SYSTEMATIC LISTING AND THE MULTIPLICATION PRINCIPLE FOR AMC 8: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (1997 AJHSME, Problem #5) There are many two-digit multiples of 7, but only two of the multiples have a digit sum of 10. The sum of these two multiples of 7 is

(A) 119 **(B)** 126 **(C)** 140 **(D)** 175 **(E)** 189

Strategy. Make an organized list of the two-digit multiples of 7, written as $7 \times m$ in order of m, and find the digit sum of each one.

Solution. Since $7 \times 1 = 7$ has one digit and $7 \times 2 = 14$ has two, the list starts at 7×2. Since $7 \times 14 = 98$ and $7 \times 15 = 105$, it stops at 7×14. So there are $14 - 2 + 1 = 13$ two-digit multiples of 7. The table lists them in order, each with its digit sum.

Multiple	14	21	28	35	42	49	56	63	70	77	84	91	98
Digit sum	5	3	10	8	6	13	11	9	7	14	12	10	17

Only 28 and 91 have digit sum 10, and their sum is $28 + 91 = 119$.

Answer: (A) 119. □

2. (2004 AMC 10B, Problem #1) Each row of the Misty Moon Amphitheater has 33 seats. Rows 12 through 22 are reserved for a youth club. How many seats are reserved for this club?

(A) 297 **(B)** 330 **(C)** 363 **(D)** 396 **(E)** 726

Strategy. The reserved rows are the integers from 12 to 22, inclusive, so count them with $b - a + 1$. Then a reserved seat is a choice of a row followed by a choice of a seat in that row.

Solution. The reserved rows are $12, 13, \ldots, 22$, so there are $22 - 12 + 1 = 11$ of them. Taking $22 - 12 = 10$ rows would be a fencepost error: 10 is the number of steps from row 12 to row 22, and the list of rows has one more number than it has steps.

A reserved seat is chosen by choosing its row, in 11 ways, and then its seat in that row, in 33 ways. Different choices give different seats, so by the Multiplication Principle the number of reserved seats is

$$11 \times 33 = 363.$$

Answer: (C) 363. □

3. (1988 AJHSME, Problem #19) What is the 100th number in the arithmetic sequence 1, 5, 9, 13, 17, 21, 25, ...?

(A) 397 **(B)** 399 **(C)** 401 **(D)** 403 **(E)** 405

Strategy. The list goes up by 4 each time, and a list has one more number than it has steps. So the 100th number comes 99 steps of 4 after the first.

Solution. Write the numbers as

$$1 + 0 \times 4, \quad 1 + 1 \times 4, \quad 1 + 2 \times 4, \quad \ldots,$$

with multipliers of 4 that start at 0. The integers from 0 to 99 are $99 - 0 + 1 = 100$ multipliers, so the first 100 numbers use the multipliers $0, 1, \ldots, 99$. The 100th number therefore has multiplier 99, and it is

$$1 + 99 \times 4 = 397.$$

As a check, a list that starts at 1, goes up by 4 and ends at 397 has $\frac{397-1}{4}+1 = 99+1 = 100$ numbers.

Answer: (A) 397. □

4. (2021 Fall AMC 10A, Problem #6) Elmer the emu takes 44 equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in 12 equal leaps. The telephone poles are evenly spaced, and the 41st pole along this road is exactly one mile (5280 feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?

(A) 6 **(B)** 8 **(C)** 10 **(D)** 11 **(E)** 15

Strategy. First find the distance between consecutive poles. From the first pole to the 41st there are 40 gaps, not 41.

Solution. The 41 poles have 40 gaps between them, so each gap is $\frac{5280}{40} = 132$ feet. Elmer's stride is $\frac{132}{44} = 3$ feet and Oscar's leap is $\frac{132}{12} = 11$ feet. The leap is longer by $11 - 3 = 8$ feet.

Answer: (B) 8. □

5. (1991 AJHSME, Problem #11) There are several sets of three different numbers whose sum is 15 which can be chosen from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many of these sets contain a 5?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Strategy. A set that contains 5 is decided by its other two numbers. Make an organized list of these pairs, each written in one fixed form with the smaller number first so that no set is listed twice, and sort by the smaller number.

