PERMUTATIONS AND ARRANGEMENTS FOR AMC 8: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (2004 AMC 8, Problem #2) How many different four-digit numbers can be formed by rearranging the four digits in 2004?

(A) 4 **(B)** 6 **(C)** 16 **(D)** 24 **(E)** 81

Strategy. The leading digit cannot be 0, so choose it first. The three digits left include two 0s, so count their orders as arrangements with a repeated letter.

Solution. The digits of 2004 are 2, 0, 0 and 4. A four-digit number does not begin with 0, so the leading digit is 2 or 4: 2 ways. The other three places hold the two 0s and the nonzero digit that is left. Three digits of which two are equal have $3! \div 2! = 3$ arrangements, just like the letters of a three-letter word with one letter repeated. Whichever digit leads, there are 3 ways to finish the number, so by the Multiplication Principle there are $2 \times 3 = 6$ numbers: 2004, 2040, 2400, 4002, 4020 and 4200.

Answer: (B) 6. □

2. Four friends take seats in a row of 7 empty chairs, one friend to a chair. In how many ways can they be seated?

Strategy. Seat the friends one at a time. A seating chooses 4 of the 7 chairs in order, one for each friend, so the count is $P(7, 4)$.

Solution. Seat the friends in a fixed order, for example alphabetically by name. The first friend can take any of the 7 chairs, the second any of the 6 chairs still empty, the third any of the remaining 5, and the fourth any of the remaining 4. Which chairs are empty depends on the earlier choices, but how many are empty does not, and different choices give different seatings. By the Multiplication Principle there are

$$P(7, 4) = 7 \times 6 \times 5 \times 4 = 840$$

seatings. Listing the chairs of the four friends in that fixed order gives an ordered choice of 4 of the 7 chairs, which is why the count is $P(7, 4)$. In factorial form it is $7! \div 3!$.

Answer: 840. □

3. (2019 AMC 10A, Problem #2) What is the hundreds digit of $(20! - 15!)$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 5

Strategy. Factor out the smaller factorial, 15!, and show that 15! is a multiple of 1000.

Solution. Since $20! = 20 \times 19 \times 18 \times 17 \times 16 \times 15!$, factoring out 15! gives

$$20! - 15! = (20 \times 19 \times 18 \times 17 \times 16 - 1) \times 15!.$$

Among the factors of $15! = 1 \times 2 \times \cdots \times 15$ are 2, 4, 5, 10 and 15, and their product is $2 \times 4 \times 5 \times 10 \times 15 = 6000 = 6 \times 1000$. So 15! is a multiple of 1000, and therefore so is $20! - 15!$. A positive multiple of 1000 ends in the digits 000, so the hundreds digit of $20! - 15!$ is 0.

Answer: (A) 0. □

4. (1993 AJHSME, Problem #12) If each of the three operation signs, $+$, $-$, $\times$, is used exactly **once** in one of the blanks in the expression

$$5 _ 4 _ 6 _ 3$$

then the value of the result could equal

(A) 9 **(B)** 10 **(C)** 15 **(D)** 16 **(E)** 19

Strategy. The three different signs fill the three blanks in $3! = 6$ orders. List the orders in an organized way and evaluate each expression, doing the multiplication before the addition and subtraction.

Solution. The signs $+$, $-$ and $\times$ are three distinct objects arranged in the three blanks, so there are $3! = 6$ expressions. The table lists them sorted by the first sign and then by the second, with the multiplication done first.

Expression	After multiplying	Value
$5 + 4 - 6 \times 3$	$5 + 4 - 18$	-9
$5 + 4 \times 6 - 3$	$5 + 24 - 3$	26
$5 - 4 + 6 \times 3$	$5 - 4 + 18$	19
$5 - 4 \times 6 + 3$	$5 - 24 + 3$	-16
$5 \times 4 + 6 - 3$	$20 + 6 - 3$	23
$5 \times 4 - 6 + 3$	$20 - 6 + 3$	17

The possible values are -9, 26, 19, -16, 23 and 17. Of the five choices, only 19 is among them, from $5 - 4 + 6 \times 3$.

Answer: (E) 19. □

5. (2020 AMC 10A, Problem #6) How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by 5?

(A) 80 **(B)** 100 **(C)** 125 **(D)** 200 **(E)** 500

Strategy. A multiple of 5 ends in 0 or 5, and 5 is odd. Fix the units digit first, then count the choices for the other digits.

Solution. The units digit must be 0, since 5 is odd. The thousands digit can be 2, 4, 6, or 8 (it cannot be 0), which is 4 choices. The hundreds and tens digits can each be 0, 2, 4, 6, or 8, which is 5 choices each. By the Multiplication Principle there are $4 \cdot 5 \cdot 5 \cdot 1 = 100$ such integers.

