TRIGONOMETRIC IDENTITIES AND EQUATIONS FOR AMC 12: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (2020 AMC 12B, Problem #7) Two nonhorizontal, non-vertical lines in the xy-coordinate plane intersect to form a $45°$ angle. One line has slope equal to 6 times the slope of the other line. What is the greatest possible value of the product of the slopes of the two lines?

(A) $\frac{1}{6}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{2}$ **(D)** 3 **(E)** 6

Strategy. A line of slope s makes an angle with the x-axis whose tangent is s. The angle between two lines is the difference of these angles, so a $45°$ angle makes the tangent of the difference equal to 1 or -1.

Solution. Let the slopes be m and $6m$, where $m \neq 0$ because the lines are not horizontal. A line of slope s is parallel to the line through the origin and $(1, s)$, which lies along the terminal ray of the angle φ with $-90° < \varphi < 90°$ and $\tan\varphi = s$. Let α and β be these angles for the slopes $6m$ and m. The two lines are parallel to the lines through the origin at the angles α and β, so they meet at the angles $|\alpha - \beta|$ and $180° - |\alpha - \beta|$, where $0° < |\alpha - \beta| < 180°$. Hence they form a $45°$ angle exactly when $|\alpha - \beta|$ is $45°$ or $135°$, that is, exactly when $\tan(\alpha - \beta)$ is 1 or -1.

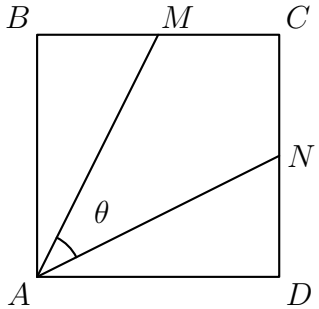
The formula for the tangent of a sum, with $-\beta$ in place of β and $\tan(-\beta) = -\tan\beta$, applies because $\tan\alpha\tan(-\beta) = -6m^2$ is not 1:

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{6m - m}{1 + 6m^2} = \frac{5m}{1 + 6m^2}.$$

So the condition is $5m = 1 + 6m^2$ or $5m = -(1 + 6m^2)$, that is, $(2m - 1)(3m - 1) = 0$ or $(2m + 1)(3m + 1) = 0$. Thus m is $\pm\frac{1}{2}$ or $\pm\frac{1}{3}$. The slopes multiply to $6m^2$, and this equals $\frac{3}{2}$ when $m = \pm\frac{1}{2}$ and $\frac{2}{3}$ when $m = \pm\frac{1}{3}$. The greatest possible value is $\frac{3}{2}$, attained, for example, by the slopes $\frac{1}{2}$ and 3.

Answer: (C) $\frac{3}{2}$. □

2. (1987 AHSME, Problem #14) $ABCD$ is a square and M and N are the midpoints of BC and CD, respectively. Then $\sin\theta =$



(A) $\frac{\sqrt{5}}{5}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{\sqrt{10}}{5}$ **(D)** $\frac{4}{5}$ **(E)** none of these

Strategy. The angle θ is the difference of the angles that AN and AM make with AB. Each of these angles comes from a right triangle with legs 1 and 2, so its sine and cosine are known, and the difference formula for sine gives $\sin\theta$.

Solution. Let the side of the square be 2, so $BM = DN = 1$. The right triangle ABM has legs $AB = 2$ and $BM = 1$ and hypotenuse $AM = \sqrt{2^2 + 1^2} = \sqrt{5}$, so

$$\sin\angle BAM = \frac{1}{\sqrt{5}}, \qquad \cos\angle BAM = \frac{2}{\sqrt{5}}.$$

In the same way, the right triangle ADN gives $\sin\angle DAN = \frac{1}{\sqrt{5}}$ and $\cos\angle DAN = \frac{2}{\sqrt{5}}$. Since $\angle BAN = 90^\circ - \angle DAN$, the complementary-angle identities give $\sin\angle BAN = \frac{2}{\sqrt{5}}$ and $\cos\angle BAN = \frac{1}{\sqrt{5}}$. As the figure shows, the ray AM lies between the rays AB and AN, so $\theta = \angle BAN - \angle BAM$. By the difference formula for sine,

$$\begin{aligned}\sin\theta &= \sin\angle BAN\cos\angle BAM - \cos\angle BAN\sin\angle BAM \\ &= \frac{2}{\sqrt{5}}\cdot\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}} = \frac{4}{5} - \frac{1}{5} = \frac{3}{5}.\end{aligned}$$

Answer: (B) $\frac{3}{5}$. □

3. (1996 AHSME, Problem #25) Given that $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value that $3x + 4y$ can have?