Solution. The other two numbers of the set are different, neither of them is 5, and their sum is $15 - 5 = 10$. Call them a and b with $a < b$. The smaller of two different numbers with sum 10 is less than half of 10, so a is 1, 2, 3 or 4, and then $b = 10 - a$. This gives the sets

$$\{1, 5, 9\}, \quad \{2, 5, 8\}, \quad \{3, 5, 7\}, \quad \{4, 5, 6\}.$$

Each of them consists of three different numbers from 1 to 9 with sum 15, so there are 4 such sets.

Answer: (B) 4. □

Intermediate.

6. (2015 AMC 8, Problem #10) How many integers between 1000 and 9999 have four distinct digits?

(A) 3024 **(B)** 4536 **(C)** 5040 **(D)** 6480 **(E)** 6561

Strategy. Choose the digits from left to right. After the first digit, the number of options at each step is the same whichever digits were used before, so the Multiplication Principle applies.

Solution. The integers from 1000 to 9999 are exactly the four-digit numbers. Neither 1000 nor 9999 has four distinct digits, so it makes no difference whether these two endpoints are counted. Choose the digits from left to right.

- The thousands digit cannot be 0, so it is one of $1, 2, \ldots, 9$: 9 ways.
- The hundreds digit is any of the 10 digits other than the thousands digit, 0 included: 9 ways.

- The tens digit is any digit other than the two already used: 8 ways.
- The units digit is any digit other than the three already used: 7 ways.

Which digits are left at each step depends on the earlier choices, but the number of them does not, and different choices give different numbers. By the Multiplication Principle, the number of such integers is

$$9 \times 9 \times 8 \times 7 = 4536.$$

Answer: (B) 4536. □

7. (2022 AMC 8, Problem #13) How many positive integers can fill the blank in the sentence below?
"One positive integer is _____ more than twice another, and the sum of the two numbers is 28"

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Strategy. Write the first integer and the number in the blank in terms of the second integer. The condition that the blank is positive then becomes a range of integers for the second integer.

Solution. Let y be the second integer, the one called "another", and let x be the first, so that $x + y = 28$. If k is the number in the blank, then $x = 2y + k$, so

$$k = x - 2y = (28 - y) - 2y = 28 - 3y.$$

The blank must hold a positive integer, so $28 - 3y > 0$, that is, $y < \frac{28}{3} = 9\frac{1}{3}$. The integers in this range that are positive are $1, 2, \ldots, 9$, and each of them works: $x = 28 - y$ is then a positive integer and $k = 28 - 3y$ is positive. So there are $9 - 1 + 1 = 9$ possible values of y, and they give different blanks. The blanks are $1, 4, 7, \ldots, 25$ (for $y = 9, 8, 7, \ldots, 1$), a list that goes up by 3 and has $\frac{25-1}{3} + 1 = 9$ numbers.

Answer: (D) 9. □

8. (2006 AMC 10A, Problem #9) How many sets of two or more consecutive positive integers have a sum of 15?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. Make an organized list sorted by the smallest number of the set. For each smallest number, add the next integers one at a time and stop as soon as the sum reaches or passes 15.

Solution. A set of consecutive integers is fixed by its smallest number and its size. If the smallest number is 8 or more, the two smallest numbers alone add up to at least $8+9 = 17$, which is more than 15. So the smallest number is one of $1, 2, \ldots, 7$, and the list stops there. For each of these, the table gives the sum of the first two numbers, then of the first three, and so on, until the sum reaches or passes 15.

Smallest number	Sums	Set with sum 15
1	$3, 6, 10, 15$	$\{1, 2, 3, 4, 5\}$
2	$5, 9, 14, 20$	none
3	$7, 12, 18$	none
4	$9, 15$	$\{4, 5, 6\}$
5	$11, 18$	none
6	$13, 21$	none
7	15	$\{7, 8\}$

Exactly 3 sets have sum 15.