Answer: (B) 100. □

Intermediate.

6. (2003 AMC 8, Problem #16) Ali, Bonnie, Carlo and Dianna are going to drive together to a nearby theme park. The car they are using has four seats: one driver's seat, one front passenger seat and two back seats. Bonnie and Carlo are the only two who can drive the car. How many possible seating arrangements are there?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 12 **(E)** 24

Strategy. The driver's seat is the restricted position, so fill it first. Then the other three people fill the other three seats in any order.

Solution. Only Bonnie and Carlo can drive, so the driver's seat can be filled in 2 ways. Whoever drives, the other three people take the front passenger seat and the two back seats. These are three different seats, so the three people can sit in them in $3! = 6$ ways. By the Multiplication Principle there are $2 \times 6 = 12$ seating arrangements. Filling the driver's seat last would not work with a single product, because the number of people left who can drive would depend on whether Bonnie and Carlo had already been seated.

Answer: (D) 12. □

7. (2018 AMC 8, Problem #16) Professor Chang has nine different language books lined up on a bookshelf: two Arabic, three German, and four Spanish. How many ways are there to arrange the nine books on the shelf keeping the Arabic books together and keeping the Spanish books together?

(A) 1440 **(B)** 2880 **(C)** 5760 **(D)** 182,440 **(E)** 362,880

Strategy. Make two blocks, one of the Arabic books and one of the Spanish books. Arrange the blocks together with the three German books, and then order the books inside each block.

Solution. The Arabic books form one block, and the Spanish books form a second block. These two blocks and the three German books are 5 units, which can be put in order in $5! = 120$ ways. Inside the Arabic block the books can be ordered in $2! = 2$ ways, and inside the Spanish block in $4! = 24$ ways. Each allowed arrangement comes from exactly one order of the units and one order inside each block, so by the Multiplication Principle there are

$$5! \times 2! \times 4! = 120 \times 2 \times 24 = 5760$$

arrangements.

Answer: (C) 5760. □

8. (1968 AHSME, Problem #21) If $S = 1! + 2! + 3! + \cdots + 99!$, then the units digit in the value of S is:

(A) 9 **(B)** 8 **(C)** 5 **(D)** 3 **(E)** 0

Strategy. From 5! on, every factorial contains the factors 2 and 5, so it ends in 0.

Solution. For $n \geq 5$, the product $n!$ contains the factors 2 and 5, so it is a multiple of 10 and its units digit is 0. Hence the units digit of S is the units digit of

$$1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33,$$

which is 3.

Answer: (D) 3. □

9. (2018 AMC 10A, Problem #4) How many ways can a student schedule 3 mathematics courses — algebra, geometry, and number theory — in a 6-period day if no two mathematics courses can be taken in consecutive periods?
(What courses the student takes during the other 3 periods is of no concern here.)

(A) 3 **(B)** 6 **(C)** 12 **(D)** 18 **(E)** 24

Strategy. Keep the mathematics courses apart with gaps. The three other periods are all alike for this count, so they form a row in only 1 way, and the three different courses go into different gaps of that row.

Solution. Think of the day as a row of 6 periods, and mark each of the 3 periods without a mathematics course with an X. What happens in these periods does not matter, so the X's are all alike and their row can be formed in only 1 way. It has 4 gaps:

$$__ \text{X} __ \text{X} __ \text{X} __$$

Two mathematics courses in the same gap would be in consecutive periods, while two courses in different gaps have an X between them. So the condition says exactly that the

three courses go into three different gaps. Algebra can go into any of the 4 gaps, geometry into any of the other 3, and number theory into any of the remaining 2, which gives

$$P(4,3) = 4 \times 3 \times 2 = 24$$

placements. Reading the finished row from left to right gives periods 1 to 6, so each placement is exactly one schedule, and there are 24 schedules.

As a check, the possible sets of periods for the three courses are $\{1,3,5\}$, $\{1,3,6\}$, $\{1,4,6\}$ and $\{2,4,6\}$, and each set holds the three courses in $3! = 6$ orders, which again gives $4 \times 6 = 24$.

Answer: (E) 24. □

10. (2013 AMC 10A, Problem #13) How many three-digit numbers are not divisible by 5, have digits that sum to less than 20, and have the first digit equal to the third digit?

(A) 52 **(B)** 60 **(C)** 66 **(D)** 68 **(E)** 70

Strategy. The first digit is also the units digit, so it is restricted twice: it is not 0, and it is not 5, since a number that ends in 5 is a multiple of 5. Split into cases by this digit, and count the middle digits that keep the digit sum below 20.

