

SYSTEMATIC LISTING AND THE MULTIPLICATION PRINCIPLE FOR AMC 8: SOLUTIONS

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CONTENTS

1. Solutions	2
2. Answer Key	7

1. SOLUTIONS

Beginner.

1. (1996 AJHSME, Problem #1) How many positive factors of 36 are also multiples of 4?
 (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Strategy. Make an organized list of the factors of 36 by writing 36 as a product of two factors in every possible way. Then pick out the multiples of 4.

Solution. Sort the products by the smaller factor, going through $1, 2, \dots, 6$ and skipping 5, which does not divide 36:

$$1 \times 36, \quad 2 \times 18, \quad 3 \times 12, \quad 4 \times 9, \quad 6 \times 6.$$

The list stops at 6×6 , since the smaller factor of a pair cannot be larger than 6 when $6 \times 6 = 36$. So the positive factors of 36 are

$$1, 2, 3, 4, 6, 9, 12, 18, 36.$$

The ones that are multiples of 4 are 4, 12 and 36, so there are 3 of them.

Answer: (B) 3. □

2. (2000 AMC 8, Problem #3) How many whole numbers lie in the interval between $\frac{5}{3}$ and 2π ?
 (A) 2 (B) 3 (C) 4 (D) 5 (E) infinitely many

Strategy. Find the smallest and the largest whole numbers in the interval, and then count from one to the other.

Solution. Since $\frac{5}{3} = 1\frac{2}{3}$, the smallest whole number greater than $\frac{5}{3}$ is 2. Since $3 < \pi < 3.5$, it follows that $6 < 2\pi < 7$, so the largest whole number less than 2π is 6. The whole numbers in the interval are 2, 3, 4, 5, 6, and there are $6 - 2 + 1 = 5$ of them.

Answer: (D) 5. □

3. (2011 AMC 8, Problem #8) Bag A contains three chips labeled 1, 3, and 5. Bag B contains three chips labeled 2, 4, and 6. If one chip is drawn from each bag, how many different values are possible for the sum of the two numbers on the chips?
 (A) 4 (B) 5 (C) 6 (D) 7 (E) 9

Strategy. By the Multiplication Principle there are $3 \times 3 = 9$ ways to draw the two chips, but different draws can give the same sum. Make a table of all nine sums and count the different values.

Solution. In the table below, each row is a chip from Bag A, each column is a chip from Bag B, and each entry is the sum of the two chips.

Bag A	Bag B		
	2	4	6
1	3	5	7
3	5	7	9
5	7	9	11

The sums that appear are 3, 5, 7, 9 and 11, so there are 5 different values.

Answer: (B) 5. □

4. (2022 AMC 8, Problem #11) Henry the donkey has a very long piece of pasta. He takes a number of bites of pasta, each time eating 3 inches of pasta from the middle of one piece. In the end, he has 10 pieces of pasta whose total length is 17 inches. How long, in inches, was the piece of pasta he started with?

(A) 34 (B) 38 (C) 41 (D) 44 (E) 47

Strategy. Count the bites by tracking the number of pieces. A bite from the middle of a piece splits that piece into two.

Solution. Henry starts with 1 piece, and each bite increases the number of pieces by exactly 1. To end with 10 pieces he must take $10 - 1 = 9$ bites. This is the fencepost idea: if the pieces are put back in their original positions, the 10 pieces are separated by 9 gaps, one for each bite. He eats $9 \times 3 = 27$ inches in all, so the piece he started with was

$$17 + 27 = 44$$

inches long.

Answer: (D) 44. □

Intermediate.

5. (2016 AMC 8, Problem #11) Determine how many two-digit numbers satisfy the following property:

When the number is added to the number obtained by reversing its digits, the sum is 132.

(A) 5 (B) 7 (C) 9 (D) 11 (E) 12

Strategy. Write the number in terms of its digits. The sum depends only on the sum of the two digits, so list the numbers whose digits have the right sum.

