INTRODUCTION TO NUMBER THEORY FOR AMC 8: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (1987 AJHSME, Problem #9) When finding the sum

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7},$$

what is the least common denominator used?

(A) 120 **(B)** 210 **(C)** 420 **(D)** 840 **(E)** 5040

Strategy. The common denominators of the six fractions are the common multiples of 2, 3, 4, 5, 6 and 7, so the least one is their least common multiple. Find it from the prime factorizations.

Solution. A fraction $\frac{1}{k}$ can be written with denominator D exactly when k divides D. So the least common denominator is $\text{lcm}(2,3,4,5,6,7)$. The prime factorizations of the denominators are

$$2=2, \qquad 3=3, \qquad 4=2^2, \qquad 5=5, \qquad 6=2\cdot 3, \qquad 7=7.$$

The primes that appear are 2, 3, 5 and 7, and their highest powers are 2^2 (from 4), 3, 5 and 7. Taking each prime to its highest power gives

$$\text{lcm}(2,3,4,5,6,7)=2^2\cdot 3\cdot 5\cdot 7=420.$$

The product of the denominators, 5040, is also a common denominator, but it is not the least one.

Answer: (C) 420. □

2. (1990 AJHSME, Problem #4) Which of the following could **not** be the units digit [one's digit] of the square of a whole number?

(A) 1 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

Strategy. A whole number is congruent modulo 10 to its last digit, so its square is congruent to the square of that digit. Square the digits 0 through 9 and collect the units digits of the results.

Solution. The units digit of a whole number is its remainder modulo 10. If n has units digit d, then $n\equiv d \pmod{10}$, so $n^2\equiv d^2 \pmod{10}$, and n^2 has the same units digit as d^2. The squares of the digits 0 to 9 are

$$0,\quad 1,\quad 4,\quad 9,\quad 16,\quad 25,\quad 36,\quad 49,\quad 64,\quad 81,$$

with units digits 0, 1, 4, 9, 6, 5, 6, 9, 4 and 1. So the units digit of a square is always one of 0, 1, 4, 5, 6 and 9. The digits 1, 4, 5 and 6 do occur, for example in the squares 1, 4, 25 and 16, but 8 never does.

Answer: (E) 8. □

3. (1994 AJHSME, Problem #10) For how many positive integer values of N ($N>0$) is the expression

$$\frac{36}{N+2}$$

an integer?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 12

Strategy. The fraction is a whole number precisely when $N+2$ divides 36. Count the divisors of 36 with the formula for the number of divisors, then remove those that are too small to equal $N+2$.

Solution. Since N is a positive integer, $\frac{36}{N+2}$ is an integer exactly when $N+2$ is a positive divisor of 36. The prime factorization $36 = 2^2 \cdot 3^2$ gives

$$(2+1)(2+1) = 9$$

positive divisors: 1, 2, 3, 4, 6, 9, 12, 18 and 36. Since $N \geq 1$, the number $N+2$ is at least 3, so it can be any of these divisors except 1 and 2. That leaves 7 divisors, which give $N = 1$, 2, 4, 7, 10, 16 and 34.

Answer: (A) 7. □

4. (2019 AMC 10B, Problem #2) Consider the statement, "If n is not prime, then $n-2$ is prime." Which of the following values of n is a counterexample to this statement?

(A) 11 **(B)** 15 **(C)** 19 **(D)** 21 **(E)** 27

Strategy. A counterexample makes the "if" part true and the "then" part false. So a counterexample is a value of n for which neither n nor $n-2$ is prime.

Solution. The numbers 11 and 19 are prime, so they cannot be counterexamples. For $n = 15$, the number $n-2 = 13$ is prime, and for $n = 21$, the number $n-2 = 19$ is prime. For $n = 27 = 3^3$, which is not prime, the number $n-2 = 25 = 5^2$ is not prime either. So 27 is a counterexample.

Answer: (E) 27. □

5. (2014 AMC 8, Problem #8) Eleven members of the Middle School Math Club each paid the same amount for a guest speaker to talk about problem solving at their math club meeting. They paid their guest speaker $\$\underline{1A2}$. What is the missing digit A of this 3-digit number?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Strategy. The eleven members paid equal shares, so the total is a multiple of 11. Apply the rule for 11 to the digits 1, A and 2.

