THE LAWS OF SINES AND COSINES FOR AMC 12

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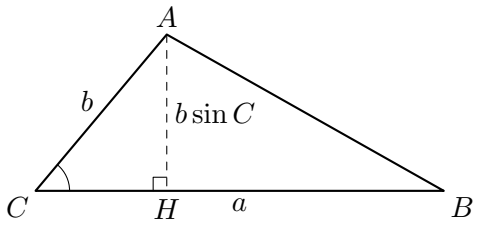
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1. Area and the Law of Sines

This section finds the area of a triangle from two sides and the angle between them, and uses it to relate the sides of a triangle to the sines of the opposite angles.

1.1. **The Area Formula.** In triangle ABC, the side lengths are written $a = BC$, $b = CA$, and $c = AB$, so each side is named after the opposite vertex. The letters A, B, and C also stand for the measures of the angles at those vertices. The area of a polygon P is written $[\boldsymbol{P}]$; for example, $[ABC]$ is the area of triangle ABC.



Theorem 1.1.1 (Area Formula)**.** *The area of a triangle is half the product of two sides and the sine of the angle between them:*

$$[ABC] = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B.$$

Proof. Let H be the foot of the perpendicular from A to line BC, as in the figure. By the previous handout, $AH = b\sin C$ whether C is acute or obtuse; when C is obtuse, H lies on the extension of BC past C. If $C = 90°$, then $H = C$ and $AH = b = b\sin C$. In every case the altitude to the base BC has length $b\sin C$, so

$$[ABC] = \frac{1}{2}\cdot BC\cdot AH = \frac{1}{2}ab\sin C.$$

The same argument with the altitudes from B and from C gives the other two forms. □

Because $\sin(180° - \theta) = \sin\theta$, two triangles with the same two sides and supplementary angles between them have the same area. For example, the triangles with sides 8 and 5 enclosing an angle of 45° or an angle of 135° both have area $\frac{1}{2}\cdot 8\cdot 5\cdot\frac{\sqrt{2}}{2} = 10\sqrt{2}$, although they are not congruent.

The formula also gives the area of a polygon that can be cut into triangles with a common vertex. In particular, the segments from the center of a circle of radius r to the vertices of an inscribed regular n-gon cut the n-gon into n congruent isosceles triangles, each with two sides r and the angle $\frac{360°}{n}$ between them.

Examples.

Problem 1.1.2 (1962 AHSME, Problem #18)**.** A regular dodecagon (12 sides) is inscribed in a circle with radius r inches. The area of the dodecagon, in square inches, is:

(A) $3r^2$ **(B)** $2r^2$ **(C)** $\dfrac{3r^2\sqrt{3}}{4}$ **(D)** $r^2\sqrt{3}$ **(E)** $3r^2\sqrt{3}$

Strategy. Draw the radii to the twelve vertices. As in the paragraph before these examples, they cut the dodecagon into congruent isosceles triangles, and the Area Formula gives the area of each.

Solution. The radii to the 12 vertices cut the dodecagon into 12 congruent isosceles triangles. Each has two sides of length r, and the angle between them is $\frac{360°}{12} = 30°$. By the Area Formula, each of these triangles has area

$$\frac{1}{2}\cdot r\cdot r\cdot\sin 30° = \frac{r^2}{4},$$

so the dodecagon has area $12 \cdot \frac{r^2}{4} = 3r^2$.

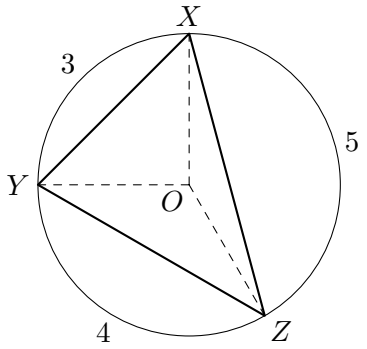
Answer: (A) $3r^2$. □

Problem 1.1.3 (1989 AHSME, Problem #19)**.** A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of lengths 3, 4, and 5. What is the area of the triangle?

(A) 6 **(B)** $\frac{18}{\pi^2}$ **(C)** $\frac{9}{\pi^2}(\sqrt{3}-1)$ **(D)** $\frac{9}{\pi^2}(\sqrt{3}+1)$ **(E)** $\frac{9}{\pi^2}(\sqrt{3}+3)$

Strategy. The arcs together make up the whole circle, so their total length gives the radius, and the length of each arc divided by the radius is its central angle in radians. The radii to the vertices then cut the triangle into pieces, each with two sides equal to the radius and a known angle between them.

Solution. Call the vertices X, Y, and Z, with arc XY of length 3, arc YZ of length 4, and arc ZX of length 5, and let O be the center and r the radius of the circle.



