INTRODUCTION TO NUMBER THEORY FOR AMC 8

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1. Divisibility

An integer n is **divisible** by a nonzero integer d if n/d is an integer, that is, if dividing n by d leaves remainder 0. Then d is a **divisor**, or **factor**, of n, and n is a **multiple** of d.

The rules in the table decide divisibility by small numbers from the digits alone, without long division.

Divisor	Rule
2	The last digit is even (0, 2, 4, 6 or 8).
3	The sum of the digits is divisible by 3.
4	The number formed by the last two digits is divisible by 4.
5	The last digit is 0 or 5.
6	The number is divisible by both 2 and 3.
9	The sum of the digits is divisible by 9.
10	The last digit is 0.
11	The alternating sum of the digits (from the right: add the first digit, subtract the second, add the third, and so on) is divisible by 11.

The rule for 4 works because $100 = 4 \cdot 25$. For example, $5732 = 57 \cdot 100 + 32$, and $57 \cdot 100$ is a multiple of 4. In general, a number leaves the same remainder as the number formed by its last two digits when divided by 4.

The rules for 3 and 9 work because each power of 10 is 1 more than a multiple of 9: $10 = 9 + 1$, $100 = 99 + 1$, $1000 = 999 + 1$, and so on. For example,

$$4725 = 4 \cdot 1000 + 7 \cdot 100 + 2 \cdot 10 + 5 = (4 \cdot 999 + 7 \cdot 99 + 2 \cdot 9) + (4 + 7 + 2 + 5).$$

The first group is a multiple of 9, and therefore also a multiple of 3. In general, a number and the sum of its digits leave the same remainder when divided by 9, and also when divided by 3.

To test divisibility by a number such as 6 or 12, write it as a product of two factors that have no common divisor greater than 1, and combine the rules for those two factors. A number is divisible by 6 if and only if it is divisible by both 2 and 3, and it is divisible by 12 if and only if it is divisible by both 3 and 4. The two factors must have no common divisor greater than 1: the number 6 is divisible by both 2 and 6, but not by 12.

Examples.

Problem 1.1 (2021 Fall AMC 10A, Problem #5)**.** The six-digit number $\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}\,\underline{0}\,\underline{A}$ is prime for only one digit A. What is A?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

Strategy. A number that has a positive divisor other than 1 and itself is not prime. For every digit A except one, a divisibility rule gives such a divisor.

Solution. Call the number N. A prime has no positive divisors other than 1 and itself, so N is not prime whenever it is divisible by 2, 3, 5 or 11, all of which are smaller than N.

If A is even, then N is divisible by 2. If $A = 5$, then N is divisible by 5. This leaves $A = 1$, 3, 7 and 9.

The digit sum of N is $2 + 0 + 2 + 1 + 0 + A = 5 + A$. For $A = 1$ and $A = 7$ it is 6 and 12, both multiples of 3, so 202101 and 202107 are divisible by 3. Indeed, $202101 = 3 \cdot 67367$ and $202107 = 3 \cdot 67369$.

For $A = 3$, the alternating sum of the digits of 202103, from the right, is

$$3 - 0 + 1 - 2 + 0 - 2 = 0,$$

which is divisible by 11. So 202103 is divisible by 11, and indeed $202103 = 11 \cdot 18373$.

Thus N is not prime for any digit other than 9. Since N is prime for one digit, that digit is 9.

Answer: (E) 9. □

Problem 1.2 (2018 AMC 12B, Problem #15)**.** How many 3-digit positive odd multiples of 3 do not include the digit 3?

(A) 96 **(B)** 97 **(C)** 98 **(D)** 102 **(E)** 120

Strategy. Count the choices for the hundreds digit and the units digit. Then use the rule for 3 to count the tens digits that complete each choice.