Answer: (C) 3. □

9. (1994 AJHSME, Problem #24) A 2 by 2 square is divided into four 1 by 1 squares. Each of the small squares is to be painted either green or red. In how many different ways can the painting be accomplished so that no green square shares its top or right side with any red square? There may be as few as zero or as many as four small green squares.

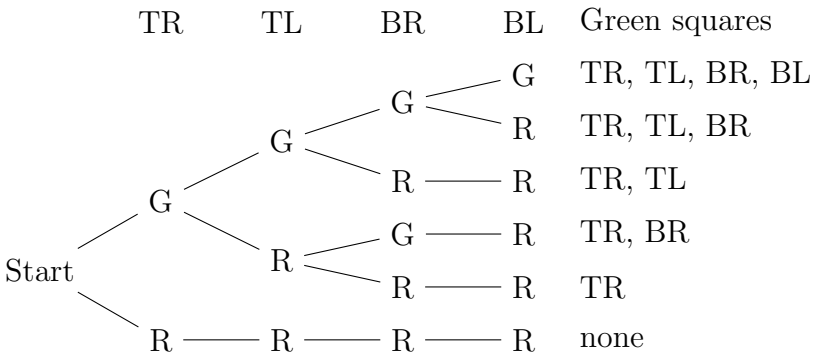
(A) 4 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 16

Strategy. A green square leaves no choice for its upper and right-hand neighbors: both must be green. Choose the colors one square at a time in a tree diagram; some branches split in two and others have only one option.

Solution. Write TL, TR, BL and BR for the top left, top right, bottom left and bottom right squares. By the condition, a green square has no red square on top of it or on its right. So whenever a square is green, its upper neighbor and its right-hand neighbor, where these exist, are green too:

- if TL is green, then TR is green;
- if BL is green, then TL and BR are green;
- if BR is green, then TR is green.

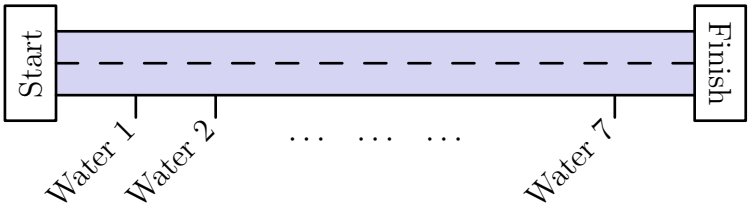
Nothing lies above TR or to its right, so a green TR forces no other square to be green. Choose the colors in the order TR, TL, BR, BL. If TR is red, then TL and BR must be red, and then BL must be red too. If TR is green, then TL and BR can each be green or red, and BL can be green only when TL and BR are both green. The tree diagram shows these choices, with G for green and R for red, and lists the green squares at each end.



The tree has 6 ends, so the painting can be done in 6 ways. The branches do not all split the same number of times, so the count comes from the ends of the tree, not from a product.

Answer: (B) 6. □

10. (2023 AMC 8, Problem #13) Along the route of a bicycle race, 7 water stations are evenly spaced between the start and finish lines, as shown in the figure below. There are also 2 repair stations evenly spaced between the start and finish lines. The 3rd water station is located 2 miles after the 1st repair station. How long is the race in miles?



(A) 8 **(B)** 16 **(C)** 24 **(D)** 48 **(E)** 96

Strategy. The stations lie strictly between the start and finish lines, so count those two lines as posts as well. Then n evenly spaced posts cut the race into $n-1$ equal gaps.

