TRIGONOMETRIC IDENTITIES AND EQUATIONS FOR AMC 12

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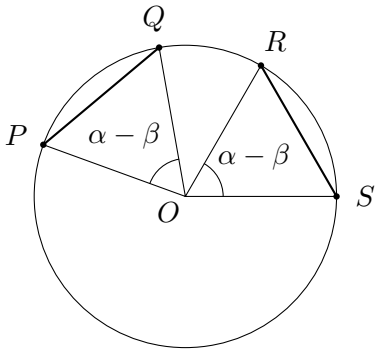
1. Sum and Difference Formulas

This section expresses the sine and cosine of a sum or a difference of two angles, and the tangent of a sum, in terms of the values at the two angles.

1.1. **Sine and Cosine.** The formulas in this handout hold for all real angles, in degrees or radians, wherever both sides are defined. The first comes from two equal chords of the unit circle.



Theorem 1.1.1 (Cosine of a Difference)**.** *For all angles α and β,*

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Proof. Let O be the origin, and on the unit circle let $P = (\cos\alpha, \sin\alpha)$, $Q = (\cos\beta, \sin\beta)$, $R = \big(\cos(\alpha-\beta), \sin(\alpha-\beta)\big)$, and $S = (1, 0)$, as in the figure. Turning the ray OQ through the angle $\alpha - \beta$ gives the ray OP, and turning the ray OS through the same angle gives the ray OR. So the chords PQ and SR have the same central angle, and they have the same length. (If $\alpha - \beta$ is a multiple of 360°, then $P = Q$ and $R = S$, and both lengths are 0.) By the distance formula and the Pythagorean identity,

$$PQ^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta),$$
$$SR^2 = \big(\cos(\alpha-\beta) - 1\big)^2 + \sin^2(\alpha-\beta) = 2 - 2\cos(\alpha-\beta).$$

Setting $PQ^2 = SR^2$ gives the formula. □

Theorem 1.1.2 (Sum and Difference Formulas)**.** *For all angles α and β,*

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Proof. The last formula is the theorem above. Replacing β by $-\beta$ and using $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$ gives the formula for $\cos(\alpha+\beta)$. For the sine, the complementary-angle identities give

$$\begin{aligned}\sin(\alpha+\beta) &= \cos\big((90^\circ - \alpha) - \beta\big) = \cos(90^\circ - \alpha)\cos\beta + \sin(90^\circ - \alpha)\sin\beta \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta,\end{aligned}$$

and replacing β by $-\beta$ gives the formula for $\sin(\alpha-\beta)$. □

With the exact values at 30°, 45°, and 60°, these formulas give exact values at angles such as $15^\circ = 45^\circ - 30^\circ$ and $75^\circ = 45^\circ + 30^\circ$. They are also used from right to left, to collapse a sum of two products into a single sine or cosine.

Examples.

Problem 1.1.3. Find $\sin 15^\circ$ and $\cos 15^\circ$ in simplest radical form.

Strategy. Write 15° as $45^\circ - 30^\circ$, a difference of two angles with known exact values.

Solution. By the difference formulas,

$$\sin 15^\circ = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4},$$
$$\cos 15^\circ = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

As a check, $(\sqrt{6}-\sqrt{2})^2+(\sqrt{6}+\sqrt{2})^2 = (8-4\sqrt{3})+(8+4\sqrt{3}) = 16$, so $\sin^2 15^\circ + \cos^2 15^\circ = \frac{16}{16} = 1$.

Answer: $\sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$. □

Problem 1.1.4 (2007 AMC 12A, Problem #17)**.** Suppose that $\sin a + \sin b = \sqrt{\frac{5}{3}}$ and $\cos a + \cos b = 1$. What is $\cos(a - b)$?

(A) $\sqrt{\frac{5}{3}} - 1$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** 1

Strategy. The difference formula for cosine contains $\cos a \cos b + \sin a \sin b$. Squaring the two given equations and adding produces these products, while the squares combine by the Pythagorean identity.

Solution. Squaring the given equations gives

$$\sin^2 a + 2 \sin a \sin b + \sin^2 b = \frac{5}{3}, \qquad \cos^2 a + 2 \cos a \cos b + \cos^2 b = 1.$$

Adding them and using $\sin^2 a + \cos^2 a = 1$ and $\sin^2 b + \cos^2 b = 1$ gives

$$2 + 2(\cos a \cos b + \sin a \sin b) = \frac{8}{3}.$$

By the difference formula, the expression in parentheses is $\cos(a - b)$. Hence $2 + 2\cos(a - b) = \frac{8}{3}$, and $\cos(a - b) = \frac{1}{3}$.