Solution. Write the number as $\underline{a}\,\underline{b}\,\underline{c}$. The hundreds digit a is neither 0 nor 3, so it is one of $1, 2, 4, 5, 6, 7, 8, 9$, which gives 8 choices. The number is odd and has no digit 3, so the units digit c is one of $1, 5, 7, 9$, which gives 4 choices.

The tens digit b is one of the nine digits other than 3. Sort these digits by their remainders when divided by 3:

$$\{0, 6, 9\} \text{ (remainder 0)}, \qquad \{1, 4, 7\} \text{ (remainder 1)}, \qquad \{2, 5, 8\} \text{ (remainder 2)}.$$

By the rule for 3, the number is a multiple of 3 exactly when $a + b + c$ is a multiple of 3. Once a and c are chosen, this happens exactly when b has one particular remainder, so exactly one of the three groups works, and there are 3 choices for b.

Each of the $8 \cdot 4$ choices of a and c goes with 3 choices of b, so the count is

$$8 \cdot 4 \cdot 3 = 96.$$

Answer: (A) 96. □

2. Primes and Prime Factorization

A **prime** is a positive integer greater than 1 whose only positive divisors are 1 and itself. The first few primes are $2, 3, 5, 7, 11, 13$. An integer greater than 1 that is not prime is **composite**. The number 1 is neither prime nor composite.

Theorem 2.1 (Fundamental Theorem of Arithmetic)**.** *Every integer greater than* 1 *can be written as a product of primes in exactly one way, apart from the order of the factors. This product is the* ***prime factorization*** *of the number.*

For example, $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2$, and no other list of primes has product 72. To find a prime factorization, split the number into two factors, and repeat with each factor until only primes remain.

Proposition 2.2 (Number of Divisors)**.** If $N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, \ldots, p_k$ are different primes, then the number of positive divisors of N is

$$(a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

Proof. If d is a positive divisor of N, then $N = d \cdot e$ for a positive integer e, and the prime factorizations of d and e together give the prime factorization of N, which is unique. So d has no prime factors other than $p_1, \ldots, p_k$, and the exponent of each p_i in d is one of $0, 1, \ldots, a_i$, which gives $a_i + 1$ choices. Every choice of the exponents gives a divisor of N, and different choices give different divisors because prime factorizations are unique. Multiplying the numbers of choices for the k exponents gives the count. □

For example, $72 = 2^3 \cdot 3^2$ has $(3 + 1)(2 + 1) = 12$ positive divisors: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.

Examples.

Problem 2.3 (2016 AMC 8, Problem #9)**.** What is the sum of the distinct prime integer divisors of 2016?

(A) 9 **(B)** 12 **(C)** 16 **(D)** 49 **(E)** 63

Strategy. Factor 2016 into primes, then add the distinct primes.

Solution. Divide by 2 until the result is odd, then factor the odd part:

$$2016 = 2 \cdot 1008 = 2^2 \cdot 504 = 2^3 \cdot 252 = 2^4 \cdot 126 = 2^5 \cdot 63, \qquad 63 = 3^2 \cdot 7.$$

Thus $2016 = 2^5 \cdot 3^2 \cdot 7$. The distinct prime divisors are 2, 3 and 7, and their sum is $2+3+7 = 12$.

Answer: (B) 12. □

Problem 2.4 (2018 AMC 8, Problem #18)**.** How many positive factors does 23,232 have?

(A) 9 **(B)** 12 **(C)** 28 **(D)** 36 **(E)** 42

Strategy. Find the prime factorization, then count the divisors from the exponents.