Solution. Let the number have tens digit a and units digit b . Then the number is $10a + b$, its reversal is $10b + a$, and their sum is

$$(10a + b) + (10b + a) = 11a + 11b = 11(a + b).$$

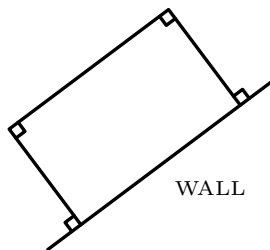
This equals 132 exactly when $a + b = 12$. Each digit is at most 9, so each is at least $12 - 9 = 3$. Sorting by the tens digit, the numbers are

$$39, \quad 48, \quad 57, \quad 66, \quad 75, \quad 84, \quad 93,$$

so there are 7 of them.

Answer: (B) 7. □

6. (1986 AJHSME, Problem #18) A rectangular grazing area is to be fenced off on three sides using part of a 100 meter rock wall as the fourth side. Fence posts are to be placed every 12 meters along the fence, including the two posts where the fence meets the rock wall. What is the fewest number of posts required to fence an area 36 m by 60 m?



(A) 11 (B) 12 (C) 13 (D) 14 (E) 16

Strategy. The fence is one path that leaves the wall, runs along three sides of the rectangle, and returns to the wall. The posts stand at 0, 12, 24, ... meters along this path, so count them as an evenly spaced list. Try both ways of placing the rectangle against the wall.

Solution. Either side of the rectangle can lie along the wall, since the wall is 100 meters long.

- If the 60 m side lies along the wall, the fence is $36 + 60 + 36 = 132$ meters long. The posts stand at 0, 12, 24, ..., 132 meters, so there are $\frac{132-0}{12} + 1 = 12$ posts. The two corners are at 36 and 96 meters, which are multiples of 12, so each corner gets a post.
- If the 36 m side lies along the wall, the fence is $60 + 36 + 60 = 156$ meters long, and it needs $\frac{156-0}{12} + 1 = 14$ posts.

The fewest number of posts is 12.

Answer: (B) 12. □

7. (2014 AMC 8, Problem #16) The “Middle School Eight” basketball conference has 8 teams. Every season, each team plays every other conference team twice (home and away), and each team also plays 4 games against non-conference opponents. What is the total number of games in a season involving the “Middle School Eight” teams?

(A) 60 (B) 88 (C) 96 (D) 144 (E) 160

Strategy. Count the conference games and the non-conference games separately. Every conference game has a home team and an away team, which gives a sequence of two choices.

Solution. Two conference teams play one game at each team’s home, so a conference game is determined by its home team and its away team. The home team can be any of the 8 teams, and the away team any of the other 7. Different choices give different games, so by the Multiplication Principle there are $8 \times 7 = 56$ conference games.

Each of the 8 teams also plays 4 non-conference games. Each of these games has only one conference team in it, so no game is counted twice, and there are $8 \times 4 = 32$ of them. The total is

$$56 + 32 = 88.$$

Answer: (B) 88. □

8. (1995 AJHSME, Problem #23) How many four-digit whole numbers are there such that the leftmost digit is odd, the second digit is even, and all four digits are different?

(A) 1120 (B) 1400 (C) 1800 (D) 2025 (E) 2500

Strategy. Choose the digits from left to right. The first two digits have conditions of their own, and the last two only have to differ from the digits already used.

Solution.

- The leftmost digit is odd: 1, 3, 5, 7 or 9, which is 5 ways. None of these is 0, so the number has four digits.
- The second digit is even: 0, 2, 4, 6 or 8, which is 5 ways. An even digit can never equal the odd first digit, so all 5 are allowed.
- The third digit can be any of the $10 - 2 = 8$ digits not yet used.
- The fourth digit can be any of the 7 digits not yet used.

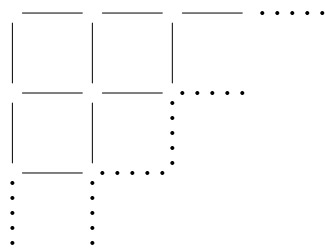
By the Multiplication Principle, the number of such numbers is

$$5 \times 5 \times 8 \times 7 = 1400.$$

Answer: (B) 1400. □

Advanced.

9. (2013 AMC 8, Problem #22) Toothpicks are used to make a grid that is 60 toothpicks long and 32 toothpicks high. How many toothpicks are used altogether?



- (A) 1920 (B) 1952 (C) 1980 (D) 2013 (E) 3932

Strategy. Count the horizontal toothpicks and the vertical toothpicks separately. A grid that is 32 squares high has 33 horizontal lines, not 32.