Solution. The total $\underline{1}\,\underline{A}\,\underline{2}$ is 11 times the amount each member paid, so it is a multiple of 11. By the rule for 11, the alternating sum of its digits from the right,

$$2 - A + 1 = 3 - A,$$

is divisible by 11. Since A is a digit from 0 to 9, the number $3-A$ lies between -6 and 3, and the only multiple of 11 in this range is 0. So $3 - A = 0$, that is, $A = 3$. Indeed, $132 = 11 \cdot 12$, so each member paid \$12.

Answer: (D) 3. □

Intermediate.

6. (2020 AMC 10B, Problem #7) How many positive even multiples of 3 less than 2020 are perfect squares?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 12

Strategy. An even multiple of 3 is a multiple of 6. If 6 divides k^2, then 2 and 3 divide k, so the squares in question are the squares of multiples of 6.

Solution. Let the number be k^2 with k a positive integer. The primes 2 and 3 divide k^2, and the prime factors of k^2 are the same as the prime factors of k, so 2 and 3 divide k. Hence $k = 6m$ and $k^2 = 36m^2$ for a positive integer m. Conversely, every number $36m^2$ is an even multiple of 3 and a perfect square. The condition is

$$36m^2 < 2020 \iff m^2 < \frac{2020}{36} = 56\tfrac{1}{9}.$$

Since $7^2 = 49 < 56\frac{1}{9} < 64 = 8^2$, the possible values are $m = 1, 2, \ldots, 7$. There are 7 such squares, from 36 to $36 \cdot 49 = 1764$.

Answer: (A) 7. □

7. (2013 AMC 8, Problem #13) When Clara totaled her scores, she inadvertently reversed the units digit and the tens digit of one score. By which of the following might her incorrect sum have differed from the correct one?

(A) 45 **(B)** 46 **(C)** 47 **(D)** 48 **(E)** 49

Strategy. Swapping two digits does not change the digit sum. A number and its digit sum leave the same remainder upon division by 9, so the true score and the miscopied score differ by a multiple of 9.

Solution. Only one score changed, so the incorrect sum differs from the correct one by the difference between that score and the number obtained by reversing its last two digits. These two numbers are made of the same digits, so they have the same digit sum. A number and its digit sum leave the same remainder when divided by 9, so the two numbers leave the same remainder when divided by 9; that is, they are congruent modulo 9. Hence 9 divides their difference, and the two sums differ by a multiple of 9.

Of the choices, only $45 = 9 \cdot 5$ is a multiple of 9. This difference can occur: if a score of 72 is written as 27, the sum drops by $72 - 27 = 45$.

Answer: (A) 45. □

8. (2000 AMC 8, Problem #11) The number 64 has the property that it is divisible by its unit digit. How many whole numbers between 10 and 50 have this property?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 20

Strategy. Sort the numbers from 11 to 49 by their units digit d, and test the four numbers ending in each d for divisibility by d, with the rules for 2, 3, 5, 6 and 9, and directly for 4, 7 and 8.

Solution. Divisibility is defined only for a nonzero divisor, so a number with units digit 0 does not have the property. This rules out 20, 30 and 40. For each other units digit d, four of the numbers end in d, and each of them is tested for divisibility by d.

Every integer is divisible by 1. By the rules for 2 and 5, every number ending in 2 is even and every number ending in 5 is a multiple of 5. The numbers ending in 3 have digit sums 4, 5, 6 and 7, so by the rule for 3 only 33 works. Of 14, 24, 34 and 44, only $24 = 4 \cdot 6$ and $44 = 4 \cdot 11$ are multiples of 4. The numbers ending in 6 are all even, and their digit sums are 7, 8, 9 and 10, so by the rule for 6 only 36 works. The multiples of 7 between 10 and 50 are 14, 21, 28, 35, 42 and 49, and none of them ends in 7. The multiples of 8 in this range are 16, 24, 32, 40 and 48, and only 48 ends in 8. The numbers ending in 9 have digit sums 10, 11, 12 and 13, so by the rule for 9 none of them works. The table collects the results.

Units digit d	Numbers divisible by d	Count
1	11, 21, 31, 41	4
2	12, 22, 32, 42	4
3	33	1
4	24, 44	2
5	15, 25, 35, 45	4
6	36	1
7	none	0
8	48	1
9	none	0

The total is $4+4+1+2+4+1+0+1+0=17$.

Answer: (C) 17. □

9. (2022 AMC 8, Problem #17) If n is an even positive integer, the double factorial notation $n!!$ represents the product of all the even integers from 2 to n. For example:

$$8!! = 2 \times 4 \times 6 \times 8$$

What is the units digit of the following sum?

$$2!! + 4!! + 6!! + \cdots + 2018!! + 2020!! + 2022!!$$

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Strategy. From 10!! on, every term contains the factor 10, so its units digit is 0. Only the units digits of the first four terms matter.

