PERCENTS AND PERCENT CHANGE FOR AMC 8

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1. Percents

This section converts between percents, fractions and decimals, finds a percent of a quantity, and finds a whole from one of its parts.

1.1. Percents, Fractions, and Decimals.

The word percent means "per hundred." A percent compares a part with a whole, as a part-to-whole ratio does, but with the whole scaled to 100: a part that is 35 percent of a whole is 35 for every 100 of the whole.

Definition 1.1.1. For a number p, "p **percent**," written $p\%$, means $\frac{p}{100}$. A part is $p\%$ of a whole when the ratio of the part to the whole is $p : 100$.

So 100% of a quantity is the whole quantity, and 50% of it is half. A percent greater than 100 gives more than the whole: 250% of a quantity is 2.5 times the quantity. A percent less than 1 gives less than one hundredth of the whole: 0.4% means $\frac{0.4}{100} = 0.004$.

Percents, decimals and fractions are converted as follows.

- *Percent to decimal:* divide by 100, which moves the decimal point two places to the left. For example, $35\% = 0.35$, $6\% = 0.06$, $0.4\% = 0.004$ and $150\% = 1.5$.
- *Decimal to percent:* multiply by 100. For example, $0.08 = 8\%$ and $1.25 = 125\%$.
- *Percent to fraction:* write the percent over 100 and reduce. For example, $35\% = \frac{35}{100} = \frac{7}{20}$.
- *Fraction to percent:* rewrite the fraction with denominator 100, or divide to get a decimal. For example, $\frac{3}{20} = \frac{15}{100} = 15\%$ and $\frac{3}{8} = 0.375 = 37.5\%$. The decimal $0.8333\ldots$ for $\frac{5}{6}$ does not end, so $\frac{5}{6}$ is about 83.3%.

The table lists the equivalents that are used most often.

Fraction	Decimal	Percent
1/2	0.5	50%
1/3	$0.333\ldots$	$33\frac{1}{3}\%$
2/3	$0.666\ldots$	$66\frac{2}{3}\%$
1/4	0.25	25%
3/4	0.75	75%
1/5	0.2	20%
1/8	0.125	12.5%
1/10	0.1	10%
1/20	0.05	5%

Example.

Problem 1.1.2 (1999 AMC 8, Problem #12)**.** The ratio of the number of games won to the number of games lost (no ties) by the Middle School Middies is $\frac{11}{4}$. To the nearest whole percent, what percent of its games did the team lose?

(A) 24% **(B)** 27% **(C)** 36% **(D)** 45% **(E)** 73%

Strategy. The ratio 11 : 4 compares the games won with the games lost. Find the part-to-whole ratio of the games lost to all the games, then write it as a percent.

Solution. The games won and the games lost are in the ratio $11 : 4$, so of every $11+4 = 15$ games, the team lost 4. It lost $\frac{4}{15}$ of its games, and

$$\frac{4}{15} = 0.2666\ldots = 26.66\ldots\%.$$

To the nearest whole percent, the team lost 27% of its games.

Answer: (B) 27%. □

1.2. **Percent of a Quantity.**

Proposition 1.2.1 (Percent of a Quantity)**.** For any positive quantity N, the amount $p\%$ of N equals $\frac{p}{100} \cdot N$. A number A is $\left(\frac{A}{B} \cdot 100\right)\%$ of a positive number B.

Proof. If a part is $p\%$ of N, then $\frac{\text{part}}{N} = \frac{p}{100}$ by the definition, so the part is $\frac{p}{100} \cdot N$. If A is $q\%$ of B, then $A/B = q/100$ in the same way, so $q = (A/B) \cdot 100$. □

By the proposition, a percent of a quantity is a product: 15% of 80 is $0.15 \cdot 80 = 12$, and 250% of 12 is $2.5 \cdot 12 = 30$. To find what percent 12 is of 48, divide: $\frac{12}{48} = 0.25 = 25\%$.

When a percent is taken of a part that is itself a percent of a whole, multiply the two percents. For example, if 40% of the students in a class are boys and 50% of the boys play chess, then the boys who play chess make up $0.5 \cdot 0.4 = 0.2$, or 20%, of the class.

