PERMUTATIONS AND ARRANGEMENTS FOR AMC 8: SOLUTIONS

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1. Solutions

Beginner.

1. (2017 AMC 8, Problem #5) What is the value of the expression

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1+2+3+4+5+6+7+8}?$$

(A) 1020 **(B)** 1120 **(C)** 1220 **(D)** 2240 **(E)** 3360

Strategy. The numerator is 8!, and the eight numbers in the denominator form four pairs with equal sums. Write 8! as a product in which the value of the denominator is one of the factors.

Solution. The numbers in the denominator pair up as $1+8$, $2+7$, $3+6$ and $4+5$, each with sum 9, so the denominator is $4 \times 9 = 36$. In the numerator, $8! = 8 \times 7 \times 6!$ and $6! = 720 = 20 \times 36$, so

$$\frac{8!}{36} = \frac{8 \times 7 \times 20 \times 36}{36} = 8 \times 7 \times 20 = 1120.$$

As a check, $1120 \times 36 = 40{,}320$, which is 8!.

Answer: (B) 1120. □

2. (2020 AMC 8, Problem #6) Aaron, Darren, Karen, Maren, and Sharon rode on a small train that has five cars that seat one person each. Maren sat in the last car. Aaron sat directly behind Sharon. Darren sat in one of the cars in front of Aaron. At least one person sat between Karen and Darren. Who sat in the middle car?

(A) Aaron **(B)** Darren **(C)** Karen **(D)** Maren **(E)** Sharon

Strategy. Sharon sits directly in front of Aaron, so the two form a block with a fixed order, and Darren also sits in front of Aaron. Use these conditions and Maren's car to find the cars Aaron can take, and then check each one against the condition on Karen and Darren.

Solution. Number the cars 1 (front) to 5 (last). Maren is in car 5, so the other four people fill cars 1 to 4. Sharon and Darren both sit in front of Aaron, so Aaron is in car 3 or car 4.

- If Aaron is in car 3, Sharon is in car 2. Darren must sit in front of Aaron, and car 1 is the only free car there, so Darren takes car 1 and Karen takes car 4. Two people sit between Karen and Darren, so all the conditions hold.
- If Aaron is in car 4, Sharon is in car 3, and Darren and Karen take cars 1 and 2. Then no one sits between them, which is not allowed.

So from front to back the order is Darren, Sharon, Aaron, Karen, Maren, and Aaron sits in the middle car.

Answer: (A) Aaron. □

3. (1994 AJHSME, Problem #8) For how many three-digit whole numbers does the sum of the digits equal 25?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 10

Strategy. Compare each digit with 9, the largest digit, to find which three digits are possible. Then count the orders of the digits as with repeated letters.

Solution. Each digit is at most 9, so the three digits fall short of 9 by amounts that add to $27 - 25 = 2$. Either one digit falls short by 2, so the digits are 7, 9, 9, or two digits fall

short by 1 each, so the digits are 8, 8, 9. In each case one digit appears twice, so each case gives $3! \div 2! = 3$ arrangements:

$$799, \quad 979, \quad 997 \qquad \text{and} \qquad 889, \quad 898, \quad 988.$$

None of these digits is 0, so every arrangement is a three-digit number. In total there are $3 + 3 = 6$ numbers.

Answer: (C) 6. □

4. (2024 AMC 10B, Problem #2) What is $10! - 7! \cdot 6!$?

(A) -120 **(B)** 0 **(C)** 120 **(D)** 600 **(E)** 720

Strategy. Divide 10! by 6!, and compare the quotient with 7!.

Solution. Dividing 10! by 6! leaves the product of the integers from 7 to 10:

$$\frac{10!}{6!} = 10 \times 9 \times 8 \times 7 = 5040 = 7!.$$

So $10! = 7! \cdot 6!$, and the difference $10! - 7! \cdot 6!$ is 0.

Answer: (B) 0. □

Intermediate.

5. (2002 AMC 8, Problem #19) How many whole numbers between 99 and 999 contain exactly one 0?

