrangements.

Answer: 60. □

Problem 4.3 (1992 AJHSME, Problem #7)**.** The digit-sum of 998 is $9 + 9 + 8 = 26$. How many 3-digit whole numbers, whose digit-sum is 26, are even?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. First find which digits can be used, and then arrange them. Repeated digits behave like repeated letters.

Solution. Each digit is at most 9, so the three digits fall short of 9 by amounts that add to $27 - 26 = 1$. Hence one digit is 8 and the other two are 9. By Theorem 4.1, the digits 9, 9, 8 have $3! \div 2! = 3$ arrangements: 899, 989 and 998. A number is even exactly when its units digit is even, and of these three numbers only 998 ends in an even digit.

Answer: (A) 1. □

Problem 4.4 (Adapted from 2008 AMC 10B, Problem #22)**.** Three red beads, two white beads, and one blue bead are placed in a line. In how many ways can this be done so that no two neighboring beads are the same color? (Beads of the same color are identical.)

Strategy. The red beads are the most restricted: no two of them may be neighbors. Make an organized list of the possible places for the three red beads, and then fill the other three places.

Solution. Only the red and the white beads come in more than one copy, so the condition says that no two red beads are neighbors and that the two white beads are not neighbors. Number the places 1 to 6. Sorted by the place of the first red bead, then the second, the possible places for the three red beads are

$$\{1, 3, 5\}, \quad \{1, 3, 6\}, \quad \{1, 4, 6\}, \quad \{2, 4, 6\}.$$

A first red bead in place 3 or later leaves no room for two more, and a second red bead in place 5 or later leaves no room for the third. So with the first red bead in place 1 the second is in place 3 or 4, and with the first in place 2 the second is in place 4. The third red bead then takes any later place that is not next to the second. The other three places get the two white beads and the blue bead.

- Red in places 1, 3, 5: places 2, 4, 6 are left, and no two of them are neighbors, so the white, white and blue beads can go there in $3! \div 2! = 3$ ways.
- Red in places 1, 3, 6: places 2, 4, 5 are left. Places 4 and 5 are neighbors, so they cannot both hold white beads. The blue bead takes place 4 or place 5, and the white beads fill the other two places: 2 ways.
- Red in places 1, 4, 6: places 2, 3, 5 are left. Places 2 and 3 are neighbors, so the blue bead takes place 2 or place 3: 2 ways.
- Red in places 2, 4, 6: places 1, 3, 5 are left, and no two of them are neighbors: 3 ways.

In total there are $3 + 2 + 2 + 3 = 10$ ways. Without the condition, Theorem 4.1 would give $6! \div (3! \times 2!) = 60$ arrangements of the colors.

Answer: 10. □

5. Digits and Numbers

Writing a number means placing digits in its places, so the methods of the previous sections apply to numbers as well. Two extra rules come into play.

First, a number with two or more digits cannot begin with 0: its first digit, the **leading digit**, is one of $1, 2, \ldots, 9$. Second, whether a number is even, odd or a multiple of 5 depends only on its last digit, the units digit: a number is even when its units digit is 0, 2, 4, 6 or 8, and it is a multiple of 5 when its units digit is 0 or 5. Unless a problem says that the digits are different, a digit may appear more than once.

So choose the most restricted digits first, usually the leading digit and the units digit, and fill the other places afterward. Sometimes the two restrictions get in each other's way. For example, when the digits must be different, 0 may be the units digit of an even number but not its leading digit, so the number of choices for the leading digit depends on whether 0 has already been used. In that case split the numbers into cases, such as "the units digit is 0" and "the units digit is not 0", count each case, and add. The cases do not overlap and together include every number, so their sum is the total.

Examples.

Problem 5.1. How many three-digit numbers have three different digits?

Strategy. The leading digit is the only restricted place, so choose it first.

Solution. The leading digit is one of $1, 2, \ldots, 9$: 9 ways. The tens digit can be any of the 9 digits other than the leading digit, now including 0: 9 ways. The units digit can be any of the 8 digits not yet used. So there are $9 \times 9 \times 8 = 648$ such numbers. Choosing the units digit first would make the number of choices for the leading digit depend on whether the units digit was 0.

Answer: 648. □

Problem 5.2 (Adapted from 2017 AMC 8, Problem #20)**.** How many integers between 1000 and 9999, inclusive, are odd integers whose digits are all distinct?