Examples.

Problem 1.2.2 (1995 AJHSME, Problem #14)**.** A team won 40 of its first 50 games. How many of the remaining 40 games must this team win so it will have won exactly 70% of its games for the season?

(A) 20 **(B)** 23 **(C)** 28 **(D)** 30 **(E)** 35

Strategy. The 70% is a percent of all the games of the season, so the games the team does not win are the other 30% of the season. Count those games first.

Solution. Altogether there are $50 + 40 = 90$ games. Winning exactly 70% of them means that exactly 30% of them, $0.3 \cdot 90 = 27$ games, are not won. The team has already failed to win $50 - 40 = 10$ games, so exactly $27 - 10 = 17$ of the remaining 40 games are not won, and the team wins the other $40 - 17 = 23$. As a check, its wins then total $40 + 23 = 63$, and $\frac{63}{90} = 0.7 = 70\%$.

Answer: (B) 23. □

Problem 1.2.3 (1992 AHSME, Problem #2)**.** An urn is filled with coins and beads, all of which are either silver or gold. Twenty percent of the objects in the urn are beads. Forty percent of the coins in the urn are silver. What percent of the objects in the urn are gold coins?

(A) 40% **(B)** 48% **(C)** 52% **(D)** 60% **(E)** 80%

Strategy. The split into silver and gold is given as a percent of the coins, not of all the objects. Find the percent of the coins that are gold, then take that percent of the percent of the objects that are coins.

Solution. Beads make up 20% of the objects, so coins make up $100\% - 20\% = 80\%$ of them. Silver coins are 40% of the coins, so gold coins are 60% of the coins. Gold coins are therefore 60% of 80% of the objects, and

$$0.6 \cdot 0.8 = 0.48 = 48\%.$$

As a check, an urn with 100 objects holds 20 beads and 80 coins, and of the coins 32 are silver and 48 are gold.

Answer: (B) 48%. □

1.3. **Finding the Whole.** Sometimes a part and its percent are known, and the whole is not.

Proposition 1.3.1 (Finding the Whole)**.** If P is $p\%$ of N, where $p \neq 0$, then

$$N = \frac{P}{p/100} = \frac{100P}{p}.$$

Proof. By the previous proposition, $P = \frac{p}{100} \cdot N$. Multiplying both sides by $\frac{100}{p}$ gives $N = \frac{100P}{p}$. □

The same computation can be organized in two other ways. First, 1% of N is $\frac{P}{p}$, and N is 100 times that. For example, if 36 is 45% of a number, then 1% of the number is $\frac{36}{45} = 0.8$, and the number is $100 \cdot 0.8 = 80$. Second, the proportion $\frac{P}{N} = \frac{p}{100}$ can be solved for N by cross-multiplication.

Remark 1.3.2 (A Part That Stays Fixed)**.** When objects of one kind are added to a collection or removed from it, the number of objects of every other kind does not change. If the new percent of a kind that stays fixed is known, the proposition gives the new total.

Examples.

Problem 1.3.3 (2005 AMC 10A, Problem #1)**.** While eating out, Mike and Joe each tipped their server \$2. Mike tipped 10% of his bill and Joe tipped 20% of his bill. What was the difference, in dollars, between their bills?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 10 **(E)** 20

Strategy. Each tip is a known percent of an unknown bill. Find each bill as the whole from its part.

Solution. Both tips are $P = 2$ dollars. By the proposition on finding the whole, a bill of which \$2 is $p\%$ comes to $\frac{100 \cdot 2}{p} = \frac{200}{p}$ dollars. With $p = 10$, Mike's bill was $\frac{200}{10} = \$20$, and with $p = 20$, Joe's bill was $\frac{200}{20} = \$10$. The bills differ by $20 - 10 = 10$ dollars.

Answer: (D) 10. □

Problem 1.3.4 (2010 AMC 8, Problem #12)**.** Of the 500 balls in a large bag, 80% are red and the rest are blue. How many of the red balls must be removed so that 75% of the remaining balls are red?

(A) 25 **(B)** 50 **(C)** 75 **(D)** 100 **(E)** 150

Strategy. Only red balls leave the bag, so by the remark on a part that stays fixed, the blue balls determine the new total: their number is known, and so is their new percent.

