INTRODUCTION TO NUMBER THEORY FOR AMC 8: SOLUTIONS

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1. Solutions

Beginner.

1. What is the sum of the distinct prime divisors of 924?

(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

Strategy. Find the prime factorization of 924, then add the distinct primes.

Solution. Divide by small primes:

$$924 = 2 \cdot 462 = 2^2 \cdot 231, \qquad 231 = 3 \cdot 77 = 3 \cdot 7 \cdot 11.$$

So $924 = 2^2 \cdot 3 \cdot 7 \cdot 11$. The distinct prime divisors are 2, 3, 7 and 11, and their sum is $2 + 3 + 7 + 11 = 23$.

Answer: (C) 23. □

2. The product of two positive integers is 90 and their greatest common divisor is 3. What is their least common multiple?

(A) 15 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 45

Strategy. Use the identity $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$.

Solution. Let the integers be a and b. By the identity,

$$3 \cdot \operatorname{lcm}(a, b) = 90,$$

so $\operatorname{lcm}(a, b) = 30$. For example, $a = 6$ and $b = 15$ have product 90, greatest common divisor 3 and least common multiple 30.

Answer: (D) 30. □

3. (2002 AMC 8, Problem #5) Carlos Montado was born on Saturday, November 9, 2002. On what day of the week will Carlos be 706 days old?

(A) Monday **(B)** Wednesday **(C)** Friday **(D)** Saturday **(E)** Sunday

Strategy. Two days fall on the same day of the week exactly when the number of days from one to the other is a multiple of 7. Write 706 as a multiple of 7 plus a remainder.

Solution. Carlos is 706 days old on the day 706 days after his birth, which was a Saturday. Days a multiple of 7 apart fall on the same day of the week. Since $706 = 7 \cdot 100 + 6$, that is, $706 \equiv 6 \pmod 7$, the day 706 days after a Saturday falls on the same day of the week as the day 6 days after a Saturday. Counting Sunday, Monday, Tuesday, Wednesday, Thursday and Friday, that day is a Friday.

Answer: (C) Friday. □

4. (1994 AJHSME, Problem #6) The unit's digit (one's digit) of the product of any six consecutive positive whole numbers is

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Strategy. Show that the product is divisible by both 2 and 5, and therefore by 10. Then use the rule for 10.

Solution. From one whole number to the next, the remainder upon division by 5 goes up by 1, and after 4 it returns to 0. So any five consecutive whole numbers leave the remainders 0, 1, 2, 3 and 4, one each, and one of them is a multiple of 5. Hence six consecutive positive whole numbers include a multiple of 5. They also include an even number, since of any two consecutive whole numbers one is even. So their product is divisible by both 5 and 2.

Since $10 = 2 \cdot 5$ and 2 and 5 have no common divisor greater than 1, the product is divisible by 10, and by the rule for 10 its units digit is 0. For example, $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$.

Answer: (A) 0. □

5. (2003 AMC 8, Problem #19) How many integers between 1000 and 2000 have all three of the numbers 15, 20, and 25 as factors?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. A number with all three as factors is a common multiple of 15, 20 and 25. Find their least common multiple and count its multiples in the range.

Solution. The prime factorizations are $15 = 3 \cdot 5$, $20 = 2^2 \cdot 5$ and $25 = 5^2$, so

$$\text{lcm}(15, 20, 25) = 2^2 \cdot 3 \cdot 5^2 = 300.$$

Every common multiple of 15, 20 and 25 is a multiple of 300, and every multiple of 300 is divisible by all three. The multiples of 300 between 1000 and 2000 are 1200, 1500 and 1800, so there are 3 such integers.

Answer: (C) 3. □

Intermediate.

6. The 5-digit number $\underline{37A4B}$ is divisible by 6, where A and B are digits. What is the smallest possible value of $A + B$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. A number is divisible by 6 if and only if it is divisible by both 2 and 3. Apply both rules.

Solution. For divisibility by 2, the last digit B must be even. For divisibility by 3, the digit sum $3 + 7 + A + 4 + B = 14 + A + B$ must be a multiple of 3.

If $A + B = 0$, the digit sum is 14, which is not a multiple of 3. If $A + B = 1$, the digit sum is 15, which is a multiple of 3, and $A = 1$, $B = 0$ gives an even last digit. The number is then $37140 = 6 \cdot 6190$. So the smallest possible value of $A + B$ is 1.

Answer: (A) 1. □

7. Two positive integers have greatest common divisor 12 and least common multiple 420. One of the integers is 60. What is the other integer?

(A) 70 **(B)** 72 **(C)** 84 **(D)** 96 **(E)** 108

Strategy. Use the identity $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$ with $a = 60$.