Solution. For an even $n \geq 10$, the product $n!! = 2 \times 4 \times 6 \times 8 \times 10 \times \cdots \times n$ contains the factor 10, so it is a multiple of 10 and its units digit is 0. The units digit of a sum depends only on the units digits of the terms, so these terms do not change the units digit of the sum, and only the first four terms count:

$$2!! = 2, \qquad 4!! = 8, \qquad 6!! = 48, \qquad 8!! = 384.$$

Their units digits add up to $2+8+8+4=22$, so the sum ends in 2.

Answer: (B) 2. □

10. (2012 AMC 8, Problem #13) Jamar bought some pencils costing more than a penny each at the school bookstore and paid \$1.43. Sharona bought some of the same pencils and paid \$1.87. How many more pencils did Sharona buy than Jamar?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Strategy. In cents, the price of one pencil divides both 143 and 187. Compare the prime factorizations of the two amounts to find their common divisors.

Solution. Let one pencil cost p cents, where $p > 1$. Jamar paid 143 cents for a whole number of pencils, so p divides 143. In the same way, p divides 187. The prime factorizations are

$$143 = 11 \cdot 13, \qquad 187 = 11 \cdot 17.$$

The only prime in both factorizations is 11, with lowest power 11^1, so $\gcd(143, 187) = 11$. A common divisor of the two numbers contains only primes that appear in both factorizations, each to at most the lowest power with which it appears, so it divides the GCD. Hence the

only common divisors are 1 and 11, and since $p > 1$, the price is $p = 11$ cents. Jamar bought $143/11 = 13$ pencils and Sharona bought $187/11 = 17$, which is $17 - 13 = 4$ more.

Answer: (C) 4. □

Advanced.

11. (2001 AMC 10, Problem #12) Suppose that n is the product of three consecutive integers and that n is divisible by 7. Which of the following is not necessarily a divisor of n?

(A) 6 **(B)** 14 **(C)** 21 **(D)** 28 **(E)** 42

Strategy. Three consecutive integers include a multiple of 2 and a multiple of 3. Combine factors that have no common divisor greater than 1 to find a number that always divides n, and test the remaining choice on a small example.

Solution. Of any two consecutive integers, one is even, so one of the three integers is a multiple of 2. Upon division by 3, consecutive integers leave the remainders 0, 1, 2, 0, 1, 2, and so on in turn, so one of the three integers is a multiple of 3. Hence n is divisible by 2 and by 3, and since 2 and 3 have no common divisor greater than 1, it is divisible by 6. It is also divisible by 7, and 6 and 7 have no common divisor greater than 1, so n is divisible by $6 \cdot 7 = 42$. Then n is also divisible by 14 and 21, which divide 42.

The remaining choice, 28, does not always divide n. For example, $n = 5 \cdot 6 \cdot 7 = 210$ is a product of three consecutive integers and is divisible by 7. By the rule for 4, the number 210 is not divisible by 4, because 10 is not. Since $28 = 4 \cdot 7$ is a multiple of 4, the number 210 is not divisible by 28 either.

Answer: (D) 28. □

12. (2006 AMC 8, Problem #23) A box contains gold coins. If the coins are equally divided among six people, four coins are left over. If the coins are equally divided among five people, three coins are left over. If the box holds the smallest number of coins that meets these two conditions, how many coins are left when equally divided among seven people?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 5

Strategy. Both remainders are 2 less than the number of people, so two more coins would split evenly both ways. Use the least common multiple of 6 and 5.

Solution. Let N be the number of coins. Dividing among six people leaves 4 coins, so $N = 6q + 4$ for a whole number q, and $N + 2 = 6(q + 1)$ is a multiple of 6. Dividing among five people leaves 3 coins, so $N = 5r + 3$ for a whole number r, and $N + 2 = 5(r + 1)$ is a multiple of 5. So $N + 2$ is a common multiple of 6 and 5, and therefore a multiple of

$$\operatorname{lcm}(6, 5) = 30.$$

Conversely, if $N + 2 = 30k$ for a positive integer k, then $N = 6(5k - 1) + 4 = 5(6k - 1) + 3$, so N meets both conditions. The smallest such number is $N = 30 - 2 = 28$. Since $28 = 7 \cdot 4$, dividing 28 coins equally among seven people leaves 0 coins.