Solution. Since 80% of the balls are red, the other 20% are blue, and 20% of 500 is $0.2 \cdot 500 = 100$. All 100 blue balls are still in the bag at the end. Once red balls make up 75% of the bag, blue balls make up the other $100\% - 75\% = 25\%$, so by the proposition on finding the whole, with $P = 100$ and $p = 25$, the bag then holds

$$N = \frac{100 \cdot 100}{25} = 400$$

balls. So $500 - 400 = 100$ red balls are removed. As a check, $400 - 100 = 300$ red balls remain, and $\frac{300}{400} = 75\%$.

Answer: (D) 100. □

2. Percent Change

A percent change measures a change as a percent of the starting amount. This section computes percent changes, writes them as multipliers, and combines and reverses them.

2.1. Percent Increase and Decrease.

Definition 2.1.1. Suppose a quantity changes from an original value O to a new value N, where $O > 0$. If $N > O$, the **percent increase** is

$$\frac{N-O}{O} \cdot 100\%,$$

and if $N < O$, the **percent decrease** is

$$\frac{O-N}{O} \cdot 100\%.$$

Either one is a **percent change**. In both cases the change is divided by the original value.

A statement such as "A is $p\%$ more than B" means that $A - B$ is $p\%$ of B, and "A is $p\%$ less than B" means that $B - A$ is $p\%$ of B. The quantity that the percent is taken of is the **base**. In both statements the base is B.

The same change is a smaller percent of a larger base. A change from 40 to 50 is an increase of $\frac{10}{40} = 25\%$, while a change from 50 to 40 is a decrease of $\frac{10}{50} = 20\%$. So 50 is 25% more than 40, while 40 is 20% less than 50.

Example.

Problem 2.1.2 (1997 AJHSME, Problem #18)**.** At the grocery store last week, small boxes of facial tissue were priced at 4 boxes for \$5. This week they are on sale at 5 boxes for \$4. The percent decrease in the price per box during the sale was closest to

(A) 30% **(B)** 35% **(C)** 40% **(D)** 45% **(E)** 65%

Strategy. Compare the unit prices. The percent decrease is the drop in the price per box divided by last week's price per box.

Solution. Last week a box cost $\frac{5}{4} = \$1.25$, and this week it costs $\frac{4}{5} = \$0.80$. The price per box dropped by $1.25 - 0.80 = \$0.45$, so the percent decrease is

$$\frac{0.45}{1.25} \cdot 100\% = 36\%,$$

which is closest to 35%. Dividing the drop by the new price instead gives $\frac{0.45}{0.80} \approx 56\%$, which measures the change against the wrong base.

Answer: (B) 35%. □

2.2. Multipliers.

Theorem 2.2.1 (Multipliers)**.** *Increasing a quantity by $p\%$ multiplies it by $1 + \frac{p}{100}$. Decreasing a quantity by $p\%$ multiplies it by $1 - \frac{p}{100}$.*

Proof. For a quantity Q, an increase of $p\%$ adds $\frac{p}{100}Q$, which gives $Q + \frac{p}{100}Q = \left(1 + \frac{p}{100}\right)Q$. A decrease of $p\%$ subtracts $\frac{p}{100}Q$, which gives $Q - \frac{p}{100}Q = \left(1 - \frac{p}{100}\right)Q$. □

The number $1 + \frac{p}{100}$ or $1 - \frac{p}{100}$ is the **multiplier** of the change. It is the scale factor from the old amount to the new one. An increase of 8% has multiplier 1.08, a decrease of 35% has multiplier 0.65, and an increase of 100% has multiplier 2.

Conversely, multiplying by a number $m > 1$ is an increase of $(m-1) \cdot 100\%$, and multiplying by a number m with $0 < m < 1$ is a decrease of $(1-m) \cdot 100\%$. For example, multiplying by 2.5 is an increase of 150%, and multiplying by 0.9 is a decrease of 10%.

