

SYSTEMATIC LISTING AND THE MULTIPLICATION PRINCIPLE FOR AMC 8

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1. ORGANIZED LISTS

Counting problems ask how many objects of a certain kind there are. When the answer is small, the most reliable method is to write down every object and count them. A list written in no particular order makes it easy to miss an object or to write one twice, so the order of the list should be chosen before the list is started.

Definition 1.1. An **organized list** is a list of all the objects to be counted, written in an order fixed in advance so that no object is missed and no object appears twice.

The usual way to fix the order is to sort the objects, the way a dictionary sorts words.

- (1) Decide what a single object looks like: a number, a pair of numbers, a group of items, and so on.
- (2) Choose a feature to sort by first, such as the first digit, the smallest number, or the number of copies of one item, and go through its possible values from smallest to largest.
- (3) Within each group, sort by a second feature in the same way, and so on.
- (4) If the order of the parts of an object does not matter, as in a group of numbers, write each object in one fixed form, for example with its numbers in increasing order. Then no object can appear twice.
- (5) Check where each group starts and where it stops, and why.

For example, to find all two-digit numbers whose digits have product 12, sort by the tens digit and go through 1, 2, ..., 9. The units digit is 12 divided by the tens digit, and it must be a whole number from 0 to 9. This happens only for the tens digits 2, 3, 4 and 6, so the list is

26, 34, 43, 62,

and there are 4 such numbers.

Examples.

Problem 1.2 (2002 AMC 8, Problem #2). How many different combinations of \$5 bills and \$2 bills can be used to make a total of \$17? Order does not matter in this problem.

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Strategy. Sort the combinations by the number of \$5 bills. Once that number is fixed, the rest of the \$17 must be paid in \$2 bills, so the amount left must be even.

Solution. Four \$5 bills already make \$20, which is more than \$17, so a combination uses 0, 1, 2 or 3 of them. The table lists each case.

\$5 bills	Amount left	\$2 bills
0	\$17	not possible
1	\$12	6
2	\$7	not possible
3	\$2	1

An odd amount cannot be paid in \$2 bills, so exactly two combinations work: one \$5 bill with six \$2 bills, and three \$5 bills with one \$2 bill.

Answer: (A) 2. □

Problem 1.3 (1990 AJHSME, Problem #12). There are twenty-four 4-digit whole numbers that use each of the four digits 2, 4, 5, and 7 exactly once. Listed in numerical order from smallest to largest, the number in the 17th position in the list is

(A) 4527 (B) 5724 (C) 5742 (D) 7245 (E) 7524

Strategy. A list in numerical order is sorted by the first digit, then by the second digit, and so on. Count how many numbers begin with each digit, and then write out only the group that contains the 17th number.

Solution. The numbers that begin with 2 use the digits 4, 5 and 7 in the last three places. Sorted by the second digit and then by the third, they are

$$2457, \quad 2475, \quad 2547, \quad 2574, \quad 2745, \quad 2754.$$

There are 6 of them, and for every first digit the other three digits can be arranged in the same 6 ways. So positions 1 to 6 hold the numbers that begin with 2, positions 7 to 12 hold those that begin with 4, and positions 13 to 18 hold those that begin with 5. The 17th number is therefore the fifth number that begins with 5. That group, in order, is

$$5247, \quad 5274, \quad 5427, \quad 5472, \quad 5724, \quad 5742,$$

and its fifth number is 5724.

Answer: (B) 5724. □

Problem 1.4 (2022 AMC 8, Problem #3). When three positive integers a , b , and c are multiplied together, their product is 100. Suppose $a < b < c$. In how many ways can the numbers be chosen?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Strategy. The condition $a < b < c$ already writes each choice of three numbers in one fixed form, so no choice can be listed twice. Sort by the smallest number a , and for each value of a list the pairs $b < c$.

Solution. If $a \geq 5$, then $b \geq 6$ and $c \geq 7$, so $abc \geq 5 \times 6 \times 7 = 210 > 100$. Hence $a \leq 4$. Also a divides 100, so a is 1, 2 or 4.