Solution. Let the race be L miles long. The start line, the 7 water stations and the finish line are 9 evenly spaced points, so they cut the race into $9-1=8$ equal gaps of $\frac{L}{8}$ miles. The 3rd water station is 3 gaps after the start line, at $\frac{3L}{8}$ miles. In the same way, the start line, the 2 repair stations and the finish line are 4 evenly spaced points with 3 gaps of $\frac{L}{3}$ miles, so the 1st repair station is at $\frac{L}{3}$ miles. The 3rd water station is 2 miles after it, so

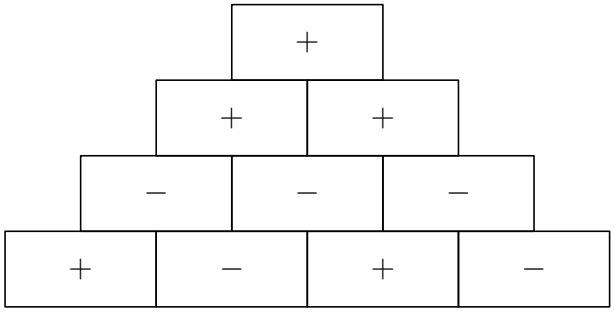
$$\frac{3L}{8}-\frac{L}{3}=\frac{9L-8L}{24}=\frac{L}{24}=2,$$

and $L=48$. Dividing the race into 7 parts for the water stations, or 2 parts for the repair stations, would be a fencepost error.

Answer: (D) 48. □

Advanced.

11. (2018 AMC 8, Problem #19) In a sign pyramid a cell gets a "+" if the two cells below it have the same sign, and it gets a "-" if the two cells below it have different signs. The diagram below illustrates a sign pyramid with four levels. How many possible ways are there to fill the four cells in the bottom row to produce a "+" at the top of the pyramid?



(A) 2 **(B)** 4 **(C)** 8 **(D)** 12 **(E)** 16

Strategy. Fill the bottom row from left to right. Changing the sign of the rightmost bottom cell always changes the sign at the top, so whatever the first three signs are, exactly one sign for the fourth cell gives + at the top.

Solution. A cell's sign depends only on the two cells below it. If exactly one of those two cells changes sign, the pair changes from "same" to "different" or back, so the cell above changes sign as well.

Number the rows 1 to 4 from the bottom, and change the sign of the rightmost cell of row 1 while keeping the other three. In row 2, only the rightmost cell sits on that cell; exactly one of the two cells below it has changed, so it changes, and the other two cells of row 2 stay the same. In row 3, the right cell sits on one changed cell and one unchanged cell, so it changes, and the left cell stays the same. The top cell then sits on one changed cell and one unchanged cell, so it changes too.

Hence, for each choice of the first three bottom cells, one of the two signs for the fourth cell gives + at the top and the other gives −. Each of the first three cells has 2 options, and the fourth has exactly 1. Which sign works depends on the first three, but the number of signs that work is always 1, so by the Multiplication Principle the number of ways is

$$2 \times 2 \times 2 \times 1 = 8.$$

Answer: (C) 8. □

12. (2022 AMC 10B, Problem #8) Consider the following 100 sets of 10 elements each:

$$\{1, 2, 3, \ldots, 10\},$$
$$\{11, 12, 13, \ldots, 20\},$$
$$\{21, 22, 23, \ldots, 30\},$$
$$\vdots$$
$$\{991, 992, 993, \ldots, 1000\}.$$

How many of these sets contain exactly two multiples of 7?

(A) 40 **(B)** 42 **(C)** 43 **(D)** 49 **(E)** 50

Strategy. Count all the multiples of 7 from 1 to 1000. Each set contains one or two of them, so the total shows how many sets contain two.

Solution. Together the 100 sets contain each integer from 1 to 1000 exactly once. Since $7 \times 142 = 994$ and $7 \times 143 = 1001$, the multiples of 7 in this range are $7 \times 1, 7 \times 2, \ldots, 7 \times 142$, so there are 142 of them.

Each set contains one or two of these multiples.

- At least one: two consecutive multiples of 7 differ by 7, so only $7 - 1 = 6$ integers lie strictly between them. Hence any 7 consecutive integers include a multiple of 7, and so does each set of 10 consecutive integers.
- At most two: if a set contained three multiples of 7, the largest would be at least 14 more than the smallest, but any two numbers in a set differ by at most 9.

Suppose k of the sets contain two multiples of 7. The other $100 - k$ sets contain one each, so

$$2k + (100 - k) = 142, \quad \text{that is,} \quad 100 + k = 142,$$

and $k = 42$.