Solution. Let the other integer be b. By the identity,

$$12 \cdot 420 = 60 \cdot b,$$

so $b = \frac{5040}{60} = 84$. As a check, $60 = 2^2 \cdot 3 \cdot 5$ and $84 = 2^2 \cdot 3 \cdot 7$, so $\gcd(60, 84) = 2^2 \cdot 3 = 12$ and $\text{lcm}(60, 84) = 2^2 \cdot 3 \cdot 5 \cdot 7 = 420$.

Answer: (C) 84. □

8. What is the units digit of 8^{2026}?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Strategy. Work modulo 10. Find the cycle of units digits of the powers of 8, then use the remainder of the exponent upon division by the length of the cycle.

Solution. Each units digit in the list of powers of 8 comes from the previous one by multiplying by 8 and keeping the units digit. The first five units digits are

$$8, \quad 4, \quad 2, \quad 6, \quad 8,$$

since $8 \cdot 8 = 64$, $4 \cdot 8 = 32$, $2 \cdot 8 = 16$ and $6 \cdot 8 = 48$. The fifth power has the same units digit as the first, so the units digits repeat in the cycle $8, 4, 2, 6$ of length 4. Since $2026 = 4 \cdot 506 + 2$, the units digit of 8^{2026} is the second one in the cycle, which is 4.

Answer: (C) 4. □

9. (1997 AJHSME, Problem #11) Let $\boxed{N}$ mean the number of whole number divisors of N. For example, $\boxed{3} = 2$ because 3 has two divisors, 1 and 3. Find the value of

$$\boxed{\boxed{11} \times \boxed{20}}.$$

(A) 6 **(B)** 8 **(C)** 12 **(D)** 16 **(E)** 24

Strategy. Work from the inside out. Count the divisors of 11 and of 20, multiply the two counts, and then count the divisors of the product.

Solution. Since 11 is prime, its only divisors are 1 and 11, so $\boxed{11} = 2$. Since $20 = 2^2 \cdot 5$, it has $(2+1)(1+1) = 6$ divisors, so $\boxed{20} = 6$.

Therefore $\boxed{11} \times \boxed{20} = 2 \cdot 6 = 12$. Since $12 = 2^2 \cdot 3$, it has $(2+1)(1+1) = 6$ divisors, namely $1, 2, 3, 4, 6, 12$. So the value is 6.

Answer: (A) 6. □

10. (2016 AMC 8, Problem #5) The number N is a two-digit number with the following properties:

- When N is divided by 9, the remainder is 1.
- When N is divided by 10, the remainder is 3.

What is the remainder when N is divided by 11?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 5 **(E)** 7

Strategy. The remainder upon division by 10 is the units digit. A number and its digit sum leave the same remainder upon division by 9, which gives the tens digit.

Solution. The remainder upon division by 10 is 3, so the units digit is 3. Let A be the tens digit, so $1 \leq A \leq 9$. The number N and its digit sum $A+3$ leave the same remainder when divided by 9, so $A + 3$ leaves remainder 1. Since $4 \leq A + 3 \leq 12$, this means $A + 3 = 10$, so $A = 7$ and $N = 73$. Now $73 = 11 \cdot 6 + 7$, so the remainder upon division by 11 is 7.

Answer: (E) 7. □

Advanced.

11. (2000 AMC 8, Problem #14) What is the units digit of $19^{19} + 99^{99}$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 8 **(E)** 9

Strategy. Only units digits matter. Find the units digit of each power from the cycle of units digits of the powers of 9.

Solution. Both 19 and 99 have units digit 9, so 19^{19} has the same units digit as 9^{19}, and 99^{99} has the same units digit as 9^{99}. Since $9 \cdot 9 = 81$, the units digits of the powers of 9 alternate: $9, 1, 9, 1, \ldots$. An odd power of 9 has units digit 9. Both 19 and 99 are odd, so both powers have units digit 9. Then $9 + 9 = 18$, so the sum has units digit 8.

Answer: (D) 8. □

12. (2014 AMC 8, Problem #21) The 7-digit numbers $\underline{74A52B1}$ and $\underline{326AB4C}$ are each multiples of 3. Which of the following could be the value of C?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 8

Strategy. Apply the digit-sum rule for 3 to both numbers. The first number gives information about $A + B$, which is also part of the digit sum of the second number.

Solution. Each number is a multiple of 3, so each digit sum is a multiple of 3.

The digit sum of the first number is $7 + 4 + A + 5 + 2 + B + 1 = 19 + A + B$. Since $19 \equiv 1 \pmod 3$, this gives $A + B \equiv 2 \pmod 3$.

The digit sum of the second number is $3 + 2 + 6 + A + B + 4 + C = 15 + A + B + C$. Since $15 \equiv 0 \pmod 3$, this gives $A + B + C \equiv 0 \pmod 3$.