- $a = 1$: then $bc = 100$ with $1 < b < c$. Written with the smaller factor first, the ways to write 100 as a product of two factors are 1×100 , 2×50 , 4×25 , 5×20 and 10×10 . The first fails because $b > 1$, and the last fails because $b < c$. This leaves the pairs $(2, 50)$, $(4, 25)$ and $(5, 20)$.
- $a = 2$: then $bc = 50$ with $2 < b < c$. The ways to write 50 as a product of two factors are 1×50 , 2×25 and 5×10 , and only 5×10 has $b > 2$. This gives the pair $(5, 10)$.
- $a = 4$: then $bc = 25$ with $4 < b < c$. The only ways to write 25 as a product are 1×25 and 5×5 , and neither works.

In total there are $3 + 1 + 0 = 4$ ways.

Answer: (E) 4. □

2. TREE DIAGRAMS

When each object is built by a sequence of choices, a tree diagram draws the organized list as a picture.

Definition 2.1. A **tree diagram** starts at a single point. From this point, draw one branch for each option for the first choice. From the end of each branch, draw one branch for each option for the next choice, and continue until every object is complete. Each path from the starting point to an end of the tree gives exactly one object, so the number of objects equals the number of ends.

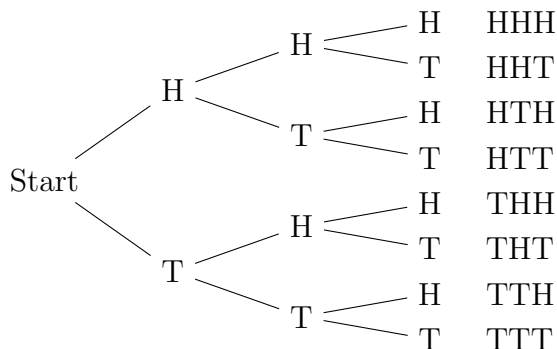
Reading the ends of a tree from top to bottom gives an organized list. A tree also works when the options at a step depend on what happened earlier, or when the process can stop early. In that case some branches split more than others, or end sooner.

Examples.

Problem 2.2. A coin is tossed three times, and the result of each toss, heads (H) or tails (T), is recorded in order. How many different sequences of results are possible?

Strategy. Draw a tree with one level of branches for each toss.

Solution. Each toss gives every branch two new branches, H and T.



The tree has 2 ends after the first toss, $2 \times 2 = 4$ after the second, and $2 \times 2 \times 2 = 8$ after the third. Read from top to bottom, the ends give the organized list

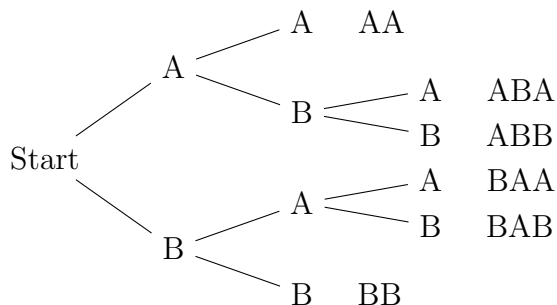
HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.

Answer: 8. □

Problem 2.3. Teams A and B play a series of games, and the first team to win two games wins the series. No game ends in a tie. A record of the series lists the winner of each game in order. For example, ABA means that A won the first and third games and B won the second. How many different records are possible?

Strategy. Draw a tree with one level for each game, and end a branch as soon as one team has won two games.

Solution. If one team wins both of the first two games, the series is over, so that branch stops after two games. Otherwise each team has one win, and a third game decides the series.



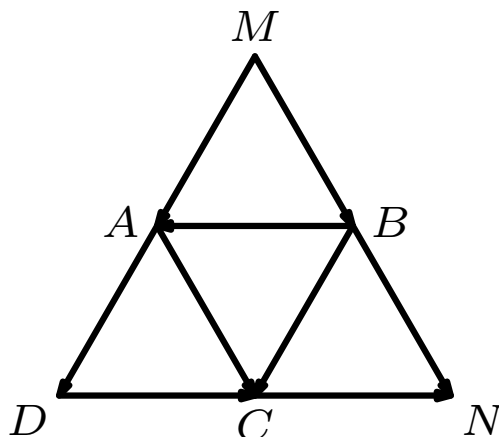
The ends of the tree give the records

AA, ABA, ABB, BAA, BAB, BB.