In particular, "A is $p\%$ more than B" means $A = \left(1 + \frac{p}{100}\right)B$, and "$A$ is $p\%$ less than B" means $A = \left(1 - \frac{p}{100}\right)B$. To compare two quantities that are each described relative to a third, write both with multipliers and divide.

Example.

Problem 2.2.2 (1997 AHSME, Problem #4)**.** If a is 50% larger than c, and b is 25% larger than c, then a is what percent larger than b?

(A) 20% **(B)** 25% **(C)** 50% **(D)** 100% **(E)** 200%

Strategy. The question asks for the percent change from b to a, whose multiplier is $\frac{a}{b}$. Both a and b are known multiples of c, so c cancels in this quotient.

Solution. By the theorem on multipliers, $b = 1.25c$ and $a = 1.5c$. The multiplier from b to a is therefore

$$\frac{a}{b} = \frac{1.5c}{1.25c} = \frac{1.5}{1.25} = 1.2,$$

which is an increase of $(1.2 - 1) \cdot 100\% = 20\%$. Subtracting the percents, $50\% - 25\% = 25\%$, measures the gap against the wrong base: $a - b = 0.25c$ is 25% of c, but only $\frac{0.25c}{1.25c} = 20\%$ of b.

Answer: (A) 20%. □

2.3. **Successive Percent Changes.**

Theorem 2.3.1 (Successive Percent Changes)**.** *If a quantity undergoes a percent change and the result then undergoes a second percent change, the quantity is multiplied by the product of the two multipliers. The same holds for any number of changes in a row, and the order of the changes does not matter.*

Proof. Suppose the first change has multiplier m_1 and the second has multiplier m_2. The first change turns Q into m_1Q, and the second turns m_1Q into m_2m_1Q. Each further change multiplies by its own multiplier in the same way. A product of numbers does not depend on the order of its factors. □

Percent changes in a row do not add, because each one is a percent of a different amount. Two increases of 10% multiply a quantity by $1.1 \cdot 1.1 = 1.21$, which is an increase of 21%, not 20%. An increase of 50% followed by a decrease of 50% multiplies it by $1.5 \cdot 0.5 = 0.75$, which is a decrease of 25%.

Corollary 2.3.2. *For $0 < p < 100$, an increase of $p\%$ and a decrease of $p\%$, in either order, multiply a quantity by $1 - \left(\frac{p}{100}\right)^2$. Together they are a decrease of $p^2/100$ percent.*

Proof. Write $x = \frac{p}{100}$. The two multipliers are $1 + x$ and $1 - x$, and

$$(1 + x)(1 - x) = 1 - x + x - x^2 = 1 - x^2.$$

Multiplying by $1 - x^2$ is a decrease of $100x^2 = p^2/100$ percent. □

With $p = 50$, the corollary gives the decrease of 25% found above.

Examples.

Problem 2.3.3 (2003 AMC 8, Problem #11)**.** Business is a little slow at Lou's Fine Shoes, so Lou decides to have a sale. On Friday, Lou increases all of Thursday's prices by 10%. Over the weekend, Lou advertises the sale: "Ten percent off the listed price. Sale starts Monday." How much does a pair of shoes cost on Monday that cost \$40 on Thursday?

(A) \$36 **(B)** \$39.60 **(C)** \$40 **(D)** \$40.40 **(E)** \$44

Strategy. The sale takes ten percent off the listed price, which on Monday is Friday's raised price. So the increase and the sale are successive percent changes, with multipliers 1.1 and 0.9.

Solution. Friday's price is $40 \cdot 1.10 = \$44$, and Monday's price is $44 \cdot 0.90 = \$39.60$. In one step,

$$40 \cdot 1.1 \cdot 0.9 = 40 \cdot 0.99 = 39.60.$$

By the corollary with $p = 10$, the two changes together are a decrease of $10^2/100 = 1$ percent.
Answer: (B) \$39.60. □

Problem 2.3.4 (1985 AJHSME, Problem #21)**.** Mr. Green receives a 10% raise every year. His salary after four such raises has gone up by what percent?
(A) less than 40% **(B)** 40% **(C)** 44% **(D)** 45% **(E)** more than 45%

Strategy. The four raises are successive percent changes. Find the multiplier of all four together, then convert it back to a percent increase.