Answer: (B) $\frac{1}{3}$. □

1.2. **Tangent.**

Theorem 1.2.1 (Tangent of a Sum)**.** *Suppose* $\tan \alpha$ *and* $\tan \beta$ *are defined. Then* $\tan(\alpha + \beta)$ *is defined exactly when* $\tan \alpha \tan \beta \neq 1$*, and then*

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

Proof. Since $\tan \alpha$ and $\tan \beta$ are defined, $\cos \alpha \cos \beta \neq 0$. By the sum formula, $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos \alpha \cos \beta \, (1 - \tan \alpha \tan \beta)$, which is nonzero exactly when $\tan \alpha \tan \beta \neq 1$. In that case, dividing the numerator and the denominator of

$$\tan(\alpha + \beta) = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}$$

by $\cos \alpha \cos \beta$ gives the formula. □

The formula needs only the sum $\tan \alpha + \tan \beta$ and the product $\tan \alpha \tan \beta$.

Examples.

Problem 1.2.2 (1985 AHSME, Problem #16)**.** If $A = 20^\circ$ and $B = 25^\circ$, then the value of $(1 + \tan A)(1 + \tan B)$ is

(A) $\sqrt{3}$ **(B)** 2 **(C)** $1 + \sqrt{2}$ **(D)** $2(\tan A + \tan B)$ **(E)** none of these

Strategy. The angles add to 45°, and $\tan 45^\circ = 1$. The tangent of the sum then relates $\tan A + \tan B$ to $\tan A \tan B$, the two terms that appear when the product is expanded.

Solution. Since $\tan(A + B) = \tan 45^\circ$ is defined, the formula for the tangent of a sum applies:

$$1 = \tan 45^\circ = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

so $\tan A + \tan B = 1 - \tan A \tan B$. Expanding the product gives

$$\begin{aligned}(1 + \tan A)(1 + \tan B) &= 1 + (\tan A + \tan B) + \tan A \tan B \\ &= 1 + (1 - \tan A \tan B) + \tan A \tan B = 2.\end{aligned}$$

Answer: (B) 2. □

Problem 1.2.3 (1986 AIME, Problem #3)**.** If $\tan x + \tan y = 25$ and $\cot x + \cot y = 30$, what is $\tan(x + y)$?

Strategy. The formula for $\tan(x + y)$ needs $\tan x + \tan y$, which is given, and $\tan x \tan y$. Writing $\cot x + \cot y$ over a common denominator gives the product.

Solution. Both $\tan x$ and $\cot x$ are defined, so $\tan x$ is a nonzero number and $\cot x = \frac{1}{\tan x}$; the same holds for y. Hence

$$30 = \cot x + \cot y = \frac{1}{\tan x} + \frac{1}{\tan y} = \frac{\tan x + \tan y}{\tan x \tan y} = \frac{25}{\tan x \tan y},$$

so $\tan x \tan y = \frac{25}{30} = \frac{5}{6}$. This is not 1, so $\tan(x + y)$ is defined, and

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{25}{1 - \frac{5}{6}} = 150.$$

Answer: 150. □

2. Double-Angle and Half-Angle Formulas

Setting $\beta = \alpha$ in the sum formulas gives formulas for twice an angle. Solving them for $\sin^2\theta$ and $\cos^2\theta$ gives formulas for squares and for half angles.

2.1. **Double Angles.**

Theorem 2.1.1 (Double-Angle Formulas)**.** *For every angle θ,*

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta, \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.\end{aligned}$$

Proof. Take $\alpha = \beta = \theta$ in the formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$. The last two forms of $\cos 2\theta$ come from replacing $\sin^2\theta$ by $1 - \cos^2\theta$, or $\cos^2\theta$ by $1 - \sin^2\theta$. □

The form of $\cos 2\theta$ to use depends on which of $\sin\theta$ and $\cos\theta$ is known.

Example.

Problem 2.1.2 (1976 AHSME, Problem #17)**.** If θ is an acute angle and $\sin 2\theta = a$, then $\sin\theta + \cos\theta$ equals

(A) $\sqrt{a+1}$ **(B)** $(\sqrt{2}-1)a+1$ **(C)** $\sqrt{a+1}-\sqrt{a^2-a}$ **(D)** $\sqrt{a+1}+\sqrt{a^2-a}$ **(E)** $\sqrt{a+1}+a^2-a$

Strategy. Square the required sum. The square contains $\sin^2\theta+\cos^2\theta$ and the product $2\sin\theta\cos\theta$, which is $\sin 2\theta$.