(A) 72 **(B)** 73 **(C)** 74 **(D)** 75 **(E)** 76

Strategy. Completing the squares shows that the points (x, y) form a circle. Write its points with the cosine and sine of an angle t; then $3x + 4y$ is a constant plus $32\sin t + 24\cos t$, and the amplitude-phase form gives the largest value of that sum.

Solution. Moving every term to the left side and completing the squares turns the equation into

$$(x - 7)^2 + (y - 3)^2 = 6 + 49 + 9 = 64.$$

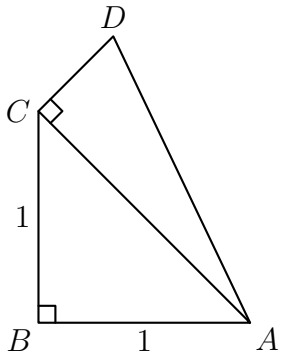
So (x, y) satisfies the equation exactly when $\left(\frac{x-7}{8}, \frac{y-3}{8}\right)$ lies on the unit circle, that is, exactly when $x = 7 + 8\cos t$ and $y = 3 + 8\sin t$ for some angle t. Then

$$3x + 4y = 21 + 24\cos t + 12 + 32\sin t = 33 + 32\sin t + 24\cos t.$$

For the amplitude-phase form with $a = 32$ and $b = 24$, $R = \sqrt{32^2 + 24^2} = \sqrt{1600} = 40$, and $32\sin t + 24\cos t = 40\sin(t + \varphi)$, where $\cos\varphi = \frac{4}{5}$ and $\sin\varphi = \frac{3}{5}$. Its largest value 40 occurs when $t + \varphi = 90^\circ + 360^\circ k$; then $\sin t = \cos\varphi = \frac{4}{5}$ and $\cos t = \sin\varphi = \frac{3}{5}$. So the largest value of $3x + 4y$ is $33 + 40 = 73$, attained at $(x, y) = \left(\frac{59}{5}, \frac{47}{5}\right)$. As a check, $3\cdot\frac{59}{5} + 4\cdot\frac{47}{5} = \frac{177+188}{5} = 73$.

Answer: (B) 73. □

4. (2019 AMC 12B, Problem #12) Right triangle ACD with right angle at C is constructed outwards on the hypotenuse $\overline{AC}$ of isosceles right triangle ABC with leg length 1, as shown, so that the two triangles have equal perimeters. What is $\sin(2\angle BAD)$?



(A) $\frac{1}{3}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{7}{9}$ **(E)** $\frac{\sqrt{3}}{2}$

Strategy. Find CD from the equal perimeters. The angle BAD is 45° more than the angle CAD, whose sine is then known; doubling turns the 45° into 90°, and the form of the double-angle formula for cosine in terms of sine finishes the computation.

Solution. The hypotenuse of triangle ABC is $AC = \sqrt{2}$, so its perimeter is $2 + \sqrt{2}$. Let $x = CD$. The angle at C is right, so $AD = \sqrt{2 + x^2}$, and the equal perimeters give $\sqrt{2} + x + \sqrt{2 + x^2} = 2 + \sqrt{2}$, that is, $\sqrt{2 + x^2} = 2 - x$. Squaring gives $2 + x^2 = 4 - 4x + x^2$, so $x = \frac{1}{2}$. Then $AD = \sqrt{2 + \frac{1}{4}} = \frac{3}{2}$, which is indeed $2 - x$.

Let $c = \angle CAD$. In the right triangle ACD, $\sin c = \frac{CD}{AD} = \frac{1}{3}$. The triangle ACD lies on the other side of AC from B, so the ray AC is inside the angle BAD, and $\angle BAD = \angle BAC + c = 45° + c$. Hence $2\angle BAD = 90° + 2c$. The sum formula gives $\sin(90° + u) = \sin 90° \cos u + \cos 90° \sin u = \cos u$, so by the double-angle formula

$$\sin(2\angle BAD) = \cos 2c = 1 - 2\sin^2 c = 1 - \frac{2}{9} = \frac{7}{9}.$$

Answer: (D) $\frac{7}{9}$. □

5. (1999 AHSME, Problem #18) How many zeros does $f(x) = \cos(\log x)$ have on the interval $0 < x < 1$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 10 **(E)** infinitely many

Strategy. Cosine is 0 exactly at the odd multiples of $\frac{\pi}{2}$. Count how many of these numbers $\log x$ can equal when $0 < x < 1$.