(A) 72 **(B)** 90 **(C)** 144 **(D)** 162 **(E)** 180

Strategy. Check first whether the endpoints 99 and 999 can change the count. Then place the 0, which cannot be the leading digit, and fill the other two places.

Solution. Neither 99 nor 999 contains a 0, so it does not matter whether they count as "between", and it is enough to count the three-digit numbers from 100 to 999 with exactly one 0. In such a number the 0 is the tens digit or the units digit, since the leading digit is not 0: 2 ways. The other two places hold digits from 1 to 9, which may repeat: $9 \times 9 = 81$ ways. So there are $2 \times 81 = 162$ such numbers.

Answer: (D) 162. □

6. (2018 AMC 10A, Problem #3) A unit of blood expires after

$$10! = 10 \cdot 9 \cdot 8 \cdots 1$$

seconds. Yasin donates a unit of blood at noon on January 1. On what day does his unit of blood expire?

(A) January 2 **(B)** January 12 **(C)** January 22 **(D)** February 11
(E) February 12

Strategy. One day lasts $24 \times 60 \times 60$ seconds, and $24 = 4!$. Divide 10! by the number of seconds in a day, and then count that many days forward from January 1.

Solution. One day lasts $24 \times 60 \times 60$ seconds. Since $24 = 4!$, dividing 10! by 24 leaves the product of the integers from 5 to 10:

$$\frac{10!}{4!} = 10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151{,}200.$$

Dividing this by $60 \times 60 = 3600$ gives $151{,}200 \div 3600 = 1512 \div 36 = 42$. So 10! seconds is exactly 42 days. From noon on January 1 to noon on February 1 is 31 days, and 11 more days reach noon on February 12.

Answer: (E) February 12. □

7. (2015 AMC 10A, Problem #10) How many rearrangements of *abcd* are there in which no two adjacent letters are also adjacent letters in the alphabet? For example, no such rearrangements could include either *ab* or *ba*.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Strategy. Place the most restricted letters first. The letters b and c each have only one allowed neighbor.

Solution. The pairs that may not stand next to each other are a, b and b, c and c, d. So b may stand next to d only, and c may stand next to a only. A letter in the middle of the row has two neighbors, so b and c both stand at the ends of the row, with d next to b and a next to c. This gives *bdac* and *cadb*. In both, the middle pair d, a is allowed. So there are 2 rearrangements.

Answer: (C) 2. □

8. (2009 AMC 8, Problem #22) How many whole numbers between 1 and 1000 do **not** contain the digit 1?

(A) 512 **(B)** 648 **(C)** 720 **(D)** 728 **(E)** 800

Strategy. Both 1 and 1000 contain the digit 1, so it does not matter whether they count as "between". Count the one-, two- and three-digit numbers separately. The leading digit can be neither 0 nor 1.

Solution.

- One-digit numbers: $2, 3, \ldots, 9$, which is 8 numbers.
- Two-digit numbers: the leading digit has 8 choices (2 to 9) and the units digit has 9 choices (any digit but 1). This gives $8 \times 9 = 72$ numbers.
- Three-digit numbers: in the same way, $8 \times 9 \times 9 = 648$ numbers.

In total there are $8 + 72 + 648 = 728$ numbers.

Answer: (D) 728. □

Advanced.

9. (2022 AMC 8, Problem #14) In how many ways can the letters in **BEEKEEPER** be rearranged so that two or more **E**'s do not appear together?

(A) 1 **(B)** 4 **(C)** 12 **(D)** 24 **(E)** 120

Strategy. Apart from its five E's, BEEKEEPER has one copy each of B, K, P and R. Place these four letters in a row first, and then put the E's into the gaps, at most one E in each gap.

Solution. The four letters other than E have $4! = 24$ orders, and each order has 5 gaps for the E's. No gap may hold two E's, and there are 5 E's for 5 gaps, so each gap gets exactly one E. The E's are identical, so this can be done in only 1 way. So there are $24 \times 1 = 24$ arrangements. In each of them the E's are the 1st, 3rd, 5th, 7th and 9th letters.