Solution. By the Pythagorean identity and the double-angle formula,

$$(\sin\theta+\cos\theta)^2 = \sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta = 1+\sin 2\theta = a+1.$$

Since θ is acute, $\sin\theta$ and $\cos\theta$ are positive, so $\sin\theta+\cos\theta$ is the positive square root of $a+1$.

Answer: (A) $\sqrt{a+1}$. □

2.2. **Power Reduction.**

Theorem 2.2.1 (Power Reduction)**.** *For every angle θ,*

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

Proof. Solve $\cos 2\theta = 1-2\sin^2\theta$ for $\sin^2\theta$, and $\cos 2\theta = 2\cos^2\theta - 1$ for $\cos^2\theta$. □

These formulas replace a square by a cosine of twice the angle. Together with $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, they rewrite any product of two of $\sin\theta$ and $\cos\theta$ in terms of $\sin 2\theta$ and $\cos 2\theta$, whose graphs have known amplitude and period.

Example.

Problem 2.2.2. Find the largest and smallest values of $f(x) = \sin^4 x+\cos^4 x$, and its least period.

Strategy. Write f in terms of $\cos 4x$. The square of $\sin^2 x+\cos^2 x$ is 1 and equals $\sin^4 x+\cos^4 x$ plus half the square of $\sin 2x$, which power reduction rewrites with $\cos 4x$.

Solution. Squaring $\sin^2 x+\cos^2 x = 1$ gives $\sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x = 1$, so

$$f(x) = 1-2\sin^2 x\cos^2 x = 1-\frac{1}{2}(2\sin x\cos x)^2 = 1-\frac{1}{2}\sin^2 2x.$$

By power reduction with $\theta = 2x$, $\sin^2 2x = \frac{1-\cos 4x}{2}$. Hence

$$f(x) = 1-\frac{1-\cos 4x}{4} = \frac{3}{4}+\frac{1}{4}\cos 4x.$$

This function has amplitude $\frac{1}{4}$ and least period $\frac{2\pi}{4} = \frac{\pi}{2}$. Its values range from $\frac{3}{4}-\frac{1}{4} = \frac{1}{2}$ to $\frac{3}{4}+\frac{1}{4} = 1$; the largest value occurs at $x = 0$ and the smallest at $x = \frac{\pi}{4}$.

Answer: Largest value 1, smallest value $\frac{1}{2}$, and least period $\frac{\pi}{2}$. □

2.3. **Half Angles.**

Theorem 2.3.1 (Half-Angle Formulas)**.** *For every angle θ,*

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}, \qquad \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2},$$

and if $\sin\theta \neq 0$*,*

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}.$$

Proof. The first two formulas are power reduction with $\frac{\theta}{2}$ in place of θ. If $\sin\theta \neq 0$, then $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ are nonzero, because their product is $\frac{1}{2}\sin\theta$. Multiplying the numerator and the denominator of $\tan\frac{\theta}{2} = \sin\frac{\theta}{2}/\cos\frac{\theta}{2}$ by $2\sin\frac{\theta}{2}$, and then by $2\cos\frac{\theta}{2}$ instead, gives

$$\tan\frac{\theta}{2} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{1-\cos\theta}{\sin\theta}, \qquad \tan\frac{\theta}{2} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\theta}{1+\cos\theta}.$$ □

The first two formulas give $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ only up to sign; the quadrant of $\frac{\theta}{2}$ decides the sign. The tangent formulas have no such choice. For example, with $\theta = 30°$,

$$\tan 15° = \frac{1-\cos 30°}{\sin 30°} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2-\sqrt{3},$$

which agrees with the quotient of the values of $\sin 15°$ and $\cos 15°$ found in Section 1.1:

$$\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{(\sqrt{6}-\sqrt{2})^2}{4} = \frac{8-4\sqrt{3}}{4} = 2-\sqrt{3}.$$

Example.

Problem 2.3.2 (1989 AHSME, Problem #14)**.** $\cot 10 + \tan 5 =$
(A) $\csc 5$ **(B)** $\csc 10$ **(C)** $\sec 5$ **(D)** $\sec 10$ **(E)** $\sin 15$

Strategy. The angle 5 is half of 10. The half-angle formula writes $\tan 5$ in terms of $\sin 10$ and $\cos 10$, which also make up $\cot 10$.

Solution. The angles are in radians, but the computation is the same in any unit. Since 10 is not a multiple of π, $\sin 10 \neq 0$, and the half-angle formula with $\theta = 10$ gives $\tan 5 = \frac{1-\cos 10}{\sin 10}$. Hence

$$\cot 10 + \tan 5 = \frac{\cos 10}{\sin 10} + \frac{1-\cos 10}{\sin 10} = \frac{1}{\sin 10} = \csc 10.$$

Answer: (B) $\csc 10$. □

3. Products and Sums

Adding and subtracting the sum and difference formulas turns a product of two sines or cosines into a sum, and reading the results backward turns a sum into a product.