Solution. The input $\log x$ is a real number, so the cosine is taken in radians. Cosine is 0 exactly at $\frac{\pi}{2} + k\pi$ for integers k, so $f(x) = 0$ exactly when $\log x = \frac{\pi}{2} + k\pi$ for some integer k. Let $b > 1$ be the base of the logarithm. For each real u, the equation $\log_b x = u$ has exactly one solution, $x = b^u$, and it lies in $0 < x < 1$ exactly when $u < 0$. The numbers $\frac{\pi}{2} + k\pi$ with $k = -1, -2, -3, \ldots$ are $-\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, \ldots$, which are negative and all different, so they give the different zeros $b^{-\pi/2}, b^{-3\pi/2}, b^{-5\pi/2}, \ldots$ in the interval. So f has infinitely many zeros there, whatever the base.

Answer: (E) infinitely many. □

Intermediate.

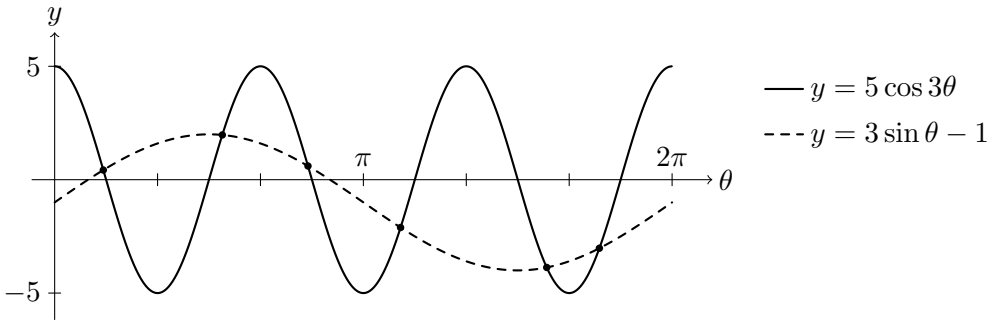
6. (2021 AMC 12B, Problem #13) How many values of θ in the interval $0 < \theta \leq 2\pi$ satisfy the following equation?

$$1 - 3\sin\theta + 5\cos 3\theta = 0$$

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

Strategy. Compare the graphs of $y = 5\cos 3\theta$ and $y = 3\sin\theta - 1$. At the multiples of $\frac{\pi}{3}$ the first is alternately 5 and -5, outside the range of the second, so the graphs cross at least six times. Writing $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$, then squaring, gives an equation of degree 6 in $\sin\theta$, so at most six values of $\sin\theta$ occur; since different solutions have different sines, there are at most six solutions.

Solution. The equation says $5\cos 3\theta = 3\sin\theta - 1$, so its solutions are the points where the graphs of $y = 5\cos 3\theta$ and $y = 3\sin\theta - 1$ meet, shown in the figure.



At least six solutions. At $\theta = \frac{k\pi}{3}$ with $k = 0, 1, \ldots, 6$, the left side is $5\cos k\pi$, which is 5 for even k and -5 for odd k, while the right side $3\sin\theta - 1$ always lies between -4 and 2. So the left side is greater than the right side at $\frac{k\pi}{3}$ for even k, and less for odd k. Both graphs are unbroken, so on each of the six intervals from $\frac{k\pi}{3}$ to $\frac{(k+1)\pi}{3}$, $k = 0, 1, \ldots, 5$, the graphs meet at a point strictly inside the interval. These six points are different and lie in $0 < \theta < 2\pi$.

At most six solutions. By the sum formula and the double-angle formulas,

$$\begin{aligned}\cos 3\theta &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\ &= (1 - 2\sin^2\theta)\cos\theta - 2\sin^2\theta\cos\theta = \cos\theta\,(1 - 4\sin^2\theta).\end{aligned}$$

Let $s = \sin\theta$. Every solution satisfies $5\cos\theta\,(1 - 4s^2) = 3s - 1$. Squaring both sides and replacing $\cos^2\theta$ by $1 - s^2$ gives

$$25(1 - s^2)(1 - 4s^2)^2 = (3s - 1)^2,$$

a polynomial equation of degree 6 in s, since the left side contains the term $-400s^6$ and the right side has degree 2. So at most six values of s occur. Two different solutions θ_1 and θ_2 in $(0, 2\pi]$ cannot have the same sine s. Otherwise their points $(\cos\theta_1, s)$ and $(\cos\theta_2, s)$ on the unit circle would be different points with the same y-coordinate, so $\cos\theta_2 = -\cos\theta_1 \neq 0$; subtracting the equations $5\cos\theta_i\,(1 - 4s^2) = 3s - 1$ would give $10\cos\theta_1\,(1 - 4s^2) = 0$, so $1 - 4s^2 = 0$ and then $3s - 1 = 0$. But $s = \frac{1}{3}$ gives $1 - 4s^2 = \frac{5}{9} \neq 0$. Hence different solutions have different sines, and there are at most six solutions.