Not every branch splits the same number of times, so this count cannot be found by multiplying. Counting the ends gives 6.

Answer: 6. □

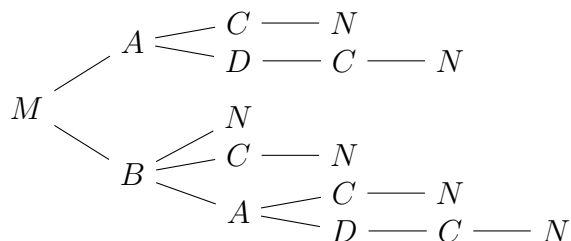
Problem 2.4 (1986 AJHSME, Problem #9). Using only the paths and the directions shown, how many different routes are there from M to N ?



- (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Strategy. Every route starts at M . Draw a tree that follows the arrows: at each point, draw one branch for each arrow leaving that point, and end a branch when it reaches N .

Solution. The arrows go from M to A and to B , from B to A , C and N , from A to C and D , from D to C , and from C to N . This gives the tree below.



The tree has 6 ends, one for each route:

$MACN$, $MADCN$, MBN , $MBCN$, $MBACN$, $MBADCN$.

Answer: (E) 6. □

3. THE MULTIPLICATION PRINCIPLE

In the coin example of the previous section, every branch split into the same number of new branches, so the number of ends could be found by multiplying. This works whenever each step has the same number of options, whatever happened in the earlier steps.

Theorem 3.1 (Multiplication Principle). *Suppose each object is made by a sequence of k steps, and different sequences of choices give different objects. If the first step can be done in n_1 ways and, however the earlier steps were done, the second step can be done in n_2 ways, the third step in n_3 ways, and so on up to the k th step in n_k ways, then the number of objects is*

$$n_1 \times n_2 \times \cdots \times n_k.$$

The tree diagram shows why. There are n_1 branches after the first step. Each of them splits into n_2 branches, so there are $n_1 \times n_2$ ends after the second step, and so on.

Before multiplying, check two things.

- (1) **The number of options at each step must not depend on the earlier choices.** The options themselves may change. If the digits of a number must be different, the second digit has 9 options whatever the first digit is, even though which 9 digits are allowed depends on the first digit. In the series example, the number of games depends on the earlier results, so the tree had to be drawn.
- (2) **Each object must come from exactly one sequence of choices.** Otherwise the product counts some objects more than once.

Examples.

Problem 3.2. A school uniform consists of a shirt, a pair of pants and a pair of shoes. There are 5 shirts, 3 pairs of pants and 2 pairs of shoes to choose from. How many different uniforms are possible?

Strategy. Choose the shirt, then the pants, then the shoes.

Solution. There are 5 ways to choose the shirt. For each shirt there are 3 ways to choose the pants, and for each shirt and pants there are 2 ways to choose the shoes. Different choices give different uniforms, so by the Multiplication Principle there are

$$5 \times 3 \times 2 = 30$$

uniforms.

Answer: 30. □

Problem 3.3. Seven runners finish a race with no ties. In how many ways can the gold, silver and bronze medals be awarded?

Strategy. Award the medals one at a time, starting with gold. A runner who already has a medal cannot win another one.

Solution. Any of the 7 runners can win gold. Silver goes to one of the other 6 runners, and bronze to one of the remaining 5. Which runners are left depends on who won gold and silver, but how many are left does not, so the Multiplication Principle applies:

$$7 \times 6 \times 5 = 210.$$

Answer: 210. □

Problem 3.4 (1985 AJHSME, Problem #22). Assume every 7-digit whole number is a possible telephone number except those that begin with 0 or 1. What fraction of telephone numbers begin with 9 and end with 0?

- (A) $\frac{1}{63}$ (B) $\frac{1}{80}$ (C) $\frac{1}{81}$ (D) $\frac{1}{90}$ (E) $\frac{1}{100}$

Strategy. Use the Multiplication Principle twice: once to count all telephone numbers, and once to count those that begin with 9 and end with 0. Then divide.