Theorem 3.1 (Product-to-Sum Formulas)**.** *For all angles α and β,*

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta),$$
$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta),$$
$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta).$$

Proof. Add the formulas for $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$ for the first. Add the formulas for $\cos(\alpha-\beta)$ and $\cos(\alpha+\beta)$ for the second, and subtract them for the third. □

Corollary 3.2 (Sum-to-Product Formulas)**.** *For all angles A and B,*

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}, \qquad \sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2},$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

Proof. Put $\alpha = \frac{A+B}{2}$ and $\beta = \frac{A-B}{2}$, so that $\alpha+\beta = A$ and $\alpha-\beta = B$, in the product-to-sum formulas. For $\sin A - \sin B$, use the first one with α and β exchanged: $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$. □

A product-to-sum formula can make a sum **telescope**: each term becomes a difference, the second part of each term cancels the first part of the next, and only the two ends remain. A sum-to-product formula turns a sum into a product, which simplifies quotients and is 0 exactly when one of its factors is 0.

Examples.

Problem 3.3 (1984 AHSME, Problem #23)**.** $\dfrac{\sin 10^\circ + \sin 20^\circ}{\cos 10^\circ + \cos 20^\circ}$ equals

(A) $\tan 10^\circ + \tan 20^\circ$ **(B)** $\tan 30^\circ$ **(C)** $\frac{1}{2}(\tan 10^\circ + \tan 20^\circ)$ **(D)** $\tan 15^\circ$ **(E)** $\frac{1}{4}\tan 60^\circ$

Strategy. Write the numerator and the denominator as products. Both contain the factor $\cos 5^\circ$.

Solution. By the sum-to-product formulas with $A = 20^\circ$ and $B = 10^\circ$,

$$\sin 20^\circ + \sin 10^\circ = 2\sin 15^\circ \cos 5^\circ, \qquad \cos 20^\circ + \cos 10^\circ = 2\cos 15^\circ \cos 5^\circ.$$

Since $\cos 5^\circ \neq 0$, the quotient is

$$\frac{2\sin 15^\circ \cos 5^\circ}{2\cos 15^\circ \cos 5^\circ} = \frac{\sin 15^\circ}{\cos 15^\circ} = \tan 15^\circ.$$

Answer: (D) $\tan 15^\circ$. □

Problem 3.4 (1999 AIME, Problem #11)**.** Given that $\sum_{k=1}^{35} \sin 5k = \tan \frac{m}{n}$, where angles are measured in degrees, and m and n are relatively prime positive integers that satisfy $\frac{m}{n} < 90$, find $m + n$.

Strategy. The angles $5^\circ, 10^\circ, \ldots, 175^\circ$ increase in steps of 5°. Multiply the sum by $2\sin 2.5^\circ$, using half the step. By the product-to-sum formula, each term becomes a difference of two cosines, and the sum telescopes.

Solution. Let S be the sum. For each k, the product-to-sum formula gives

$$2\sin 5k^\circ \sin 2.5^\circ = \cos(5k - 2.5)^\circ - \cos(5k + 2.5)^\circ.$$

In the sum over $k = 1, 2, \ldots, 35$, the term $-\cos(5k + 2.5)^\circ$ cancels the term $\cos(5(k+1) - 2.5)^\circ$ from the next value of k, so only the two ends remain:

$$2\sin 2.5^\circ \cdot S = \cos 2.5^\circ - \cos 177.5^\circ = 2\cos 2.5^\circ,$$

since $\cos 177.5^\circ = \cos(180^\circ - 2.5^\circ) = -\cos 2.5^\circ$. Hence $S = \frac{\cos 2.5^\circ}{\sin 2.5^\circ} = \frac{1}{\tan 2.5^\circ} = \tan 87.5^\circ$, by the complementary-angle identity. For $0^\circ < t < 90^\circ$, $\sin t$ increases and $\cos t$ decreases, and both are positive, so $\tan t$ increases. Since $0 < \frac{m}{n} < 90$, the equation $\tan \frac{m}{n}^\circ = \tan 87.5^\circ$ gives $\frac{m}{n} = 87.5 = \frac{175}{2}$, which is in lowest terms. Thus $m + n = 175 + 2 = 177$.

Answer: 177. □

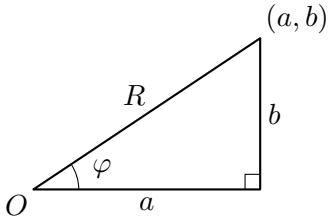
4. Combining Sine and Cosine

A sum $a\sin x + b\cos x$ is a single shifted sine. Its amplitude gives the largest and smallest values of the sum.