Solution. Divide by 2 until the result is odd:

$$23232 = 2 \cdot 11616 = 2^2 \cdot 5808 = 2^3 \cdot 2904 = 2^4 \cdot 1452 = 2^5 \cdot 726 = 2^6 \cdot 363.$$

The digit sum of 363 is 12, so 363 is divisible by 3, and $363 = 3 \cdot 121 = 3 \cdot 11^2$. Hence

$$23232 = 2^6 \cdot 3^1 \cdot 11^2.$$

The exponents are 6, 1 and 2, so the number of positive divisors is

$$(6+1)(1+1)(2+1) = 7 \cdot 2 \cdot 3 = 42.$$

Answer: (E) 42. □

3. GCD and LCM

The **greatest common divisor** (GCD) of two positive integers is the largest positive integer that divides both of them. It is also called the greatest common factor (GCF). The **least common multiple** (LCM) of two positive integers is the smallest positive integer that is a multiple of both of them. The GCD and LCM of positive integers a and b are written $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$. The same definitions apply to three or more numbers.

For small numbers, the GCD and LCM can be found by listing divisors and multiples. For larger numbers, use prime factorizations:

1. Find the prime factorization of each number.
2. For the GCD, take each prime that appears in every factorization, to the lowest power with which it appears, and multiply.
3. For the LCM, take each prime that appears in any of the factorizations, to the highest power with which it appears, and multiply.

For example, $72 = 2^3 \cdot 3^2$ and $90 = 2^1 \cdot 3^2 \cdot 5^1$. The primes in both factorizations are 2 and 3, with lowest powers 2^1 and 3^2, so $\gcd(72, 90) = 2^1 \cdot 3^2 = 18$. The primes in either factorization are 2, 3 and 5, with highest powers 2^3, 3^2 and 5^1, so $\operatorname{lcm}(72, 90) = 2^3 \cdot 3^2 \cdot 5 = 360$.

A common multiple of several numbers contains each of their primes to at least the highest power with which it appears. So every common multiple of several numbers is a multiple of their LCM.

For two positive integers a and b,

$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) = a \cdot b.$$

For each prime, the GCD takes the smaller of its two exponents in a and b, and the LCM takes the larger. The smaller plus the larger is the sum of the two exponents, which is the exponent of that prime in ab. For example, $18 \cdot 360 = 6480 = 72 \cdot 90$.

Examples.

Problem 3.1 (2013 AMC 8, Problem #10)**.** What is the ratio of the least common multiple of 180 and 594 to the greatest common factor of 180 and 594?

(A) 110 **(B)** 165 **(C)** 330 **(D)** 625 **(E)** 660

Strategy. Factor both numbers into primes. Use the lowest exponents for the GCD and the highest exponents for the LCM, then divide.

Solution. The prime factorization of 180 is

$$180 = 18 \cdot 10 = (2 \cdot 3^2) \cdot (2 \cdot 5) = 2^2 \cdot 3^2 \cdot 5.$$

For 594, first $594 = 2 \cdot 297$. The digit sum of 297 is 18, so 297 is divisible by 9, and $297 = 9 \cdot 33 = 3^3 \cdot 11$. Hence

$$594 = 2 \cdot 3^3 \cdot 11.$$

The common primes are 2 and 3, so the GCD is $\gcd(180, 594) = 2^1 \cdot 3^2 = 18$. The LCM takes the highest power of every prime present:

$$\operatorname{lcm}(180, 594) = 2^2 \cdot 3^3 \cdot 5 \cdot 11 = 5940.$$

The ratio is

$$\frac{\operatorname{lcm}(180, 594)}{\gcd(180, 594)} = \frac{2^2 \cdot 3^3 \cdot 5 \cdot 11}{2 \cdot 3^2} = 2 \cdot 3 \cdot 5 \cdot 11 = 330.$$

Answer: (C) 330. □

Problem 3.2 (2019 AMC 10B, Problem #7)**.** Each piece of candy in a shop costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the least possible value of n?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 25 **(E)** 28

Strategy. Casper's money, in cents, is a common multiple of 12, 14, 15 and 20. Every common multiple is a multiple of the least common multiple.