Strategy. Two places are restricted: the units digit must be odd, and the leading digit must not be 0. An odd units digit is never 0, so once it is chosen, the leading digit has the same number of choices, whichever odd digit was used.

Solution. The integers from 1000 to 9999 are exactly the four-digit numbers. The units digit is 1, 3, 5, 7 or 9: 5 ways. The leading digit is neither 0 nor the units digit, which is not 0 either, so it has $10 - 2 = 8$ choices. The hundreds digit can be any of the 8 digits not yet used, and the tens digit any of the remaining 7. So there are $5 \times 8 \times 8 \times 7 = 2240$ such integers.

Answer: 2240. □

Problem 5.3. How many even three-digit numbers have three different digits?

Strategy. The units digit must be even and the leading digit must not be 0. Since 0 is an even digit that cannot lead, split into cases by whether the units digit is 0.

Solution.

- Units digit 0: the leading digit is any of $1, 2, \ldots, 9$ (9 ways), and the tens digit is any of the 8 digits left. This gives $9 \times 8 = 72$ numbers.
- Units digit 2, 4, 6 or 8 (4 ways): the leading digit is neither 0 nor the units digit, so it has 8 choices, and the tens digit is any of the 8 digits not yet used, now including 0. This gives $4 \times 8 \times 8 = 256$ numbers.

The cases do not overlap, so there are $72 + 256 = 328$ numbers.

Answer: 328. □

6. Problems

Beginner.

1. (2017 AMC 8, Problem #5) What is the value of the expression

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1+2+3+4+5+6+7+8}?$$

(A) 1020 **(B)** 1120 **(C)** 1220 **(D)** 2240 **(E)** 3360

2. (2020 AMC 8, Problem #6) Aaron, Darren, Karen, Maren, and Sharon rode on a small train that has five cars that seat one person each. Maren sat in the last car. Aaron sat directly behind Sharon. Darren sat in one of the cars in front of Aaron. At least one person sat between Karen and Darren. Who sat in the middle car?

(A) Aaron **(B)** Darren **(C)** Karen **(D)** Maren **(E)** Sharon

3. (1994 AJHSME, Problem #8) For how many three-digit whole numbers does the sum of the digits equal 25?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 10

4. (2024 AMC 10B, Problem #2) What is $10! - 7! \cdot 6!$?

(A) -120 **(B)** 0 **(C)** 120 **(D)** 600 **(E)** 720

Intermediate.

5. (2002 AMC 8, Problem #19) How many whole numbers between 99 and 999 contain exactly one 0?

(A) 72 **(B)** 90 **(C)** 144 **(D)** 162 **(E)** 180

6. (2018 AMC 10A, Problem #3) A unit of blood expires after

$$10! = 10 \cdot 9 \cdot 8 \cdots 1$$

seconds. Yasin donates a unit of blood at noon on January 1. On what day does his unit of blood expire?

(A) January 2 **(B)** January 12 **(C)** January 22 **(D)** February 11 **(E)** February 12

7. (2015 AMC 10A, Problem #10) How many rearrangements of *abcd* are there in which no two adjacent letters are also adjacent letters in the alphabet? For example, no such rearrangements could include either *ab* or *ba*.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

8. (2009 AMC 8, Problem #22) How many whole numbers between 1 and 1000 do **not** contain the digit 1?

(A) 512 **(B)** 648 **(C)** 720 **(D)** 728 **(E)** 800

Advanced.

9. (2022 AMC 8, Problem #14) In how many ways can the letters in **BEEKEEPER** be rearranged so that two or more **E**'s do not appear together?

(A) 1 **(B)** 4 **(C)** 12 **(D)** 24 **(E)** 120

10. (2011 AMC 8, Problem #23) How many 4-digit positive integers have four different digits, where the leading digit is not zero, the integer is a multiple of 5, and 5 is the largest digit?
(A) 24 **(B)** 48 **(C)** 60 **(D)** 84 **(E)** 108

11. (2019 AMC 10B, Problem #6) A positive integer n satisfies the equation

$$(n+1)! + (n+2)! = n! \cdot 440.$$

What is the sum of the digits of n?
(A) 2 **(B)** 5 **(C)** 10 **(D)** 12 **(E)** 15

12. (2007 AMC 12A, Problem #16) How many three-digit numbers are composed of three distinct digits such that one digit is the average of the other two?
(A) 96 **(B)** 104 **(C)** 112 **(D)** 120 **(E)** 256