Theorem 4.1 (Amplitude-Phase Form)**.** *Let a and b be real numbers, not both 0, and let $R = \sqrt{a^2+b^2}$. Let φ be an angle whose terminal ray passes through (a,b), so that $\cos\varphi = \frac{a}{R}$ and $\sin\varphi = \frac{b}{R}$. Then for every angle x,*

$$a\sin x + b\cos x = R\sin(x+\varphi).$$

So $a\sin x + b\cos x$ takes every value from $-R$ to R. Its largest value R occurs exactly when $x+\varphi = 90^\circ + 360^\circ k$, and its smallest value $-R$ exactly when $x+\varphi = -90^\circ + 360^\circ k$, for integers k.



Proof. The point (a,b) is at distance R from the origin, so $\cos\varphi = \frac{a}{R}$ and $\sin\varphi = \frac{b}{R}$. By the sum formula,

$$a\sin x + b\cos x = R(\sin x\cos\varphi + \cos x\sin\varphi) = R\sin(x+\varphi).$$

As x varies, $\sin(x+\varphi)$ takes every value from -1 to 1; it equals 1 exactly when $x+\varphi = 90^\circ + 360^\circ k$ and -1 exactly when $x+\varphi = -90^\circ + 360^\circ k$. □

The right side is the **amplitude-phase form** of $a\sin x + b\cos x$: its graph is the graph of $y = R\sin x$ shifted left by φ, with amplitude R. In radians, replace 90° and 360° by $\frac{\pi}{2}$ and 2π.

Example.

Problem 4.2. Find the largest and smallest values of $f(x) = 5\sin x + 12\cos x$. At a point where $f(x)$ is largest, find $\sin x$ and $\cos x$.

Strategy. Write $f(x)$ in amplitude-phase form, with $R = \sqrt{5^2+12^2}$. The largest value occurs when $x+\varphi$ is a right angle, so x is then the complement of φ, up to full turns.

Solution. Here $R = \sqrt{25+144} = 13$. Let φ be an angle with $\cos\varphi = \frac{5}{13}$ and $\sin\varphi = \frac{12}{13}$. Then $f(x) = 13\sin(x+\varphi)$, so the largest value of f is 13 and the smallest is -13. The value 13 occurs exactly when $x = 90^\circ - \varphi + 360^\circ k$ for an integer k. Full turns do not change sine or cosine, so at such x,

$$\sin x = \sin(90^\circ - \varphi) = \cos\varphi = \frac{5}{13}, \qquad \cos x = \cos(90^\circ - \varphi) = \sin\varphi = \frac{12}{13}.$$

As a check, $5\cdot\frac{5}{13} + 12\cdot\frac{12}{13} = \frac{169}{13} = 13$.

Answer: Largest value 13, smallest value -13; where $f(x) = 13$, $\sin x = \frac{5}{13}$ and $\cos x = \frac{12}{13}$. □

5. Trigonometric Equations

Identities reduce most trigonometric equations to an equality of two values of one function, or to a product equal to 0. The solutions are then listed with an integer k, and those in the required interval are counted.

5.1. **Basic Equations.**

Theorem 5.1.1 (Equal Sines, Cosines, and Tangents)**.** *Let k range over the integers.*

(a) $\sin u = \sin v$ *exactly when* $u = v + 360°k$ *or* $u = 180° - v + 360°k$.

(b) $\cos u = \cos v$ *exactly when* $u = v + 360°k$ *or* $u = -v + 360°k$.

(c) If $\tan u$ *and* $\tan v$ *are defined,* $\tan u = \tan v$ *exactly when* $u = v + 180°k$.

In radians, replace $360°$ *by* 2π *and* $180°$ *by* π.

Proof. (a) The points of the unit circle at the angles v and $180° - v$ are $(\cos v, \sin v)$ and $(-\cos v, \sin v)$, since $\sin(180° - v) = \sin v$ and $\cos(180° - v) = -\cos v$. A point (x, y) of the circle with $y = \sin v$ has $x^2 = 1 - \sin^2 v = \cos^2 v$, so these are the only points of the circle with that y-coordinate. Hence $\sin u = \sin v$ exactly when the terminal ray of u is the ray of v or the ray of $180° - v$, that is, when u differs from v or from $180° - v$ by a whole number of turns.

(b) The same argument applies to the points at the angles v and $-v$, which have the same x-coordinate.