Answer: (A) 0. □

13. (2019 AMC 8, Problem #23) After Euclid High School's last basketball game, it was determined that $\frac{1}{4}$ of the team's points were scored by Alexa and $\frac{2}{7}$ were scored by Brittany. Chelsea scored 15 points. None of the other 7 team members scored more than 2 points. What was the total number of points scored by the other 7 team members?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Strategy. Alexa's and Brittany's points are whole numbers, so the team's total is a common multiple of 4 and 7. Test the multiples of $\text{lcm}(4,7) = 28$ against the limit on the other seven players.

Solution. Let the team score T points. Alexa scored $\frac{T}{4}$ points, a whole number, so 4 divides T. Brittany scored $\frac{2T}{7}$ points, so 7 divides $2T$. Then 7 divides both $4 \cdot 2T = 8T$ and $7T$, so it divides their difference, T. Hence T is a common multiple of 4 and 7, and therefore a multiple of $\text{lcm}(4,7) = 28$.

Write $T = 28k$ for a positive integer k. Then Alexa scored $7k$ points, Brittany scored $8k$ and Chelsea scored 15, so the other seven players scored

$$28k - 7k - 8k - 15 = 13k - 15$$

points in all. None of them scored more than 2 points, so this total is between 0 and $7 \cdot 2 = 14$. For $k = 1$ it is -2, which is negative. For $k = 2$ it is 11. For $k \geq 3$ it is at least $13 \cdot 3 - 15 = 24$, which is too large. So $k = 2$, the team scored 56 points, and the other seven players scored 11 points, for example 2 points each for five of them, 1 point for one and 0 for the last.

Answer: (B) 11. □

14. (2015 AMC 8, Problem #22) On June 1, a group of students is standing in rows, with 15 students in each row. On June 2, the same group is standing with all of the students in one long row. On June 3, the same group is standing with just one student in each row. On June 4, the same group is standing with 6 students in each row. This process continues through June 12 with a different number of students per row each day. However, on June 13, they cannot find a new way of organizing the students. What is the smallest possible number of students in the group?

(A) 21 **(B)** 30 **(C)** 60 **(D)** 90 **(E)** 1080

Strategy. The students can stand in rows of k exactly when k divides the number of students, so that number has exactly 12 positive divisors. It is also a common multiple of 15 and 6.

Solution. Let N be the number of students. They can stand in equal rows of k students exactly when k is a divisor of N, so each day's arrangement uses a different positive divisor of N. One long row uses the divisor N, and one student in each row uses the divisor 1. The days from June 1 to June 12 use 12 different divisors, and on June 13 no divisor is left, so N has exactly 12 positive divisors.

Rows of 15 and rows of 6 both work, so N is a common multiple of $15 = 3 \cdot 5$ and $6 = 2 \cdot 3$, and therefore a multiple of $\text{lcm}(15,6) = 2 \cdot 3 \cdot 5 = 30$. By the formula for the number of divisors, the number $30 = 2 \cdot 3 \cdot 5$ itself has only $(1+1)(1+1)(1+1) = 8$ positive divisors. The next multiple, $60 = 2^2 \cdot 3 \cdot 5$, has

$$(2+1)(1+1)(1+1) = 12$$

positive divisors: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60. So the smallest possible number of students is 60.

Answer: (C) 60. □

15. (2011 AMC 8, Problem #22) What is the tens digit of 7^{2011}?

(A) 0 **(B)** 1 **(C)** 3 **(D)** 4 **(E)** 7

Strategy. The last two digits of a number give its remainder upon division by 100. Find a power of 7 that is congruent to 1 modulo 100, and split the exponent 2011 accordingly.

Solution. A positive integer equals 100 times the number formed by its other digits plus the number formed by its last two digits. So it is congruent modulo 100 to the number formed by its last two digits, and the tens digit of 7^{2011} is the tens digit of its remainder upon division by 100.

The first four powers of 7 are 7, 49, 343 and 2401. Thus $7^4 = 2401 \equiv 1 \pmod{100}$, and since $2011 = 4 \cdot 502 + 3$,

$$7^{2011} = \left(7^4\right)^{502} \cdot 7^3 \equiv 1^{502} \cdot 343 = 343 \equiv 43 \pmod{100}.$$

The last two digits of 7^{2011} are 43, so its tens digit is 4.

Answer: (D) 4. □

2. Answer Key

Problem	Letter	Value
1	C	420
2	E	8
3	A	7
4	E	27
5	D	3
6	A	7
7	A	45
8	C	17
9	B	2
10	C	4
11	D	28
12	A	0
13	B	11
14	C	60
15	D	4