Together, the two parts show that the equation has exactly 6 solutions in $0 < \theta \leq 2\pi$.

Answer: (D) 6. □

7. (2022 AMC 12A, Problem #17) Suppose a is a real number such that the equation

$$a \cdot (\sin x + \sin(2x)) = \sin(3x)$$

has more than one solution in the interval $(0, \pi)$. The set of all such a can be written in the form $(p, q) \cup (q, r)$, where p, q, and r are real numbers with $p < q < r$. What is $p + q + r$?
(A) -4 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 4

Strategy. Factor both sides: the double-angle formula handles $\sin x + \sin 2x$, and the sum and double-angle formulas give $\sin 3x$. After division by $\sin x$, which is positive on $(0, \pi)$, the equation says that a product of two factors is 0; count the solutions each factor gives.

Solution. By the double-angle formula, $\sin x + \sin 2x = \sin x\,(1 + 2\cos x)$. By the sum formula and the double-angle formulas,

$$\begin{aligned}\sin 3x &= \sin 2x \cos x + \cos 2x \sin x \\ &= 2\sin x \cos^2 x + (2\cos^2 x - 1)\sin x = \sin x\,(4\cos^2 x - 1),\end{aligned}$$

and $4\cos^2 x - 1 = (2\cos x - 1)(2\cos x + 1)$. On $(0, \pi)$, $\sin x > 0$, so dividing by $\sin x$ shows that the equation is equivalent there to $a(2\cos x + 1) = (2\cos x - 1)(2\cos x + 1)$, that is,

$$(2\cos x + 1)(a + 1 - 2\cos x) = 0.$$

As x increases from 0 to π, $\cos x$ decreases from 1 to -1, so on $(0, \pi)$ it takes each value strictly between -1 and 1 exactly once, and no other values. The first factor is 0 exactly when $\cos x = -\frac{1}{2}$, which happens only at $x = \frac{2\pi}{3}$, whatever a is. The second factor is 0 exactly when $\cos x = \frac{a+1}{2}$; this has one solution in $(0, \pi)$ if $-1 < \frac{a+1}{2} < 1$, that is, if $-3 < a < 1$, and none otherwise. That solution differs from $\frac{2\pi}{3}$ unless $\frac{a+1}{2} = -\frac{1}{2}$, that is, unless $a = -2$.

So the equation has two solutions in $(0, \pi)$ when $-3 < a < 1$ and $a \neq -2$, and only one otherwise. The set of such a is $(-3, -2) \cup (-2, 1)$, so $p = -3$, $q = -2$, $r = 1$, and $p + q + r = -4$.

Answer: (A) -4. □

8. (1982 AHSME, Problem #23) The lengths of the sides of a triangle are consecutive integers, and the largest angle is twice the smallest angle. The cosine of the smallest angle is
(A) $\frac{3}{4}$ **(B)** $\frac{7}{10}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{9}{14}$ **(E)** none of these

Strategy. Write the sides as $n - 1$, n, and $n + 1$. The Law of Sines and the double-angle formula make the ratio of the longest side to the shortest equal to $2\cos A$, where A is the smallest angle, and the Law of Cosines gives $\cos A$ in terms of n.

Solution. Let the sides be $n-1$, n, and $n+1$. The smallest angle A is opposite the shortest side $n - 1$, and the largest angle, $2A$, is opposite the longest side $n + 1$. By the Law of Sines and the double-angle formula,

$$\frac{n+1}{n-1} = \frac{\sin 2A}{\sin A} = \frac{2\sin A \cos A}{\sin A} = 2\cos A.$$

By the Law of Cosines,

$$\cos A = \frac{n^2 + (n+1)^2 - (n-1)^2}{2n(n+1)} = \frac{n^2 + 4n}{2n(n+1)} = \frac{n+4}{2(n+1)}.$$

Equating the two expressions for $2\cos A$ gives $\frac{n+1}{n-1} = \frac{n+4}{n+1}$, so $(n+1)^2 = (n+4)(n-1)$, that is, $n^2+2n+1 = n^2+3n-4$, and $n = 5$. The sides are 4, 5, and 6, and $\cos A = \frac{5+4}{2\cdot 6} = \frac{3}{4}$. As

a check, the Law of Cosines gives the largest angle the cosine $\frac{4^2+5^2-6^2}{2\cdot4\cdot5} = \frac{1}{8}$, which equals $\cos 2A = 2\cos^2 A - 1 = \frac{9}{8} - 1$.