Solution. Let Casper have M cents. A piece of red candy costs a whole number of cents, and 12 of them cost exactly M cents, so M is a multiple of 12. In the same way, M is a multiple of 14 and of 15. Also $M = 20n$, so M is a multiple of 20. Hence M is a common multiple of 12, 14, 15 and 20, and therefore a multiple of their least common multiple. Since

$$12 = 2^2 \cdot 3, \qquad 14 = 2 \cdot 7, \qquad 15 = 3 \cdot 5, \qquad 20 = 2^2 \cdot 5,$$

the highest powers of the primes are 2^2, 3, 5 and 7, and

$$\operatorname{lcm}(12, 14, 15, 20) = 2^2 \cdot 3 \cdot 5 \cdot 7 = 420.$$

So $20n = M \geq 420$, which gives $n \geq 21$. The value $n = 21$ is possible: if $M = 420$, a piece of red, green or blue candy costs $420/12 = 35$, $420/14 = 30$ or $420/15 = 28$ cents, and 21 pieces of purple candy cost $21 \cdot 20 = 420$ cents. The least possible value of n is 21.

Answer: (B) 21. □

4. Remainders and Modular Arithmetic

Modular arithmetic is arithmetic with remainders. On a 12-hour clock, 4 hours after 10 o'clock it is 2 o'clock, because $10 + 4 = 14$ leaves remainder 2 when divided by 12.

Definition 4.1. Let m be a positive integer. Integers a and b are **congruent modulo** m if m divides $a - b$. This is written

$$a \equiv b \pmod{m}.$$

Equivalently, a and b leave the same remainder when divided by m.

For example, $14 \equiv 2 \pmod{12}$, and every integer is congruent modulo m to its remainder upon division by m.

Proposition 4.2 (Sums, Products, and Powers)**.** If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a + c \equiv b + d \pmod{m}, \qquad a \cdot c \equiv b \cdot d \pmod{m}, \qquad a^k \equiv b^k \pmod{m}$$

for every positive integer k.

Proof. The differences $a - b$ and $c - d$ are multiples of m. Then $(a + c) - (b + d) = (a - b) + (c - d)$ is a multiple of m, and so is $ac - bd = a(c - d) + d(a - b)$. Applying the product rule $k - 1$ times gives $a^k \equiv b^k \pmod{m}$. □

So any term of a sum, any factor of a product and the base of a power can be replaced by its remainder modulo m, but the exponent cannot. For example, $15 \equiv 1 \pmod{7}$, so

$$15^{100} \equiv 1^{100} = 1 \pmod{7},$$

and 15^{100} leaves remainder 1 when divided by 7.

The units digit of a positive integer is its remainder modulo 10. By the proposition, the units digit of a sum or a product depends only on the units digits of the numbers added or multiplied. So the units digits of $a, a^2, a^3, \ldots$ can be found by multiplying units digits only, and they repeat in a cycle.

Examples.

Problem 4.3. What is the units digit of 3^{2021}?

Strategy. Find the first power of 3 that is congruent to 1 modulo 10. Then write 2021 as a multiple of the exponent of that power plus a remainder.

Solution. The first powers of 3 are 3, 9, 27 and 81, so $3^4 \equiv 1 \pmod{10}$, and the units digits repeat in the cycle $3, 9, 7, 1$. Since $2021 = 4 \cdot 505 + 1$,

$$3^{2021} = \left(3^4\right)^{505} \cdot 3 \equiv 1^{505} \cdot 3 = 3 \pmod{10}.$$

The units digit is 3.

Answer: 3. □

Problem 4.4 (2018 AMC 8, Problem #21)**.** How many positive three-digit integers have a remainder of 2 when divided by 6, a remainder of 5 when divided by 9, and a remainder of 7 when divided by 11?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. Each remainder is 4 less than its divisor. Add 4 to the number and use the least common multiple.