(c) If $\tan u = \tan v = t$, then the points of the unit circle at the angles u and v both lie on the line $y = tx$ through the origin. This line meets the circle at two opposite points, at the angles v and $v + 180°$, so u differs from v by a multiple of $180°$. Conversely, tangent has period $180°$. □

The cases that come up most are $\sin u = 0$ exactly when $u = 180°k$, $\cos u = 0$ exactly when $u = 90° + 180°k$, $\sin u = 1$ exactly when $u = 90° + 360°k$, and $\cos u = 1$ exactly when $u = 360°k$.

Every real number t is the tangent of exactly one angle u with $-90° < u < 90°$, namely the one whose terminal ray passes through $(1, t)$. As u increases from $-90°$ to $90°$, this ray turns counterclockwise and the height t at which it meets the line $x = 1$ rises, so tangent increases on this interval.

To count the solutions in an interval, write each family of solutions with k, find the values of k that put a solution in the interval, and check whether two families share a solution, so that no solution is counted twice.

Examples.

Problem 5.1.2 (1984 AHSME, Problem #15)**.** If $\sin 2x \sin 3x = \cos 2x \cos 3x$, then one value for x is

(A) $18°$ **(B)** $30°$ **(C)** $36°$ **(D)** $45°$ **(E)** $60°$

Strategy. Moving every term to one side gives the expansion of $\cos(2x + 3x)$.

Solution. The equation says $\cos 2x \cos 3x - \sin 2x \sin 3x = 0$, that is, $\cos 5x = 0$ by the sum formula. Cosine is 0 exactly at $90° + 180°k$, so $5x = 90° + 180°k$ and $x = 18° + 36°k$ for an integer k. The value $k = 0$ gives $x = 18°$. The other listed angles $30°$, $36°$, $45°$, and $60°$ differ from $18°$ by $12°$, $18°$, $27°$, and $42°$, none of which is a multiple of $36°$, so they are not solutions.

Answer: (A) $18°$. □

Problem 5.1.3 (1989 AHSME, Problem #28)**.** Find the sum of the roots of $\tan^2 x - 9 \tan x + 1 = 0$ that are between $x = 0$ and $x = 2\pi$ radians.

(A) $\frac{\pi}{2}$ **(B)** π **(C)** $\frac{3\pi}{2}$ **(D)** 3π **(E)** 4π

Strategy. The equation is a quadratic in $\tan x$ with two positive roots of product 1, so the angles in $(0, \frac{\pi}{2})$ with these tangents are complementary. Adding π to each gives the other two roots.

Solution. As a quadratic in $t = \tan x$, the equation $t^2 - 9t + 1 = 0$ has two distinct roots t_1 and t_2, since its discriminant is $81 - 4 = 77 > 0$. Their sum 9 and product 1 are positive, so both roots are positive. Let α be the angle in $(0, \frac{\pi}{2})$ with $\tan \alpha = t_1$. By the complementary-angle identity,

$\tan\left(\frac{\pi}{2}-\alpha\right)=\frac{1}{\tan\alpha}=\frac{1}{t_1}=t_2$, and $\frac{\pi}{2}-\alpha$ also lies in $(0,\frac{\pi}{2})$. By the theorem on equal tangents, $\tan x=t_1$ exactly when $x=\alpha+k\pi$, and $\tan x=t_2$ exactly when $x=\frac{\pi}{2}-\alpha+k\pi$. In $(0,2\pi)$ this gives the four roots α, $\alpha+\pi$, $\frac{\pi}{2}-\alpha$, and $\frac{3\pi}{2}-\alpha$, which are distinct because $t_1\neq t_2$. Their sum is

$$\alpha+(\alpha+\pi)+\left(\frac{\pi}{2}-\alpha\right)+\left(\frac{3\pi}{2}-\alpha\right)=3\pi.$$

Answer: (D) 3π. □

Problem 5.1.4 (2021 Fall AMC 12A, Problem #19)**.** Let x be the least real number greater than 1 such that $\sin(x)=\sin(x^2)$, where the arguments are in degrees. What is x rounded up to the closest integer?

(A) 10 **(B)** 13 **(C)** 14 **(D)** 19 **(E)** 20

Strategy. By the theorem on equal sines, either x^2-x or $x^2+x-180$ is a multiple of 360. Find the least root greater than 1 in each case.

Solution. By the theorem on equal sines, $\sin x^\circ=\sin(x^2)^\circ$ exactly when $x^2=x+360k$ or $x^2=180-x+360k$ for an integer k.

In the first case, $x(x-1)=360k$. For $x>1$ the left side is positive, so $k\geq 1$ and $x(x-1)\geq 360$. Since $19\cdot 18=342<360$, this forces $x>19$.