Answer: (D) 24. □

10. (2011 AMC 8, Problem #23) How many 4-digit positive integers have four different digits, where the leading digit is not zero, the integer is a multiple of 5, and 5 is the largest digit?

(A) 24 **(B)** 48 **(C)** 60 **(D)** 84 **(E)** 108

Strategy. The digits come from $0, 1, \ldots, 5$ and include 5. The units digit is 0 or 5, and the two values behave differently: with units digit 5, the digit 0 is still available and must not lead; with units digit 0, the digit 5 must take one of the first three places.

Solution.

- Units digit 5: the other three digits are different digits from 0 to 4, and the leading digit is not 0. The leading digit has 4 choices (1 to 4), and the two middle places are filled from the 4 digits left, now including 0, in $P(4,2) = 4 \times 3 = 12$ ways. This gives $4 \times 12 = 48$ numbers.
- Units digit 0: the digit 5 takes one of the first three places (3 ways), and the other two places get different digits from 1 to 4 in $P(4,2) = 12$ ways. This gives $3 \times 12 = 36$ numbers. The leading digit is never 0 here, since 0 is the units digit.

The cases do not overlap, so there are $48 + 36 = 84$ numbers.

Answer: (D) 84. □

11. (2019 AMC 10B, Problem #6) A positive integer n satisfies the equation

$$(n+1)! + (n+2)! = n! \cdot 440.$$

What is the sum of the digits of n?

(A) 2 **(B)** 5 **(C)** 10 **(D)** 12 **(E)** 15

Strategy. Every term of the equation is a multiple of $n!$. After canceling $n!$, the left side factors into two whole numbers that differ by 2. Find the pair with product 440.

Solution. Since $(n+1)! = (n+1) \times n!$ and $(n+2)! = (n+2) \times (n+1) \times n!$, dividing the equation by $n!$ gives

$$(n+1) + (n+2)(n+1) = 440.$$

The left side is $(n+1)(1+n+2) = (n+1)(n+3)$, so $(n+1)(n+3) = 440$. The two factors differ by 2, and their product is just below $21 \times 21 = 441$, so try 20 and 22: indeed $20 \times 22 = 440$. So $n = 19$ is a solution, and it is the only one, because $(n+1)(n+3)$ grows as n grows. The sum of the digits of 19 is $1 + 9 = 10$.

Answer: (C) 10. □

12. (2007 AMC 12A, Problem #16) How many three-digit numbers are composed of three distinct digits such that one digit is the average of the other two?

(A) 96 **(B)** 104 **(C)** 112 **(D)** 120 **(E)** 256

Strategy. The average of two different numbers lies between them, so the digit that is the average is the middle one, and the three digits are equally spaced. Make an organized list of these sets of digits. For each set, count the three-digit numbers it makes, choosing the leading digit first, since it is never 0.

Solution. In increasing order, the three digits are a, $a+d$ and $a+2d$ for some whole numbers $a \geq 0$ and $d \geq 1$. Since $a + 2d \leq 9$, the spacing d is 1, 2, 3 or 4, and for each d the smallest digit a runs from 0 to $9 - 2d$. That is $(9 - 2d) - 0 + 1 = 10 - 2d$ sets for each d: 8, 6, 4 and 2 sets, 20 in all. The sets that include 0 are the four with $a = 0$: $\{0,1,2\}$, $\{0,2,4\}$, $\{0,3,6\}$ and $\{0,4,8\}$.

- A set with no 0 makes $3! = 6$ three-digit numbers, and 16 of the sets have no 0.
- A set with 0 makes $2 \times 2 \times 1 = 4$ three-digit numbers: the leading digit is one of the two nonzero digits, the middle digit is either of the other two, and the last digit is the one left.

In total there are $16 \times 6 + 4 \times 4 = 96 + 16 = 112$ numbers.

Answer: (C) 112. □

2. Answer Key

Problem	Letter	Value
1	B	1120
2	A	Aaron
3	C	6
4	B	0
5	D	162
6	E	February 12
7	C	2
8	D	728
9	D	24
10	D	84
11	C	10
12	C	112