Solution. Choose the digits from left to right. The first digit can be any of $2, 3, \dots, 9$, which is 8 ways, and each of the other six digits can be any of the 10 digits. So there are

$$8 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 8,000,000$$

telephone numbers. For a number that begins with 9 and ends with 0, the first and last digits can each be chosen in only 1 way, and each of the five middle digits in 10 ways. So there are

$$1 \times 10 \times 10 \times 10 \times 10 \times 10 \times 1 = 100,000$$

such numbers. The fraction is

$$\frac{100,000}{8,000,000} = \frac{1}{80}.$$

Answer: (B) $\frac{1}{80}$. □

Problem 3.5 (1999 AMC 8, Problem #15). Bicycle license plates in Flatville each contain three letters. The first is chosen from the set $\{C, H, L, P, R\}$, the second from $\{A, I, O\}$, and the third from $\{D, M, N, T\}$.

When Flatville needed more license plates, they added two new letters. The new letters may both be added to one set or one letter may be added to one set and one to another set. What is the largest possible number of *additional* license plates that can be made by adding two letters?

(A) 24 (B) 30 (C) 36 (D) 40 (E) 60

Strategy. For any sizes of the three sets, the Multiplication Principle gives the number of plates. There are only six ways to add the two letters, so make an organized list of them.

Solution. At present there are $5 \times 3 \times 4 = 60$ plates. The two new letters can both go into one set (3 ways) or into two different sets (3 ways). The table lists all six cases.

Where the new letters go	Set sizes	Plates	Additional plates
both in the first set	7, 3, 4	84	24
both in the second set	5, 5, 4	100	40
both in the third set	5, 3, 6	90	30
one in the first, one in the second	6, 4, 4	96	36
one in the first, one in the third	6, 3, 5	90	30
one in the second, one in the third	5, 4, 5	100	40

The largest possible number of plates is 100, which is 40 more than before.

Answer: (D) 40. □

4. COUNTING INTEGERS IN A RANGE

Many counting problems come down to counting the integers in a list such as 23, 24, 25, $\dots$, 87. The difference $87 - 23 = 64$ is not the answer. It counts the steps from each number to the next, and a list always has one more number than it has steps.

4.1. Consecutive Integers.

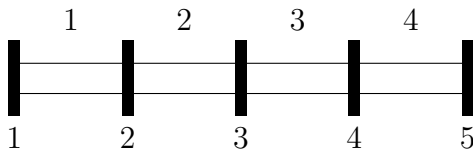
Theorem 4.1.1. *If a and b are integers with $a \leq b$, then there are $b - a + 1$ integers from a to b , inclusive (that is, counting both a and b).*

Proof. Subtract $a - 1$ from each number in the list $a, a + 1, \dots, b$. This gives the list $1, 2, \dots, b - a + 1$. The two lists have the same length, and the second one has $b - a + 1$ numbers. □

For example, there are $87 - 23 + 1 = 65$ integers from 23 to 87.

Confusing the number of posts with the number of gaps between them, for example answering $b - a$ instead of $b - a + 1$, is called a **fencepost error**. A straight fence with 4 sections needs 5

posts, as in the picture, where the sections are numbered above the fence and the posts below it. In general, n posts in a row have $n - 1$ gaps between neighboring posts.



Contest problems often say “between”, which may or may not include the endpoints. Decide separately whether each endpoint is included and whether it has the property being counted. The integers strictly between a and b are $a + 1, \dots, b - 1$, so there are $(b - 1) - (a + 1) + 1 = b - a - 1$ of them. When the ends of a range are not integers, first find the smallest and the largest integers inside the range.

Examples.

Problem 4.1.2 (2001 AMC 8, Problem #6). Six trees are equally spaced along one side of a straight road. The distance from the first tree to the fourth is 60 feet. What is the distance in feet between the first and last trees?

- (A) 90 (B) 100 (C) 105 (D) 120 (E) 140

Strategy. Count the gaps between neighboring trees, not the trees.

Solution. From the first tree to the fourth there are 3 gaps between neighboring trees, so each gap is $60 \div 3 = 20$ feet. From the first tree to the sixth, which is the last one, there are 5 gaps, so the distance is $5 \times 20 = 100$ feet. Using the numbers of trees, 4 and 6, in place of the numbers of gaps, 3 and 5, is a fencepost error.