Solution. Let N be such an integer. If $N = 6q + 2$, then $N + 4 = 6(q + 1)$, so $N + 4$ is divisible by 6. In the same way, $N + 4$ is divisible by 9 and by 11. So $N + 4$ is a common multiple of 6, 9 and 11, and therefore a multiple of

$$\operatorname{lcm}(6, 9, 11) = 2 \cdot 3^2 \cdot 11 = 198.$$

Conversely, if $N = 198k - 4$ for a positive integer k, then

$$N = 6(33k - 1) + 2 = 9(22k - 1) + 5 = 11(18k - 1) + 7,$$

so N has all three remainders. It remains to count the three-digit numbers of the form $198k - 4$. The condition $100 \leq 198k - 4 \leq 999$ means $104 \leq 198k \leq 1003$. Since $198 \cdot 5 = 990$ and $198 \cdot 6 = 1188$, this holds for $k = 1, 2, 3, 4, 5$. The integers are 194, 392, 590, 788 and 986, so there are 5 of them.

Answer: (E) 5. □

5. Problems

Beginner.

1. What is the sum of the distinct prime divisors of 924?
(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

2. The product of two positive integers is 90 and their greatest common divisor is 3. What is their least common multiple?
(A) 15 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 45

3. (2002 AMC 8, Problem #5) Carlos Montado was born on Saturday, November 9, 2002. On what day of the week will Carlos be 706 days old?
(A) Monday **(B)** Wednesday **(C)** Friday **(D)** Saturday **(E)** Sunday

4. (1994 AJHSME, Problem #6) The unit's digit (one's digit) of the product of any six consecutive positive whole numbers is
(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

5. (2003 AMC 8, Problem #19) How many integers between 1000 and 2000 have all three of the numbers 15, 20, and 25 as factors?
(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Intermediate.

6. The 5-digit number $\underline{37A4B}$ is divisible by 6, where A and B are digits. What is the smallest possible value of $A + B$?
(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

7. Two positive integers have greatest common divisor 12 and least common multiple 420. One of the integers is 60. What is the other integer?
(A) 70 **(B)** 72 **(C)** 84 **(D)** 96 **(E)** 108

8. What is the units digit of 8^{2026}?
(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

9. (1997 AJHSME, Problem #11) Let $\boxed{N}$ mean the number of whole number divisors of N. For example, $\boxed{3} = 2$ because 3 has two divisors, 1 and 3. Find the value of

$$\boxed{\boxed{11} \times \boxed{20}}.$$

(A) 6 **(B)** 8 **(C)** 12 **(D)** 16 **(E)** 24

10. (2016 AMC 8, Problem #5) The number N is a two-digit number with the following properties:

- When N is divided by 9, the remainder is 1.
- When N is divided by 10, the remainder is 3.

What is the remainder when N is divided by 11?
(A) 0 **(B)** 2 **(C)** 4 **(D)** 5 **(E)** 7

Advanced.

11. (2000 AMC 8, Problem #14) What is the units digit of $19^{19} + 99^{99}$?
(A) 0 **(B)** 1 **(C)** 2 **(D)** 8 **(E)** 9

12. (2014 AMC 8, Problem #21) The 7-digit numbers $\underline{74A52B1}$ and $\underline{326AB4C}$ are each multiples of 3. Which of the following could be the value of C?
(A) 1 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 8

13. (2020 AMC 8, Problem #19) A number is called flippy if its digits alternate between two distinct digits. For example, 2020 and 37373 are flippy, but 3883 and 123123 are not. How many five-digit flippy numbers are divisible by 15?
(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

14. (2016 AMC 8, Problem #15) What is the largest power of 2 that is a divisor of $13^4 - 11^4$?
(A) 8 **(B)** 16 **(C)** 32 **(D)** 64 **(E)** 128

15. (2011 AMC 8, Problem #24) In how many ways can 10001 be written as the sum of two primes?
(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

16. (2016 AMC 8, Problem #24) The digits 1, 2, 3, 4, and 5 are each used once to write a five-digit number $PQRST$. The three-digit number PQR is divisible by 4, the three-digit number QRS is divisible by 5, and the three-digit number RST is divisible by 3. What is P?
(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5