Solution. Call the first and third digit a and the middle digit b. As the leading digit, a is not 0. As the units digit, a decides whether the number is a multiple of 5, which happens exactly when a is 0 or 5. So a is one of 1, 2, 3, 4, 6, 7, 8, 9. The digit sum is $2a+b$, so the condition $2a+b<20$ means $b \le 19-2a$.

- If a is 1, 2, 3 or 4, then $2a+b \le 8+9 = 17$, so all 10 middle digits work. This gives $4 \times 10 = 40$ numbers.
- If a is 6, 7, 8 or 9, then b runs from 0 to $19-2a$, which allows $20-2a$ middle digits. This gives 8, 6, 4 and 2 numbers.

The cases do not overlap, so there are $40+8+6+4+2 = 60$ such numbers.

Answer: (B) 60. □

Advanced.

11. (2013 AMC 10B, Problem #18) The number 2013 has the property that its units digit is the sum of its other digits, that is $2+0+1=3$. How many integers less than 2013 but greater than 1000 have this property?

(A) 33 **(B)** 34 **(C)** 45 **(D)** 46 **(E)** 58

Strategy. Once the first three digits are chosen, the units digit is determined. Split into cases by the leading digit, 1 or 2, and count the choices of the two middle digits that keep the units digit at most 9 and the number less than 2013.

Solution. The integers greater than 1000 and less than 2013 are 1001 to 2012.

- Leading digit 1. Let the hundreds, tens and units digits be a, b and c. The property says $c = 1+a+b$, which is a digit exactly when $a+b \le 8$. So such a number is determined by its hundreds and tens digits, which can be any two digits with $a+b \le 8$. For $a = 0, 1, \ldots, 8$ the digit b runs from 0 to $8-a$, which gives $9, 8, \ldots, 1$ choices, and $9+8+\cdots+1 = 45$. Each of these numbers has units digit at least 1, so it lies between 1001 and 1999.
- Leading digit 2. The numbers from 2000 to 2012 have hundreds digit 0 and tens digit 0 or 1. With tens digit 0 the units digit must be $2+0+0=2$, which gives 2002. With tens digit 1 it must be $2+0+1=3$, which gives 2013, and 2013 is not less than 2013.

In total there are $45 + 1 = 46$ such integers.

Answer: (D) 46. □

12. (2009 AMC 8, Problem #16) How many 3-digit positive integers have digits whose product equals 24?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 24

Strategy. First find the groups of three digits whose product is 24 with an organized list. Then count the orders of each group, dividing by 2! when a digit is repeated, as with repeated letters.

Solution. The product is not 0, so no digit is 0, and every order of the three digits gives a three-digit number. Write the digits of each group in increasing order, $a \le b \le c$, with $a \times b \times c = 24$. Since $3 \times 3 \times 3 = 27 > 24$, the smallest digit a is 1 or 2.

- If $a = 1$, then $b \times c = 24$ with $b \le c \le 9$, so b and c are 3 and 8, or 4 and 6.
- If $a = 2$, then $b \times c = 12$ with $2 \le b \le c$, so b and c are 2 and 6, or 3 and 4.

So the groups of digits are given by

$$24 = 1 \times 3 \times 8 = 1 \times 4 \times 6 = 2 \times 3 \times 4 = 2 \times 2 \times 6.$$

Each of the first three groups has three different digits and $3! = 6$ orders. The last group has the digit 2 twice, so it has $3! \div 2! = 3$ orders: 226, 262 and 622. In total there are $3 \times 6 + 3 = 21$ numbers.

Answer: (D) 21. □

13. (2019 AMC 10A, Problem #17) A child builds towers using identically shaped cubes of different colors. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

(A) 24 **(B)** 288 **(C)** 312 **(D)** 1,260 **(E)** 40,320

Strategy. Split into cases by the color of the cube that is left out. In each case a tower is an arrangement of 8 cubes in which cubes of the same color are identical, so the formula for repeated letters counts it.

Solution. Cubes of the same color look alike, so a tower is given by its colors from bottom to top, like a word in the letters R, B and G. Leaving out a red cube leaves 1 red, 3 blue and 4 green cubes for the tower; leaving out a blue cube leaves 2 red, 2 blue and 4 green; and leaving out a green cube leaves 2 red, 3 blue and 3 green. By the formula for repeated letters, the numbers of towers in the three cases are

$$\begin{aligned}
\text{red left out:} \quad & \frac{8!}{1! \times 3! \times 4!} = \frac{40{,}320}{144} = 280, \\
\text{blue left out:} \quad & \frac{8!}{2! \times 2! \times 4!} = \frac{40{,}320}{96} = 420, \\
\text{green left out:} \quad & \frac{8!}{2! \times 3! \times 3!} = \frac{40{,}320}{72} = 560.
\end{aligned}$$

Towers from different cases use different numbers of cubes of some color, so the cases do not overlap, and there are $280 + 420 + 560 = 1260$ towers.