In the second case, $x^2+x=180+360k$. For $x>1$ the left side is positive, so $k\geq 0$. For $k=0$, the positive root of $x^2+x-180=0$ is $x=\frac{-1+\sqrt{721}}{2}$; for $k\geq 1$, $x^2+x\geq 540$, which gives a larger x, since x^2+x increases for $x>0$.

So the least x greater than 1 is $\frac{-1+\sqrt{721}}{2}$. Since $26^2=676<721<729=27^2$, this number lies between $\frac{25}{2}$ and 13, and rounding it up gives 13.

Answer: (B) 13. □

5.2. **Counting Solutions with Graphs.** When the two sides of an equation cannot be combined into one function, count the points where their graphs meet. Split the interval at the points where either side is undefined. On each piece, two facts make the count exact:

- If one side increases and the other decreases on the piece, the graphs meet at most once there.
- If both graphs are unbroken on the piece, and the first side is less than the second at one point and greater at another, the graphs meet between these points.

Since tangent has period 180°, the fact in Section 5.1 shows that tangent increases on each interval between consecutive odd multiples of 90° and takes every real value there. Cosine decreases as the angle increases from 0° to 180°.

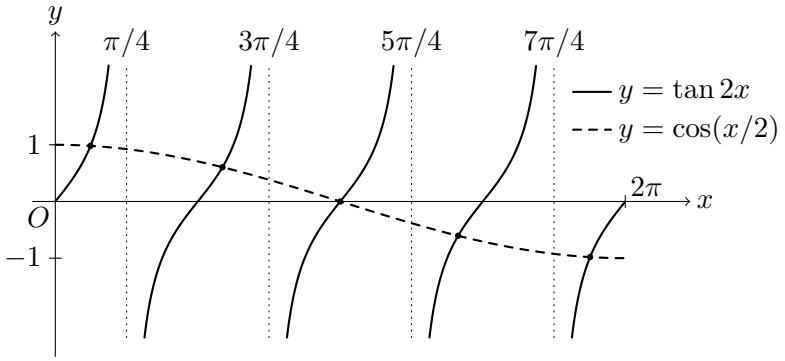
Example.

Problem 5.2.1 (2020 AMC 12A, Problem #9)**.** How many solutions does the equation $\tan(2x)=\cos\left(\frac{x}{2}\right)$ have on the interval $[0,2\pi]$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. Split $[0,2\pi]$ at the points where $\tan 2x$ is undefined. On each piece $\tan 2x$ increases while $\cos\frac{x}{2}$ decreases, so each piece contains at most one solution. Then, on each piece, find a point where the left side is less than the right side and a point where it is greater.

Solution. The function $\tan 2x$ is undefined where $2x$ is an odd multiple of $\frac{\pi}{2}$, that is, at $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. These points split $[0, 2\pi]$ into the five pieces $\left[0, \frac{\pi}{4}\right)$, $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$, $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$, and $\left(\frac{7\pi}{4}, 2\pi\right]$, shown in the figure.



On each piece, $2x$ stays between two consecutive odd multiples of $\frac{\pi}{2}$, so $\tan 2x$ increases. As x increases from 0 to 2π, $\frac{x}{2}$ increases from 0 to π, so $\cos\frac{x}{2}$ decreases from 1 to -1. Hence each piece contains at most one solution.

Since $-1 \le \cos\frac{x}{2} \le 1$, the left side is less than the right side wherever $\tan 2x = -2$, and greater wherever $\tan 2x = 2$. On each of the three middle pieces, $\tan 2x$ takes every real value, including -2 and 2, so the graphs meet once on each middle piece.

On $\left[0, \frac{\pi}{4}\right)$, the left side is less than the right side at $x = 0$, where $\tan 0 = 0 < 1 = \cos 0$, and greater at the point of the piece where $\tan 2x = 2$. On $\left(\frac{7\pi}{4}, 2\pi\right]$, it is less at the point where $\tan 2x = -2$, and greater at $x = 2\pi$, where $\tan 4\pi = 0 > -1 = \cos\pi$. So each end piece also contains one solution.

In all, the equation has 5 solutions.

Answer: (E) 5. □

6. Problems

Beginner.