Solution. A raise of 10% has multiplier 1.1, so by the theorem on successive percent changes the multiplier of all four raises is $1.1 \cdot 1.1 \cdot 1.1 \cdot 1.1 = 1.1^4$. Squaring twice,

$$1.1^2 = 1.21 \qquad \text{and} \qquad 1.1^4 = 1.21^2 = 1.4641.$$

A multiplier of 1.4641 is an increase of $(1.4641 - 1) \cdot 100\% = 46.41\%$, and $46.41\% > 45\%$. The total is more than $4 \cdot 10\% = 40\%$ because each raise is 10% of a larger salary than the one before.
Answer: (E) more than 45%. □

2.4. **Reversing a Percent Change.** Sometimes the new amount is known and the original amount is not. If a change with multiplier m turns the original amount O into the new amount N, then $N = mO$, so $O = \frac{N}{m}$: divide by the multiplier.

For example, a coat costs \$36 after a 20% discount. The discount multiplies the original price by 0.8, so the original price was $\frac{36}{0.8} = \$45$. Increasing \$36 by 20% gives \$43.20, which is wrong, because the 20% was a percent of the original price, not of \$36.

Proposition 2.4.1 (Reversing a Percent Change)**.** For a multiplier $m > 0$, the new amount N is the original amount O multiplied by m exactly when $O = \frac{N}{m}$. So a change with multiplier m is undone by the change with multiplier $\frac{1}{m}$. After several changes, the original amount is the new amount divided by the product of all the multipliers.

Proof. Divide both sides of $N = mO$ by m. For several changes, the product of the multipliers is the multiplier of the combined change, by the theorem on successive percent changes. □

For example, a decrease of 10% has multiplier 0.9. It is undone by multiplying by $\frac{1}{0.9} = \frac{10}{9}$. Since $\frac{10}{9} = 1 + \frac{1}{9}$, this is an increase of $\frac{1}{9} \cdot 100\% = 11\frac{1}{9}\%$, not 10%.

Examples.

Problem 2.4.2 (1988 AJHSME, Problem #22)**.** Tom's Hat Shoppe increased all original prices by 25%. Now the shoppe is having a sale where all prices are 20% off these increased prices. Which statement best describes the sale price of an item?
(A) The sale price is 5% higher than the original price.
(B) The sale price is higher than the original price, but by less than 5%.
(C) The sale price is higher than the original price, but by more than 5%.
(D) The sale price is lower than the original price.
(E) The sale price is the same as the original price.

Strategy. The multiplier $\frac{1}{1.25}$ undoes the increase. Compare that change with the sale.

Solution. The increase has multiplier $1.25 = \frac{5}{4}$. By the proposition, it is undone by the multiplier $\frac{4}{5} = 0.8$, a decrease of 20%. The sale takes exactly 20% off the increased prices, so it brings every price back to the original price. As a check, \$100 becomes \$125 and then \$100.

Answer: (E) The sale price is the same as the original price. □

Problem 2.4.3 (2000 AMC 10, Problem #3)**.** Each day, Jenny ate 20% of the jellybeans that were in her jar at the beginning of that day. At the end of the second day, 32 remained. How many jellybeans were in the jar originally?

(A) 40 **(B)** 50 **(C)** 55 **(D)** 60 **(E)** 75

Strategy. Each day removes 20% of the number in the jar that morning, so the two days are successive percent changes. Combine them into one multiplier, then reverse that change.

Solution. A decrease of 20% has multiplier $1 - 0.20 = 0.8$. After two days, by the theorem on successive percent changes, the jar holds $0.8 \cdot 0.8 = 0.64$ times the original number, so by the proposition on reversing a percent change the original number is

$$\frac{32}{0.64} = 50.$$

As a check, 50 jellybeans become 40 after the first day and 32 after the second. Adding 20% back each day (32, then 38.4, then 46.08) is wrong, because each day's 20% was a percent of the larger number at the start of that day.

Answer: (B) 50. □

2.5. **Percentage Points.**

Definition 2.5.1. When a quantity that is itself a percent, such as a tax rate or a share of the votes, rises from a% to b%, it rises by $b - a$ **percentage points**. Its percent increase is $\frac{b-a}{a} \cdot 100\%$, as for any other quantity. A fall is measured in the same way.