As a check, put the left-out cube on top of the tower. This turns each tower into an arrangement of all 9 cubes, and removing the top cube of such an arrangement gives back the tower. So the count is also $9! \div (2! \times 3! \times 4!) = 362{,}880 \div 288 = 1260$.

Answer: (D) 1,260. □

14. (2017 AMC 8, Problem #19) For any positive integer M, the notation $M!$ denotes the product of the integers 1 through M. What is the largest integer n for which 5^n is a factor of the sum:

$$98! + 99! + 100!$$

(A) 23 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 27

Strategy. Factor out the smallest factorial, 98!. Then count the factors of 5 in 98! and in the number that is left.

Solution. Since $99! = 99 \times 98!$ and $100! = 100 \times 99 \times 98!$, factoring out 98! gives

$$98! + 99! + 100! = 98! \times (1 + 99 + 100 \times 99) = 98! \times 10{,}000.$$

The number $10{,}000 = 10^4 = 2^4 \times 5^4$ has 4 factors of 5. In $98! = 1 \times 2 \times \cdots \times 98$, only the multiples of 5 have a factor of 5. Since $5 \times 19 = 95$ and $5 \times 20 = 100$, these are $5 \times 1, 5 \times 2, \ldots, 5 \times 19$, and each of these 19 numbers has at least one factor of 5. The multiples of 25 among them, 25, 50 and 75, have a second factor of 5, and none of the numbers up to 98 is a multiple of 125. So 98! has exactly $19 + 3 = 22$ factors of 5, and the sum has exactly $22 + 4 = 26$. Hence 5^{26} is a factor of the sum and 5^{27} is not.

Answer: (D) 26. □

15. (2024 AMC 10B, Problem #20) Three different pairs of shoes are placed in a row so that no left shoe is next to a right shoe from a different pair. In how many ways can these six shoes be lined up?

(A) 60 **(B)** 72 **(C)** 90 **(D)** 108 **(E)** 120

Strategy. Wherever a left shoe stands next to a right shoe, the two shoes must be a matching pair, so they form a block. Record the row as a word in L and R, and split into cases by the number of places where the word switches between the two letters.

Solution. Write the row as a word of three L's and three R's, with L for a left shoe and R for a right shoe. The word splits into **runs**, stretches of equal neighboring letters that cannot be made longer. A **switch** is a pair of neighboring places where one run ends and the next begins, so that one place holds an L and the other an R. By the condition, the two shoes at a switch come from the same pair, so each switch holds a block made of one pair, in the order the word gives.

Two switches never share a place. Otherwise some shoe would have shoes of the other kind on both sides, say a right shoe between two left shoes, and it would have to match both of them, which is impossible because two left shoes come from different pairs. So different switches hold different pairs, there are at most 3 switches, and every run between two switches has at least 2 letters. There is at least one switch, since the word uses both letters.

- One switch: the words are LLL RRR and RRR LLL. In LLL RRR the switch is in places 3 and 4 and holds any of the 3 pairs. The other two left shoes fill places 1 and 2 in $2! = 2$ orders, and the other two right shoes fill places 5 and 6 in $2! = 2$ orders, which gives $3 \times 2 \times 2 = 12$ rows. The word RRR LLL also gives 12, so this case gives 24 rows.
- Two switches: the word has three runs, and the two end runs use the same letter. So the middle run holds all three copies of the other letter, and the end runs have 1 and 2 letters, in either order. The words are L RRR LL, LL RRR L, R LLL RR and RR LLL R. In each word the first switch holds any of the 3 pairs and the second switch

any of the other 2. The third pair then has one place left for its left shoe and one for its right shoe. Each word gives $3 \times 2 = 6$ rows, so this case gives 24 rows.

- Three switches: the word has four runs, and the two middle runs have at least 2 letters each. The second and fourth runs use the same letter, so the second run has 2 letters and the fourth has 1. In the same way the third run has 2 letters and the first has 1. The words are L RR LL R and R LL RR L. Their switches are in places 1 and 2, 3 and 4, and 5 and 6, and they hold the three pairs in $3! = 6$ orders, which places every shoe. This case gives $2 \times 6 = 12$ rows.

In total there are $24 + 24 + 12 = 60$ rows.

Answer: (A) 60. □

2. Answer Key

Problem	Letter	Value
1	B	6
2		840
3	A	0
4	E	19
5	B	100
6	D	12
7	C	5760
8	D	3
9	E	24
10	B	60
11	D	46
12	D	21
13	D	1,260
14	D	26
15	A	60