Answer: (B) 100. □

Problem 4.1.3 (1986 AJHSME, Problem #7). How many whole numbers are between $\sqrt{8}$ and $\sqrt{80}$?

- (A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Strategy. Find the smallest and the largest whole numbers between the two square roots, and then count from one to the other.

Solution. Since $2^2 = 4 < 8 < 9 = 3^2$, it follows that $2 < \sqrt{8} < 3$, so the smallest whole number greater than $\sqrt{8}$ is 3. Since $8^2 = 64 < 80 < 81 = 9^2$, it follows that $8 < \sqrt{80} < 9$, so the largest whole number less than $\sqrt{80}$ is 8. The whole numbers between $\sqrt{8}$ and $\sqrt{80}$ are $3, 4, \dots, 8$, and there are $8 - 3 + 1 = 6$ of them.

Answer: (B) 6. □

Problem 4.1.4 (2018 AMC 8, Problem #25). How many perfect cubes lie between $2^8 + 1$ and $2^{18} + 1$, inclusive?

- (A) 4 (B) 9 (C) 10 (D) 57 (E) 58

Strategy. A cube n^3 lies in the range exactly when n lies in a range of consecutive integers. Find the smallest and the largest such n .

Solution. The lower end is $2^8 + 1 = 257$. Since $6^3 = 216 < 257 \leq 343 = 7^3$, the smallest cube in the range is 7^3 . For the upper end,

$$2^{18} = 2^6 \times 2^6 \times 2^6 = 64^3,$$

so $2^{18} + 1 = 64^3 + 1$. The cube 64^3 is in the range, and 65^3 is larger than $64^3 + 1$, so the largest cube in the range is 64^3 . The cubes in the range are $7^3, 8^3, \dots, 64^3$, one for each integer from 7 to 64, so there are $64 - 7 + 1 = 58$ of them.

Answer: (E) 58. □

4.2. Multiples in a Range. To count the numbers in a list that goes up by the same amount each time, rewrite it as a list of consecutive integers. For example, the multiples of 7 from 14 to 98 are

$$7 \times 2, \quad 7 \times 3, \quad \dots, \quad 7 \times 14,$$

so there are as many of them as there are integers from 2 to 14, namely $14 - 2 + 1 = 13$. The same idea works for any such list.

Theorem 4.2.1. *If a list starts at a , goes up by d each time, and ends at b , then it has*

$$\frac{b - a}{d} + 1$$

numbers.

Proof. Write the numbers as $a + 0 \times d, a + 1 \times d, \dots, a + m \times d$, where $a + m \times d = b$, so that $m = \frac{b-a}{d}$. The multipliers $0, 1, \dots, m$ are the integers from 0 to m , and there are $m - 0 + 1 = m + 1$ of them. □

In words, the list takes $\frac{b-a}{d}$ steps of size d , and it has one more number than steps.

Examples.

Problem 4.2.2. How many numbers are in the list $4, 7, 10, \dots, 97, 100$?

Strategy. The list goes up by 3 each time. Shift it so that it becomes a list of multiples of 3.

Solution. Subtract 1 from each number. This gives $3, 6, 9, \dots, 96, 99$, which is

$$3 \times 1, \quad 3 \times 2, \quad \dots, \quad 3 \times 33,$$

so the list has as many numbers as there are integers from 1 to 33, which is 33. The theorem gives the same count: $\frac{100-4}{3} + 1 = 32 + 1 = 33$.

Answer: 33. □

Problem 4.2.3 (2005 AMC 8, Problem #18). How many three-digit numbers are divisible by 13?

(A) 7 (B) 67 (C) 69 (D) 76 (E) 77

Strategy. Write the multiples of 13 as $13 \times m$, and find the smallest and the largest values of m that give three-digit numbers.

Solution. The three-digit numbers are 100 to 999. Since $13 \times 7 = 91$ and $13 \times 8 = 104$, the smallest three-digit multiple of 13 is 13×8 . Since $13 \times 76 = 988$ and $13 \times 77 = 1001$, the largest is 13×76 . So the three-digit multiples of 13 are $13 \times 8, 13 \times 9, \dots, 13 \times 76$, and there are $76 - 8 + 1 = 69$ of them.