Solution. Each toothpick is a side of a small square, so the grid is 60 squares wide and 32 squares high.

- The horizontal toothpicks lie on $32 + 1 = 33$ horizontal lines, each made of 60 toothpicks, so there are $33 \times 60 = 1980$ of them.
- The vertical toothpicks lie on $60 + 1 = 61$ vertical lines, each made of 32 toothpicks, so there are $61 \times 32 = 1952$ of them.

Altogether $1980 + 1952 = 3932$ toothpicks are used.

Answer: (E) 3932. □

10. (2008 AMC 8, Problem #14) Three A's, three B's, and three C's are placed in the nine spaces so that each row and column contain one of each letter. If A is placed in the upper left corner, how many arrangements are possible?

A		

- (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Strategy. Fill in the top row first, then the left column, and then count the ways to finish the grid. These choices form a tree diagram.

Solution. The top row starts with A, so its other two spaces get B and C in some order: 2 ways. In the same way, the left column is A, B, C or A, C, B from top to bottom: 2 ways. Once these are fixed, the rest of the grid is forced.

- The second row starts with B or C; call the other of these two letters Y. The row still needs A and Y. The letter Y already appears at the top of one of the two remaining columns, so Y must go in the other one, and A fills the last space of the row.
- The left column is already full. Each of the other two columns is missing one letter, and that letter goes in the third row. Each letter then appears once in every column, so three times in all, and once in each of the first two rows, so it appears exactly once in the third row.

So the tree has $2 \times 2 \times 1 = 4$ ends. The four arrangements are shown below.

A B C
B C A
C A B

A B C
C A B
B C A

A C B
B A C
C B A

A C B
C B A
B A C

Answer: (C) 4. □

11. (2008 AMC 8, Problem #22) For how many positive integer values of n are both $\frac{n}{3}$ and $3n$ three-digit whole numbers?

(A) 12 (B) 21 (C) 27 (D) 33 (E) 34

Strategy. Turn each condition into a range for n . Then count the multiples of 3 in the overlap of the two ranges.

Solution. Since $\frac{n}{3}$ is a whole number, n is a multiple of 3. Since $\frac{n}{3}$ has three digits, $100 \leq \frac{n}{3} \leq 999$, so $300 \leq n \leq 2997$. Since $3n$ has three digits, $100 \leq 3n \leq 999$. Here $3 \times 33 = 99$, $3 \times 34 = 102$ and $3 \times 333 = 999$, so $34 \leq n \leq 333$. Both conditions hold exactly when n is a multiple of 3 with $300 \leq n \leq 333$, that is,

$$n = 3 \times 100, 3 \times 101, \dots, 3 \times 111.$$

There are $111 - 100 + 1 = 12$ such values.

Answer: (A) 12. □

12. (1988 AJHSME, Problem #25) A palindrome is a whole number that reads the same forwards as backwards. If one neglects the colon, certain times displayed on a digital watch are palindromes. Three examples are 1:01, 4:44, and 12:21. How many times during a 12-hour period will be palindromes?

(A) 57 (B) 60 (C) 63 (D) 90 (E) 93

Strategy. Sort the times by their number of digits. For the three-digit times, use the Multiplication Principle. There are only three hours with four digits, so list those times.

Solution. During a 12-hour period, the watch shows each time from 1:00 to 12:59 exactly once.

- The times from 1:00 to 9:59 have three digits: the hour h , then the two digits of the minutes. Such a time is a palindrome exactly when the last digit equals h . So choose the hour (9 ways) and the tens digit of the minutes, which can be any of 0 to 5 (6 ways); the last digit is then forced. This gives $9 \times 6 = 54$ palindromes.
- The times from 10:00 to 12:59 have four digits, and the minutes of a palindrome are the hour written backwards. This gives 10:01, 11:11 and 12:21, which are all real times, so there are 3 palindromes.

In total there are $54 + 3 = 57$ palindromes.

Answer: (A) 57. □

2. ANSWER KEY

Problem	Letter	Value
1	B	3
2	D	5
3	B	5
4	D	44
5	B	7
6	B	12
7	B	88
8	B	1400
9	E	3932
10	C	4
11	A	12
12	A	57