For example, if a town's sales tax rate rises from 5% to 6%, the rate rises by 1 percentage point, but by 20% of itself, since $\frac{6-5}{5} = \frac{1}{5} = 20\%$. The tax on a \$50 item rises from \$2.50 to \$3.00: by 1% of \$50, and by 20%.

In general, the difference between a% of a quantity N and b% of the same N, where $b > a$, is $(b - a)$% of N.

Example.

Problem 2.5.2. In the first round of a student council election, 300 students voted, and Maya received 40% of the votes. In the second round, the same 300 students voted, and Maya received 50% of the votes. By how many percentage points did Maya's share of the votes increase? By what percent did the number of votes she received increase?

Strategy. Compare the two shares directly to get percentage points. Compare the numbers of votes to get the percent increase.

Solution. Her share rose from 40% to 50%, an increase of $50 - 40 = 10$ percentage points. She received $0.4 \cdot 300 = 120$ votes in the first round and $0.5 \cdot 300 = 150$ votes in the second, which is 30 more. The percent increase is

$$\frac{30}{120} \cdot 100\% = 25\%.$$

The same 25% comes from the shares alone, since $\frac{50-40}{40} = \frac{1}{4}$.

Answer: 10 percentage points; 25%. □

3. Percents and Money

Prices change by discounts, sales tax and tips, sellers make a profit or a loss, and savings earn interest. Each of these is a percent of some base amount, so the methods of the earlier sections apply once the base is clear.

3.1. Discounts, Sales Tax, and Tips.

Definition 3.1.1. A **discount** of $p\%$, also called $p\%$ off, lowers a price by $p\%$ of the price, so it multiplies the price by $1 - \frac{p}{100}$. A **sales tax** of $t\%$ adds $t\%$ of the price, so it multiplies the price by $1 + \frac{t}{100}$. A **tip** of $p\%$ is $p\%$ of the bill.

By the theorem on successive percent changes, a discount and a sales tax give the same total in either order, and two discounts in a row multiply. A 20% discount followed by a 10% discount multiplies the price by $0.8 \cdot 0.9 = 0.72$, which is a single discount of 28%, not 30%.

Two percents of the same amount add instead. When a tax of $t\%$ and a tip of $p\%$ are both computed on the price P of a meal, the total is

$$P + \frac{t}{100}P + \frac{p}{100}P = \left(1 + \frac{t+p}{100}\right)P.$$

Examples.

Problem 3.1.2 (2005 AMC 8, Problem #11)**.** The sales tax rate in Bergville is 6%. During a sale at the Bergville Coat Closet, the price of a coat is discounted 20% from its \$90.00 price. Two clerks, Jack and Jill, calculate the bill independently. Jack rings up \$90.00 and adds 6% sales tax, then subtracts 20% from this total. Jill rings up \$90.00, subtracts 20% of the price, then adds 6% of the discounted price for sales tax. What is Jack's total minus Jill's total?

(A) $-\$1.06$ **(B)** $-\$0.53$ **(C)** \$0 **(D)** \$0.53 **(E)** \$1.06

Strategy. The tax and the discount are successive percent changes, so the theorem on successive percent changes applies to each clerk's bill.

Solution. A sales tax of 6% has multiplier 1.06, and a discount of 20% has multiplier 0.80. Jack's register goes from \$90.00 to $90.00 \cdot 1.06 = \$95.40$ and then to $95.40 \cdot 0.80 = \$76.32$. Jill's goes from \$90.00 to $90.00 \cdot 0.80 = \$72.00$ and then to $72.00 \cdot 1.06 = \$76.32$. The two totals agree, as the theorem on successive percent changes guarantees, so the difference is \$0.

Answer: (C) \$0. □

Problem 3.1.3 (2012 AMC 10B, Problem #5)**.** Anna enjoys dinner at a restaurant in Washington, D.C., where the sales tax on meals is 10%. She leaves a 15% tip on the price of her meal before the sales tax is added, and the tax is calculated on the pre-tip amount. She spends a total of \$27.50 for dinner. What is the cost of her dinner without tax or tip?