Answer: (A) $\frac{3}{4}$. □

9. (1998 AHSME, Problem #28) In triangle ABC, angle C is a right angle and $CB > CA$. Point D is located on $\overline{BC}$ so that angle CAD is twice angle DAB. If $\frac{AC}{AD} = \frac{2}{3}$, then $\frac{CD}{BD} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

(A) 10 **(B)** 14 **(C)** 18 **(D)** 22 **(E)** 26

Strategy. Let $t = \angle DAB$. The right triangle ACD gives $\cos 2t$, and the half-angle formula for tangent then gives $\tan t$ with no choice of sign. The tangent of the sum $2t + t$ gives $\tan \angle CAB$, and the tangents of the angles at A give CD and CB.

Solution. Let $t = \angle DAB$, so $\angle CAD = 2t$ and $\angle CAB = 3t$, an acute angle of the right triangle ABC. Take $AC = 2$, so that $AD = 3$. In the right triangle ACD, $CD = \sqrt{3^2 - 2^2} = \sqrt{5}$, so

$$\cos 2t = \frac{AC}{AD} = \frac{2}{3}, \qquad \sin 2t = \frac{CD}{AD} = \frac{\sqrt{5}}{3}, \qquad \tan 2t = \frac{CD}{AC} = \frac{\sqrt{5}}{2}.$$

By the half-angle formula with $\theta = 2t$,

$$\tan t = \frac{\sin 2t}{1 + \cos 2t} = \frac{\sqrt{5}/3}{5/3} = \frac{1}{\sqrt{5}}.$$

Since $\tan 2t \tan t = \frac{1}{2} \neq 1$, the formula for the tangent of a sum gives

$$\tan 3t = \frac{\tan 2t + \tan t}{1 - \tan 2t \tan t} = \frac{\frac{\sqrt{5}}{2} + \frac{1}{\sqrt{5}}}{\frac{1}{2}} = \sqrt{5} + \frac{2}{\sqrt{5}} = \frac{7}{\sqrt{5}}.$$

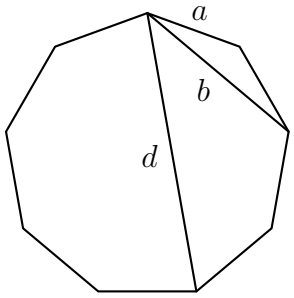
In the right triangle ABC, $CB = AC \tan 3t = \frac{14}{\sqrt{5}}$. Since D lies on $\overline{BC}$, $BD = CB - CD = \frac{14}{\sqrt{5}} - \sqrt{5} = \frac{9}{\sqrt{5}}$, and

$$\frac{CD}{BD} = \frac{\sqrt{5}}{9/\sqrt{5}} = \frac{5}{9}.$$

So $m = 5$, $n = 9$, and $m + n = 14$.

Answer: (B) 14. □

10. (1977 AHSME, Problem #30) If a, b and d are the lengths of a side, a shortest diagonal and a longest diagonal, respectively, of a regular nonagon (see adjoining figure), then



(A) $d = a + b$ **(B)** $d^2 = a^2 + b^2$ **(C)** $d^2 = a^2 + ab + b^2$ **(D)** $b = \frac{a+d}{2}$
(E) $b^2 = ad$

Strategy. The side and the diagonals are chords of the circle through the vertices, and a chord with central angle $2u$ has length $2R\sin u$. The sum-to-product formula turns $\sin 20^\circ + \sin 40^\circ$ into a single cosine.