Answer: (C) 69. □

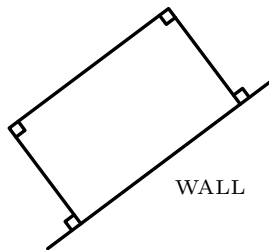
5. PROBLEMS

Beginner.

- (1996 AJHSME, Problem #1) How many positive factors of 36 are also multiples of 4?
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6
- (2000 AMC 8, Problem #3) How many whole numbers lie in the interval between $\frac{5}{3}$ and 2π ?
(A) 2 (B) 3 (C) 4 (D) 5 (E) infinitely many
- (2011 AMC 8, Problem #8) Bag A contains three chips labeled 1, 3, and 5. Bag B contains three chips labeled 2, 4, and 6. If one chip is drawn from each bag, how many different values are possible for the sum of the two numbers on the chips?
(A) 4 (B) 5 (C) 6 (D) 7 (E) 9
- (2022 AMC 8, Problem #11) Henry the donkey has a very long piece of pasta. He takes a number of bites of pasta, each time eating 3 inches of pasta from the middle of one piece. In the end, he has 10 pieces of pasta whose total length is 17 inches. How long, in inches, was the piece of pasta he started with?
(A) 34 (B) 38 (C) 41 (D) 44 (E) 47

Intermediate.

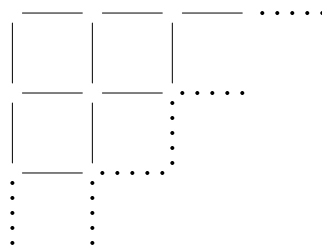
- (2016 AMC 8, Problem #11) Determine how many two-digit numbers satisfy the following property:
When the number is added to the number obtained by reversing its digits, the sum is 132.
(A) 5 (B) 7 (C) 9 (D) 11 (E) 12
- (1986 AJHSME, Problem #18) A rectangular grazing area is to be fenced off on three sides using part of a 100 meter rock wall as the fourth side. Fence posts are to be placed every 12 meters along the fence, including the two posts where the fence meets the rock wall. What is the fewest number of posts required to fence an area 36 m by 60 m?



- (A) 11 (B) 12 (C) 13 (D) 14 (E) 16
- (2014 AMC 8, Problem #16) The “Middle School Eight” basketball conference has 8 teams. Every season, each team plays every other conference team twice (home and away), and each team also plays 4 games against non-conference opponents. What is the total number of games in a season involving the “Middle School Eight” teams?
(A) 60 (B) 88 (C) 96 (D) 144 (E) 160
- (1995 AJHSME, Problem #23) How many four-digit whole numbers are there such that the leftmost digit is odd, the second digit is even, and all four digits are different?
(A) 1120 (B) 1400 (C) 1800 (D) 2025 (E) 2500

Advanced.

9. (2013 AMC 8, Problem #22) Toothpicks are used to make a grid that is 60 toothpicks long and 32 toothpicks high. How many toothpicks are used altogether?



- (A) 1920 (B) 1952 (C) 1980 (D) 2013 (E) 3932
10. (2008 AMC 8, Problem #14) Three A's, three B's, and three C's are placed in the nine spaces so that each row and column contain one of each letter. If A is placed in the upper left corner, how many arrangements are possible?

A		

- (A) 2 (B) 3 (C) 4 (D) 5 (E) 6
11. (2008 AMC 8, Problem #22) For how many positive integer values of n are both $\frac{n}{3}$ and $3n$ three-digit whole numbers?
- (A) 12 (B) 21 (C) 27 (D) 33 (E) 34
12. (1988 AJHSME, Problem #25) A palindrome is a whole number that reads the same forwards as backwards. If one neglects the colon, certain times displayed on a digital watch are palindromes. Three examples are 1:01, 4:44, and 12:21. How many times during a 12-hour period will be palindromes?
- (A) 57 (B) 60 (C) 63 (D) 90 (E) 93