(A) \$18 **(B)** \$20 **(C)** \$21 **(D)** \$22 **(E)** \$24

Strategy. Both charges are computed on the price of the meal, so they combine into one multiplier, as in the formula for a tax and a tip above. Divide the total by that multiplier.

Solution. With $t = 10$ and $p = 15$, the formula for a tax and a tip computed on the same price P gives Anna's total as

$$\left(1 + \frac{10+15}{100}\right)P = 1.25P.$$

This total is \$27.50, so dividing by the multiplier gives $P = \frac{27.50}{1.25} = 22$ dollars. As a check, $22 + 2.20 + 3.30 = 27.50$.

Answer: (D) \$22. □

3.2. **Markups, Profit, and Loss.**

Definition 3.2.1. The **cost** of an item is the amount the seller paid for it, and the **selling price** is the amount the seller receives for it. If the selling price is greater than the cost, the difference is the **profit**; if it is smaller, the difference is the **loss**. A profit of $p\%$ based on the cost means that the profit is $p\%$ of the cost, so the selling price is $\left(1+\frac{p}{100}\right)$ times the cost. Setting the price in this way is a **markup** of $p\%$. A loss of $p\%$ based on the cost gives a selling price of $\left(1-\frac{p}{100}\right)$ times the cost.

Profit and loss percents are based on the cost unless a problem says otherwise. To find the cost from the selling price, divide by the multiplier, as in Section 2.4. For example, an item that cost \$40 and is marked up 25% sells for $1.25 \cdot 40 = \$50$, while an item sold for \$50 at a loss of 20% based on its cost had cost $\frac{50}{0.8} = \$62.50$.

Example.

Problem 3.2.2 (1952 AHSME, Problem #14)**.** A house and store were sold for \$12000 each. The house was sold at a loss of 20% of the cost, and the store at a gain of 20% of the cost. The entire transaction resulted in:

(A) No loss or gain **(B)** Loss of \$1000 **(C)** Gain of \$1000 **(D)** Gain of \$2000 **(E)** None of these

Strategy. Each percent is based on that building's own cost. Reverse each sale with the proposition on reversing a percent change, then compare the loss on one building with the profit on the other.

Solution. A loss of 20% based on the cost gives a selling price of 0.8 times the cost, so by the proposition on reversing a percent change the price paid for the house was

$$\frac{12000}{0.8} = 15000$$

dollars. The house was therefore sold at a loss of \$3000. The store was sold at a gain, that is, a profit, of 20% of its cost. This is a markup of 20%, so the store's selling price was 1.2 times its cost, and the price paid for the store was

$$\frac{12000}{1.2} = 10000$$

dollars. The store was therefore sold at a profit of \$2000. The loss of \$3000 is larger than the profit of \$2000, so the whole transaction is a loss of $3000 - 2000 = 1000$ dollars. The two changes of 20% do not cancel, because they are percents of different costs.

Answer: (B) Loss of \$1000. □

3.3. **Simple Interest.**

Definition 3.3.1. The amount of money deposited in an account, or borrowed, is the **principal** P. **Simple interest** at a rate of $r\%$ per year, also written $r\%$ per annum, adds $r\%$ of the principal for each year. After t years the interest is

$$I = P \cdot \frac{r}{100} \cdot t,$$

and the account holds $P + I$.

The time t is measured in years and can be a fraction of a year: 6 months is $\frac{1}{2}$ year, and 3 months is $\frac{1}{4}$ year. Simple interest grows by the same amount every year, because it is always computed on the original principal.

Example.

Problem 3.3.2. Mr. Lee deposits \$1200 in a savings account that pays 4% simple interest per year. How much interest does the account earn in 3 years? How much interest does it earn in 9 months?

Strategy. Find the interest for one year, then multiply by the time in years.

Solution. One year's interest is 4% of \$1200, which is $0.04 \cdot 1200 = \$48$. In 3 years the account earns $3 \cdot 48 = \$144$, and it then holds \$1344. Nine months is $\frac{9}{12} = \frac{3}{4}$ of a year, so in 9 months the account earns $\frac{3}{4} \cdot 48 = \$36$.