Solution. Let O and R be the center and the radius of the circle through the nine vertices. Adjacent vertices are 40° apart as seen from O. A chord with central angle $2u$ is the base of an isosceles triangle with apex O and legs R; the perpendicular from O bisects the base and the apex angle, so the chord has length $2R\sin u$. Vertices k steps apart, for $k = 1, 2, 3, 4$, are therefore joined by a chord of length $2R\sin(20k)^\circ$, which increases with k. The side is the chord with $k = 1$, the shortest diagonal the one with $k = 2$, and the longest diagonal the one with $k = 4$:

$$a = 2R\sin 20^\circ, \qquad b = 2R\sin 40^\circ, \qquad d = 2R\sin 80^\circ.$$

By the sum-to-product formula,

$$a + b = 2R(\sin 40^\circ + \sin 20^\circ) = 2R \cdot 2\sin 30^\circ \cos 10^\circ = 2R\cos 10^\circ = 2R\sin 80^\circ = d,$$

since $\sin 30^\circ = \frac{1}{2}$ and $\cos 10^\circ = \sin 80^\circ$ by the complementary-angle identity.

Answer: (A) $d = a + b$. □

Advanced.

11. (2005 AMC 12B, Problem #24) All three vertices of an equilateral triangle are on the parabola $y = x^2$, and one of its sides has a slope of 2. The x-coordinates of the three vertices have a sum of $\dfrac{m}{n}$, where m and n are relatively prime positive integers. What is the value of $m + n$?

(A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

Strategy. The chord of $y = x^2$ between the points with x-coordinates p and q has slope $p + q$, so the slopes of the three sides add to twice the required sum. The other two sides are inclined 60° more and 60° less than the side of slope 2, and the tangent of a sum gives their slopes.

Solution. Let the vertices be (p, p^2), (q, q^2), and (r, r^2). The side joining the first two has slope $\frac{q^2-p^2}{q-p} = p + q$, so the slopes of the three sides are $p + q$, $q + r$, and $r + p$, and they add to $2(p + q + r)$.

A line of slope s is parallel to the line through the origin and $(1, s)$, so it is inclined to the x-axis at an angle whose tangent is s. Let the side of slope 2 be inclined at α, so $\tan\alpha = 2$. Each of the other two sides makes a 60° angle with this side, so it is inclined at $\alpha + 60^\circ$ or $\alpha - 60^\circ$, up to a multiple of 180°, which does not change the tangent. These two sides meet at the third vertex, so they are not parallel: one is inclined at $\alpha + 60^\circ$ and the other at $\alpha - 60^\circ$. Since $\tan 60^\circ = \sqrt{3}$ and $\tan(-60^\circ) = -\sqrt{3}$, and the products $2\sqrt{3}$ and $-2\sqrt{3}$ are not 1, the formula for the tangent of a sum gives their slopes:

$$\tan(\alpha + 60^\circ) = \frac{2 + \sqrt{3}}{1 - 2\sqrt{3}}, \qquad \tan(\alpha - 60^\circ) = \frac{2 - \sqrt{3}}{1 + 2\sqrt{3}}.$$

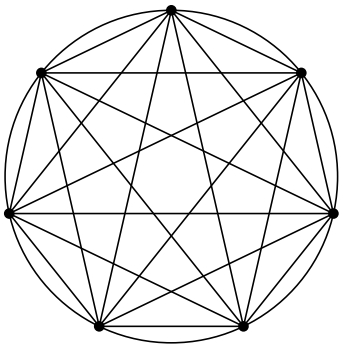
Over the common denominator $(1 - 2\sqrt{3})(1 + 2\sqrt{3}) = 1 - 12 = -11$, their sum is

$$\frac{(2 + \sqrt{3})(1 + 2\sqrt{3}) + (2 - \sqrt{3})(1 - 2\sqrt{3})}{-11} = \frac{(8 + 5\sqrt{3}) + (8 - 5\sqrt{3})}{-11} = -\frac{16}{11}.$$

Therefore $2(p+q+r) = 2 - \frac{16}{11} = \frac{6}{11}$, so $p+q+r = \frac{3}{11}$, in lowest terms, and $m+n = 3+11 = 14$.

Answer: (A) 14. □

12. (2022 AMC 12B, Problem #24) The figure below depicts a regular 7-gon inscribed in a unit circle.



What is the sum of the 4th powers of the lengths of all 21 of its edges and diagonals?

(A) 49 **(B)** 98 **(C)** 147 **(D)** 168 **(E)** 196

Strategy. Each segment is a chord of length $2\sin u$, where $2u$ is its central angle. Power reduction, used twice, writes $\sin^4 u$ in terms of $\cos 2u$ and $\cos 4u$. Both resulting sums of cosines have the terms of $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$, which telescopes after multiplication by $2\sin\frac{\pi}{7}$.