Substituting $A + B \equiv 2$ gives $2 + C \equiv 0 \pmod 3$, so $C \equiv 1 \pmod 3$. The digits with this property are 1, 4 and 7. Of the choices, only 1 is possible. For example, $A = B = 1$ gives $7415211 = 3 \cdot 2471737$ and $3261141 = 3 \cdot 1087047$.

Answer: (A) 1. □

13. (2020 AMC 8, Problem #19) A number is called flippy if its digits alternate between two distinct digits. For example, 2020 and 37373 are flippy, but 3883 and 123123 are not. How many five-digit flippy numbers are divisible by 15?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

Strategy. A number is divisible by 15 if and only if it is divisible by both 3 and 5. The rule for 5 fixes the first, third and fifth digits, and the rule for 3 then restricts the other two.

Solution. A five-digit flippy number has the form $\underline{a}\,\underline{b}\,\underline{a}\,\underline{b}\,\underline{a}$, where a and b are different digits and $a \neq 0$ because a is the leading digit.

Since $15 = 3 \cdot 5$ and 3 and 5 have no common divisor greater than 1, the number is divisible by 15 exactly when it is divisible by both 5 and 3. For divisibility by 5, the last digit a is 0 or 5. Since $a \neq 0$, $a = 5$.

For divisibility by 3, the digit sum $5 + b + 5 + b + 5 = 15 + 2b$ must be a multiple of 3. Since 15 is a multiple of 3, this holds exactly when $2b$ is a multiple of 3, that is, when b is a multiple of 3. So b is 0, 3, 6 or 9, and each of these differs from $a = 5$.

The numbers are 50505, 53535, 56565 and 59595, so there are 4 of them.

Answer: (B) 4. □

14. (2016 AMC 8, Problem #15) What is the largest power of 2 that is a divisor of $13^4 - 11^4$?

(A) 8 **(B)** 16 **(C)** 32 **(D)** 64 **(E)** 128

Strategy. Compute the difference, then divide by 2 until the result is odd.

Solution. Since $13^2 = 169$ and $11^2 = 121$,

$$13^4 - 11^4 = 169^2 - 121^2 = 28561 - 14641 = 13920.$$

Halving repeatedly gives

$$13920 \to 6960 \to 3480 \to 1740 \to 870 \to 435,$$

and 435 is odd. So $13920 = 2^5 \cdot 435$, and the largest power of 2 that divides it is $2^5 = 32$.

Answer: (C) 32. □

15. (2011 AMC 8, Problem #24) In how many ways can 10001 be written as the sum of two primes?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Strategy. The sum of two odd numbers is even, but 10001 is odd.

Solution. Every prime other than 2 is odd, because an even number greater than 2 has 2 as a divisor. If both primes were odd, their sum would be even. Since 10001 is odd, one of the primes must be 2, and the other must be $10001 - 2 = 9999$.

The digit sum of 9999 is 36, which is divisible by 9, so 9999 is divisible by 9 and is composite. Hence 10001 cannot be written as the sum of two primes.

Answer: (A) 0. □

16. (2016 AMC 8, Problem #24) The digits 1, 2, 3, 4, and 5 are each used once to write a five-digit number $PQRST$. The three-digit number PQR is divisible by 4, the three-digit number QRS is divisible by 5, and the three-digit number RST is divisible by 3. What is P?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. Apply the most restrictive rule first. The rule for 5 fixes S, the rule for 4 leaves a short list for Q and R, and the rule for 3 decides among them.

Solution. Since QRS is divisible by 5, its last digit S is 0 or 5. Only 5 is available, so $S = 5$.

Since PQR is divisible by 4, the two-digit number QR is divisible by 4. Its digits are two different digits from $\{1, 2, 3, 4\}$, and the multiples of 4 of this form are 12, 24 and 32.

Since RST is divisible by 3, the digit sum $R + 5 + T$ is a multiple of 3. Here P and T are the two digits not used for Q, R and S.

- If $QR = 12$, then $R = 2$ and T is 3 or 4, giving digit sums 10 or 11. Neither works.
- If $QR = 24$, then $R = 4$ and T is 1 or 3, giving digit sums 10 or 12. Only $T = 3$ works, and then $P = 1$.
- If $QR = 32$, then $R = 2$ and T is 1 or 4, giving digit sums 8 or 11. Neither works.

So the number is 12453, and $P = 1$.

Answer: (A) 1. □

2. Answer Key

Problem	Letter	Value
1	C	23
2	D	30
3	C	Friday
4	A	0
5	C	3
6	A	1
7	C	84
8	C	4
9	A	6
10	E	7
11	D	8
12	A	1
13	B	4
14	C	32
15	A	0
16	A	1