Answer: \$144 in 3 years; \$36 in 9 months. □

4. Problems

Beginner.

1. (1985 AJHSME, Problem #14) The difference between a 6.5% sales tax and a 6% sales tax on an item priced at \$20 before tax is

(A) \$0.01 **(B)** \$0.10 **(C)** \$0.50 **(D)** \$1 **(E)** \$10

2. (2003 AMC 8, Problem #5) If 20% of a number is 12, what is 30% of the same number?

(A) 15 **(B)** 18 **(C)** 20 **(D)** 24 **(E)** 30

3. (2001 AMC 8, Problem #10) A collector offers to buy state quarters for 2000% of their face value. At that rate how much will Bryden get for his four state quarters?

(A) \$20 **(B)** \$50 **(C)** \$200 **(D)** \$500 **(E)** \$2000

4. (2002 AMC 8, Problem #14) A merchant offers a large group of items at 30% off. Later, the merchant takes 20% off these sale prices and claims that the final price of these items is 50% off the original price. The total discount is

(A) 35% **(B)** 44% **(C)** 50% **(D)** 56% **(E)** 60%

Intermediate.

5. (2009 AMC 10B, Problem #8) In a certain year the price of gasoline rose by 20% during January, fell by 20% during February, rose by 25% during March, and fell by x% during April. The price of gasoline at the end of April was the same as it had been at the beginning of January. To the nearest integer, what is x?

(A) 12 **(B)** 17 **(C)** 20 **(D)** 25 **(E)** 35

6. (1998 AJHSME, Problem #14) At Annville Junior High School, 30% of the students in the Math Club are in the Science Club, and 80% of the students in the Science Club are in the Math Club. There are 15 students in the Science Club. How many students are in the Math Club?

(A) 12 **(B)** 15 **(C)** 30 **(D)** 36 **(E)** 40

7. (1969 AHSME, Problem #2) If an item is sold for x dollars, there is a loss of 15% based on the cost. If, however, the same item is sold for y dollars, there is a profit of 15% based on the cost. The ratio $y : x$ is:

(A) $23 : 17$ **(B)** $17y : 23$ **(C)** $23x : 17$ **(D)** dependent upon the cost
(E) none of these

8. (1997 AJHSME, Problem #16) Penni Precisely buys \$100 worth of stock in each of three companies: Alabama Almonds, Boston Beans, and California Cauliflower. After one year, AA was up 20%, BB was down 25%, and CC was unchanged. For the second year, AA was down 20% from the previous year, BB was up 25% from the previous year, and CC was unchanged. If A, B, and C are the final values of the stock, then

(A) $A = B = C$ **(B)** $A = B < C$ **(C)** $C < B = A$ **(D)** $A < B < C$
(E) $B < A < C$

Advanced.

9. (2019 AMC 8, Problem #22) A store increased the original price of a shirt by a certain percent and then decreased the new price by the same percent. Given that the resulting price was 84% of the original price, by what percent was the price increased and decreased?
(A) 16 **(B)** 20 **(C)** 28 **(D)** 36 **(E)** 40

10. (1971 AHSME, Problem #4) After simple interest for two months at 5% per annum was credited, a Boy Scout Troop had a total of $255.31 in the Council Treasury. The interest credited was a number of dollars plus the following number of cents:
(A) 11 **(B)** 12 **(C)** 13 **(D)** 21 **(E)** 31

11. (1956 AHSME, Problem #4) A man has $10,000 to invest. He invests $4,000 at 5% and $3,500 at 4%. In order to have a yearly income of $500, he must invest the remainder at:
(A) 6% **(B)** 6.1% **(C)** 6.2% **(D)** 6.3% **(E)** 6.4%

12. (2005 AMC 8, Problem #22) A company sells detergent in three different sized boxes: small (S), medium (M) and large (L). The medium size costs 50% more than the small size and contains 20% less detergent than the large size. The large size contains twice as much detergent as the small size and costs 30% more than the medium size. Rank the three sizes from best to worst buy.
(A) SML **(B)** LMS **(C)** MSL **(D)** LSM **(E)** MLS