Solution. Two vertices k steps apart around the 7-gon, for $k = 1, 2, 3$, have central angle $\frac{2k\pi}{7}$. The perpendicular from the center bisects their chord and its central angle, so the chord has length $2\sin\frac{k\pi}{7}$. Joining each vertex to the vertex k steps clockwise from it gives 7 segments for each k, all different because 7 is odd, and these $3 \cdot 7 = 21$ segments are all the edges and diagonals. So the required sum is

$$7\sum_{k=1}^{3}\Big(2\sin\frac{k\pi}{7}\Big)^4 = 7\sum_{k=1}^{3} 16\sin^4\frac{k\pi}{7}.$$

By power reduction, $\sin^2 u = \frac{1-\cos 2u}{2}$ and $\cos^2 2u = \frac{1+\cos 4u}{2}$, so

$$16\sin^4 u = 4(1-\cos 2u)^2 = 4 - 8\cos 2u + 4\cos^2 2u = 6 - 8\cos 2u + 2\cos 4u.$$

Let $S_1 = \sum_{k=1}^{3}\cos\frac{2k\pi}{7}$ and $S_2 = \sum_{k=1}^{3}\cos\frac{4k\pi}{7}$. Then

$$\sum_{k=1}^{3} 16\sin^4\frac{k\pi}{7} = 18 - 8S_1 + 2S_2.$$

By the product-to-sum formula, $2\sin\frac{\pi}{7}\cos\frac{2k\pi}{7} = \sin\frac{(2k+1)\pi}{7} - \sin\frac{(2k-1)\pi}{7}$, so multiplying S_1 by $2\sin\frac{\pi}{7}$ makes it telescope:

$$2\sin\frac{\pi}{7}\cdot S_1 = \sin\frac{7\pi}{7} - \sin\frac{\pi}{7} = -\sin\frac{\pi}{7}.$$

Since $\sin\frac{\pi}{7} \neq 0$, $S_1 = -\frac{1}{2}$. The terms of S_2 are $\cos\frac{4\pi}{7}$, $\cos\frac{8\pi}{7} = \cos\big(2\pi - \frac{6\pi}{7}\big) = \cos\frac{6\pi}{7}$, and $\cos\frac{12\pi}{7} = \cos\big(2\pi - \frac{2\pi}{7}\big) = \cos\frac{2\pi}{7}$, the terms of S_1 in another order, so $S_2 = -\frac{1}{2}$ as well. Therefore

$$\sum_{k=1}^{3} 16\sin^4\frac{k\pi}{7} = 18 + 4 - 1 = 21,$$

and the required sum is $7 \cdot 21 = 147$.

Answer: (C) 147. □

13. (1989 AIME, Problem #10) Let a, b, c be the three sides of a triangle, and let α, β, γ be the angles opposite them. If $a^2 + b^2 = 1989c^2$, find

$$\frac{\cot\gamma}{\cot\alpha + \cot\beta}.$$

Strategy. Write $\cot\alpha + \cot\beta$ over a common denominator; its numerator is $\sin(\alpha+\beta)$, which equals $\sin\gamma$. The Law of Sines and the Law of Cosines then turn the quotient into an expression in a, b, and c.

Solution. Over the common denominator $\sin\alpha\sin\beta$, the sum formula gives

$$\cot\alpha + \cot\beta = \frac{\cos\alpha\sin\beta + \sin\alpha\cos\beta}{\sin\alpha\sin\beta} = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} = \frac{\sin\gamma}{\sin\alpha\sin\beta},$$

since $\alpha + \beta = 180^\circ - \gamma$ and $\sin(180^\circ - \gamma) = \sin\gamma$. This is not 0, and by the Law of Sines

$$\frac{\cot\gamma}{\cot\alpha + \cot\beta} = \frac{\cos\gamma}{\sin\gamma} \cdot \frac{\sin\alpha\sin\beta}{\sin\gamma} = \cos\gamma \cdot \frac{\sin\alpha}{\sin\gamma} \cdot \frac{\sin\beta}{\sin\gamma} = \cos\gamma \cdot \frac{ab}{c^2}.$$

By the Law of Cosines and the given relation,

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} = \frac{1989c^2 - c^2}{2ab} = \frac{994c^2}{ab}.$$

Therefore the quotient is $\frac{994c^2}{ab} \cdot \frac{ab}{c^2} = 994$.

Answer: 994. □

14. (2002 AIME II, Problem #10) While finding the sine of a certain angle, an absent-minded professor failed to notice that his calculator was not in the correct angular mode. He was lucky to get the right answer. The two least positive real values of x for which the sine of x degrees is the same as the sine of x radians are $\frac{m\pi}{n-\pi}$ and $\frac{p\pi}{q+\pi}$, where m, n, p, and q are positive integers. Find $m+n+p+q$.