1. (1983 AHSME, Problem #11) Simplify $\sin(x-y)\cos y + \cos(x-y)\sin y$.
(A) 1 **(B)** $\sin x$ **(C)** $\cos x$ **(D)** $\sin x \cos 2y$ **(E)** $\cos x \cos 2y$

2. (2018 AMC 12A, Problem #9) Which of the following describes the largest subset of values of y within the closed interval $[0, \pi]$ for which

$$\sin(x+y) \leq \sin(x) + \sin(y)$$

for every x between 0 and π, inclusive?
(A) $y = 0$ **(B)** $0 \leq y \leq \frac{\pi}{4}$ **(C)** $0 \leq y \leq \frac{\pi}{2}$ **(D)** $0 \leq y \leq \frac{3\pi}{4}$ **(E)** $0 \leq y \leq \pi$

3. (1973 AHSME, Problem #17) If θ is an acute angle and

$$\sin\frac{1}{2}\theta = \sqrt{\frac{x-1}{2x}},$$

then $\tan\theta$ equals
(A) x **(B)** $\dfrac{1}{x}$ **(C)** $\dfrac{\sqrt{x-1}}{x+1}$ **(D)** $\dfrac{\sqrt{x^2-1}}{x}$ **(E)** $\sqrt{x^2-1}$

4. (2025 AMC 12B, Problem #16) An analog clock starts at midnight and runs for 2025 minutes before stopping. What is the tangent of the acute angle between the hour hand and the minute hand when the clock stops?
(A) 0 **(B)** $\sqrt{2}-1$ **(C)** $2-\sqrt{2}$ **(D)** $\dfrac{\sqrt{2}}{2}$ **(E)** $3-\sqrt{2}$

5. (2010 AMC 12B, Problem #13) In $\triangle ABC$, $\cos(2A-B) + \sin(A+B) = 2$ and $AB = 4$. What is BC?
(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $2\sqrt{2}$ **(E)** $2\sqrt{3}$

Intermediate.

6. (2024 AMC 12B, Problem #21) The measures of the smallest angles of three different right triangles sum to $90°$. All three triangles have side lengths that are primitive Pythagorean triples. Two of them are 3-4-5 and 5-12-13. What is the perimeter of the third triangle?
(A) 40 **(B)** 126 **(C)** 154 **(D)** 176 **(E)** 208

7. (2004 AMC 12A, Problem #21) If

$$\sum_{n=0}^{\infty} \cos^{2n}\theta = 5,$$

what is the value of $\cos 2\theta$?
(A) $\dfrac{1}{5}$ **(B)** $\dfrac{2}{5}$ **(C)** $\dfrac{\sqrt{5}}{5}$ **(D)** $\dfrac{3}{5}$ **(E)** $\dfrac{4}{5}$

8. (1975 AHSME, Problem #30) Let $x = \cos 36° - \cos 72°$. Then x equals
(A) $\frac{1}{3}$ **(B)** $\frac{1}{2}$ **(C)** $3-\sqrt{6}$ **(D)** $2\sqrt{3}-3$ **(E)** none of these

9. (1974 AHSME, Problem #22) The minimum value of $\sin\frac{A}{2} - \sqrt{3}\cos\frac{A}{2}$ is attained when A is

(A) $-180°$ **(B)** $60°$ **(C)** $120°$ **(D)** $0°$ **(E)** none of these

10. (2003 AMC 12B, Problem #23) The number of x-intercepts on the graph of $y = \sin(\frac{1}{x})$ in the interval $(0.0001, 0.001)$ is closest to

(A) 2900 **(B)** 3000 **(C)** 3100 **(D)** 3200 **(E)** 3300

11. (1996 AIME, Problem #10) Find the smallest positive integer solution to

$$\tan(19x^\circ) = \frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}.$$

Advanced.

12. (1999 AHSME, Problem #27) In triangle ABC, $3\sin A + 4\cos B = 6$ and $4\sin B + 3\cos A = 1$. Then $\angle C$ in degrees is

(A) 30 **(B)** 60 **(C)** 90 **(D)** 120 **(E)** 150

13. (2012 AIME II, Problem #9) Let x and y be real numbers such that $\frac{\sin x}{\sin y} = 3$ and $\frac{\cos x}{\cos y} = \frac{1}{2}$. The value of $\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y}$ can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

14. (1997 AIME, Problem #11) Let

$$x = \frac{\displaystyle\sum_{n=1}^{44} \cos n^\circ}{\displaystyle\sum_{n=1}^{44} \sin n^\circ}.$$

What is the greatest integer that does not exceed $100x$?

15. (2021 AMC 12A, Problem #19) How many solutions does the equation

$$\sin\left(\frac{\pi}{2}\cos x\right) = \cos\left(\frac{\pi}{2}\sin x\right)$$

have in the closed interval $[0, \pi]$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

16. (2007 AMC 12A, Problem #24) For each integer $n > 1$, let $F(n)$ be the number of solutions of the equation $\sin x = \sin nx$ on the interval $[0, \pi]$. What is $\displaystyle\sum_{n=2}^{2007} F(n)$?

(A) 2,014,524 **(B)** 2,015,028 **(C)** 2,015,033 **(D)** 2,016,532 **(E)** 2,017,033