Answer: (B) 42. □

13. (2020 AMC 8, Problem #20) A scientist walking through a forest recorded as integers the heights of 5 trees standing in a row. She observed that each tree was either twice as tall or half as tall as the one to its right. Unfortunately some of her data was lost when rain fell on her notebook. Her notes are shown below, with blanks indicating the missing numbers. Based on her observations, the scientist was able to reconstruct the lost data. What was the average height of the trees, in meters?

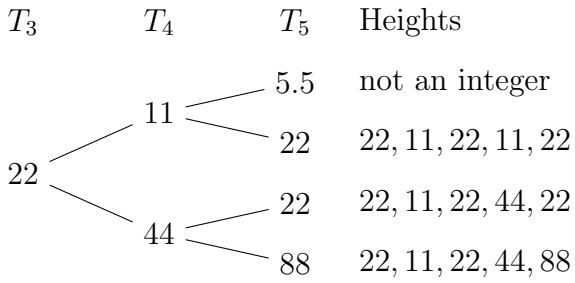
T_1	__ m
T_2	11 m
T_3	__ m
T_4	__ m
T_5	__ m
$\overline{T}$	__.2 m

(A) 22.2 **(B)** 24.2 **(C)** 33.2 **(D)** 35.2 **(E)** 37.2

Strategy. Start from the known height of 11 meters and build a tree diagram of the possible heights, one tree at a time. A branch whose height is not an integer ends at once.

Solution. Each tree is twice as tall or half as tall as the tree to its right, so two neighboring heights always differ by a factor of 2, one way or the other. Since $T_2 = 11$, the height T_1 is 22 or 5.5, and so is T_3. The heights are integers, so $T_1 = T_3 = 22$.

From $T_3 = 22$, the height T_4 is 11 or 44, and from there T_5 is half or twice T_4. The tree diagram shows these branches; the branch to 5.5 ends at once, and each other end gives a list of heights T_1, T_2, T_3, T_4, T_5.



The three lists have sums 88, 121 and 187, so their averages are $\frac{88}{5} = 17.6$, $\frac{121}{5} = 24.2$ and $\frac{187}{5} = 37.4$. The notes show an average that ends in .2, so the average height is 24.2 meters.

Answer: (B) 24.2. □

14. (1990 AJHSME, Problem #25) How many different patterns can be made by shading exactly two of the nine unit squares in a 3×3 grid? Patterns that can be matched by flips and/or turns are not considered different.

(A) 3 **(B)** 6 **(C)** 8 **(D)** 12 **(E)** 18

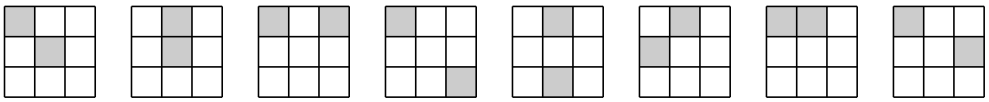
Strategy. Make an organized list of the patterns, sorted first by the kinds of the two shaded squares (corner, side or center) and then by how the two squares lie relative to each other. Turn or flip each pattern into one fixed form, so that matching patterns are listed only once.

Solution. The grid has 4 corner squares, 4 side squares (one in the middle of each side of the grid) and 1 center square. A flip or a turn moves corner squares to corner squares, side squares to side squares and the center to itself. It also keeps two squares that share a side sharing a side, and two squares in the same row or column in the same row or column

(rows and columns may trade places). So two patterns that match shade the same kinds of squares, and they agree on whether the two shaded squares share a side and on whether they lie in the same row or column. The list below sorts the patterns by these features.