Strategy. Converting x degrees to $\frac{\pi x}{180}$ radians turns the condition into $\sin\frac{\pi x}{180} = \sin x$. The theorem on equal sines gives two families of solutions; compare their least positive members.

Solution. Since x° is $\frac{\pi x}{180}$ radians, the condition is $\sin\frac{\pi x}{180} = \sin x$, with both angles in radians. By the theorem on equal sines, this holds exactly when, for some integer k,

$$\frac{\pi x}{180} = x + 2k\pi \qquad \text{or} \qquad \frac{\pi x}{180} = \pi - x + 2k\pi.$$

Multiplying by 180, the first equation becomes $(180-\pi)x = -360k\pi$, so $x = \frac{-360k\pi}{180-\pi}$, which is positive exactly when $k \le -1$. The second becomes $(180+\pi)x = 180(2k+1)\pi$, so $x = \frac{180(2k+1)\pi}{180+\pi}$, which is positive exactly when $k \ge 0$. Let

$$x_1 = \frac{360\pi}{180-\pi} \quad (k=-1), \qquad x_2 = \frac{180\pi}{180+\pi} \quad (k=0).$$

The other positive solutions are at least $2x_1$ in the first family and at least $3x_2$ in the second. Since $180\pi < 360\pi$ and $180+\pi > 180-\pi$, $x_2 < x_1$; and $x_1 < 3x_2$, because $360(180+\pi) < 540(180-\pi)$ is equivalent to $5\pi < 180$. So the two least positive solutions are x_2 and x_1, of the given forms with $m = 360$, $n = 180$, $p = 180$, and $q = 180$. Since π is irrational, no other positive integers give these values: for example, $\frac{m\pi}{n-\pi} = \frac{360\pi}{180-\pi}$

means $(360 - m)\pi = 360n - 180m$, which forces $m = 360$ and then $n = 180$. Hence $m + n + p + q = 360 + 180 + 180 + 180 = 900$.

Answer: 900. □

15. (2008 AIME II, Problem #8) Let $a = \frac{\pi}{2008}$. Find the smallest positive integer n such that

$$2[\cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2a)\sin(na)]$$

is an integer.

Strategy. By the product-to-sum formula, $2\cos(k^2a)\sin(ka)$ is a difference of two sines, and the sum telescopes to a single sine. The only integer values of sine are -1, 0, and 1, taken exactly at the multiples of $\frac{\pi}{2}$.

Solution. By the product-to-sum formula $2\sin\alpha\cos\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$ with $\alpha = ka$ and $\beta = k^2a$,

$$\begin{aligned} 2\cos(k^2a)\sin(ka) &= \sin\big((k^2+k)a\big) + \sin\big((k-k^2)a\big) \\ &= \sin\big(k(k+1)a\big) - \sin\big((k-1)ka\big). \end{aligned}$$

With $A_k = \sin\big(k(k+1)a\big)$, the term for k is $A_k - A_{k-1}$, so the sum telescopes, and the expression equals

$$A_n - A_0 = \sin\big(n(n+1)a\big) = \sin\frac{n(n+1)\pi}{2008}.$$

Since sine takes values from -1 to 1, its only possible integer values are -1, 0, and 1. Sine is 0 exactly at the multiples of π, and ±1 exactly at the odd multiples of $\frac{\pi}{2}$, so the expression is an integer exactly when $\frac{n(n+1)\pi}{2008}$ is a multiple of $\frac{\pi}{2}$, that is, when 1004 divides $n(n+1)$.

Now $1004 = 4\cdot251$, and 251 is prime, since it has no prime factor up to 13 and $17^2 > 251$. So 251 divides n or $n+1$, which requires $n \geq 250$. For $n = 250$, the product $250\cdot251$ contains only one factor of 2, so 4 does not divide it. For $n = 251$, the product $251 \cdot 252 = 1004 \cdot 63$ is a multiple of 1004. Hence the smallest n is 251.

Answer: 251. □

2. Answer Key

Problem	Letter	Value
1	C	$\frac{3}{2}$
2	B	$\frac{3}{5}$
3	B	73
4	D	$\frac{7}{9}$
5	E	infinitely many
6	D	6
7	A	-4
8	A	$\frac{3}{4}$
9	B	14
10	A	$d = a + b$
11	A	14
12	C	147
13		994
14		900
15		251