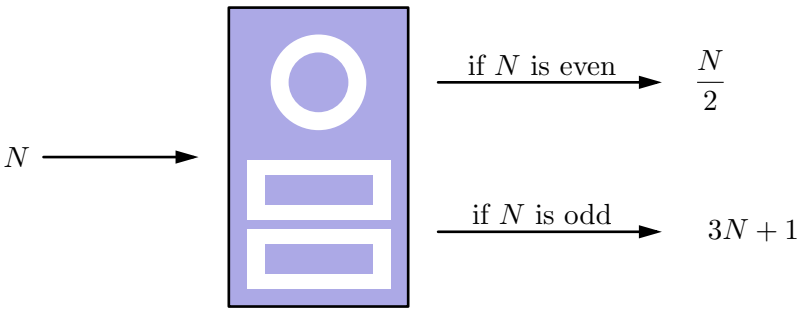
- The center and a corner: a turn moves the corner to the top left, so there is 1 pattern.
- The center and a side square: a turn moves the side square to the top, so there is 1 pattern.
- Two corners: either they lie in the same row or column, like the top left and top right corners, or they are opposite corners. A turn matches any two patterns of the same type, so there are 2 patterns.
- Two side squares: either they are opposite, like the top and bottom side squares, or they are next to the same corner, like the top and left side squares. Again a turn matches any two of the same type, so there are 2 patterns.
- A corner and a side square: turn the grid so that the corner is at the top left. The side square either shares a side with the corner (the top or the left side square) or does not (the right or the bottom side square). The flip in the diagonal through the top left corner swaps the two choices of each type, so there are 2 patterns.

In total there are $1+1+2+2+2=8$ patterns. In the order of the list, they are:



Answer: (C) 8. □

15. (2020 AMC 8, Problem #22) When a positive integer N is fed into a machine, the output is a number calculated according to the rule shown below.



For example, starting with an input of $N=7$, the machine will output $3\cdot 7+1=22$. Then if the output is repeatedly inserted into the machine five more times, the final output is 26.

$$7 \to 22 \to 11 \to 34$$
$$\to 17 \to 52 \to 26$$

When the same 6-step process is applied to a different starting value of N, the final output is 1. What is the sum of all such integers N?

$$N \to __ \to __ \to __$$
$$\to __ \to __ \to 1$$

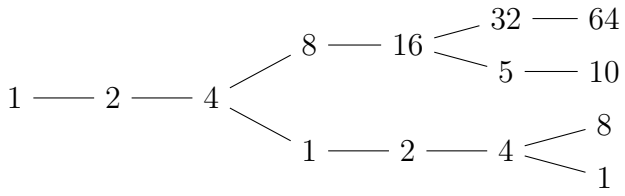
(A) 73 **(B)** 74 **(C)** 75 **(D)** 82 **(E)** 83

Strategy. Work backward from the final output 1 with a tree diagram. An output x always comes from the even input $2x$, and it also comes from an odd input when $\frac{x-1}{3}$ is an odd positive integer.

Solution. An even input n gives $\frac{n}{2}$, so the output x comes from the even input $2x$. An odd input n gives $3n+1$, so x also comes from an odd input exactly when $n = \frac{x-1}{3}$ is an odd positive integer. No other input gives x. Going back one step at a time from 1:

- 1 comes only from 2, since $\frac{1-1}{3} = 0$ is not positive;
- 2 comes only from 4, since $\frac{2-1}{3}$ is not an integer;
- 4 comes from 8, and also from 1, since $\frac{4-1}{3} = 1$ is odd;
- 8 comes only from 16, since $\frac{8-1}{3}$ is not an integer, and 1 only from 2;
- 16 comes from 32, and also from 5, since $\frac{16-1}{3} = 5$ is odd; 2 comes only from 4;
- 32 comes only from 64, 5 only from 10 (neither $\frac{31}{3}$ nor $\frac{4}{3}$ is an integer), and 4 from 8 and 1.

In the tree diagram below, the final output 1 is at the left, and each column to the right goes back one more step.



After six steps back, the ends of the tree are 64, 10, 8 and 1. These are exactly the starting values whose final output is 1, and their sum is

$$64 + 10 + 8 + 1 = 83.$$

Answer: (E) 83. □

2. Answer Key

Problem	Letter	Value
1	A	119
2	C	363
3	A	397
4	B	8
5	B	4
6	B	4536
7	D	9
8	C	3
9	B	6
10	D	48
11	C	8
12	B	42
13	B	24.2
14	C	8
15	E	83