The circumference is $3 + 4 + 5 = 12 = 2\pi r$, so $r = \frac{6}{\pi}$. An arc of length s has central angle $\frac{s}{r}$ radians, so

$$\angle XOY = \frac{3}{r} = \frac{\pi}{2}, \qquad \angle YOZ = \frac{4}{r} = \frac{2\pi}{3}, \qquad \angle ZOX = \frac{5}{r} = \frac{5\pi}{6},$$

that is, 90°, 120°, and 150°. These angles add to 360° and each is less than 180°, so the radii OX, OY, and OZ cut the triangle into three triangles, each with two sides r enclosing one of these angles. By the Area Formula,

$$[XYZ] = \frac{1}{2}r^2\left(\sin 90^\circ + \sin 120^\circ + \sin 150^\circ\right) = \frac{1}{2} \cdot \frac{36}{\pi^2} \cdot \frac{3+\sqrt{3}}{2} = \frac{9}{\pi^2}(\sqrt{3}+3).$$

Answer: (E) $\frac{9}{\pi^2}(\sqrt{3}+3)$. □

1.2. **The Law of Sines.**

Theorem 1.2.1 (Law of Sines)**.** *In triangle ABC,*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Proof. The three forms of the Area Formula are equal:

$$\frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C.$$

Dividing by $\frac{1}{2}abc$ gives $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. Each angle lies strictly between 0° and 180°, so the sines are positive, and taking reciprocals gives the theorem. □

The Law of Sines says that the sides of a triangle are proportional to the sines of the opposite angles:

$$a : b : c = \sin A : \sin B : \sin C.$$

When two angles and a side are known, the third angle comes from the angle sum, and the law gives the other two sides. Since $\sin\theta \le 1$, each side is at most the common ratio $\frac{a}{\sin A}$, with equality exactly when the opposite angle is a right angle.

The law finds an angle only through its sine, and the angles θ and $180^\circ - \theta$ have the same sine. Section 3 shows how to decide between them.

Examples.

Problem 1.2.2 (1972 AHSME, Problem #14)**.** A triangle has angles of 30° and 45°. If the side opposite the 45° angle has length 8, then the side opposite the 30° angle has length:

(A) 4 **(B)** $4\sqrt{2}$ **(C)** $4\sqrt{3}$ **(D)** $4\sqrt{6}$ **(E)** 6

Strategy. Each side is proportional to the sine of the opposite angle.

Solution. Let x be the side opposite the 30° angle. By the Law of Sines,

$$\frac{x}{\sin 30^\circ} = \frac{8}{\sin 45^\circ}, \qquad \text{so} \qquad x = \frac{8 \cdot \frac{1}{2}}{\frac{\sqrt{2}}{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}.$$

Answer: (B) $4\sqrt{2}$. □

Problem 1.2.3 (1995 AHSME, Problem #18)**.** Two rays with common endpoint O form a 30° angle. Point A lies on one ray, point B on the other ray, and $AB = 1$. The maximum possible length of OB is

(A) 1 **(B)** $\frac{1+\sqrt{3}}{\sqrt{2}}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $\frac{4}{\sqrt{3}}$

Strategy. In triangle OAB, the fixed side AB faces the fixed angle at O. Write OB with the Law of Sines.

Solution. In triangle OAB, the side $AB = 1$ is opposite the 30° angle at O, and the side OB is opposite the angle $\angle OAB$. By the Law of Sines,

$$\frac{OB}{\sin\angle OAB} = \frac{AB}{\sin 30^\circ} = \frac{1}{1/2} = 2,$$

so $OB = 2\sin\angle OAB \le 2$. Equality requires $\angle OAB = 90^\circ$, and it occurs: in the triangle with angles 30° at O, 90° at A, and 60° at B, and with $AB = 1$, the hypotenuse is $OB = \frac{AB}{\sin 30^\circ} = 2$. So the maximum is 2.

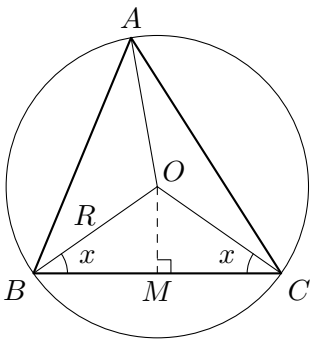
Answer: (D) 2. □

1.3. **The Circumradius.**

Definition 1.3.1. The perpendicular bisector of a segment consists of the points at equal distances from its two ends. In triangle ABC, the perpendicular bisectors of AB and BC meet at a point O at the same distance R from A, B, and C, so O lies on the perpendicular bisector of CA as well. The circle with center O and radius R is the only circle through A, B, and C. It is the **circumcircle** of the triangle, O is the **circumcenter**, and $R = OA = OB = OC$ is the **circumradius**.

Theorem 1.3.2 (Extended Law of Sines)**.** *In triangle ABC with circumradius R,*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$



Proof. Suppose first that the triangle is acute. Then O lies inside it, and the segments OA, OB, and OC, each of length R, cut it into the isosceles triangles OBC, OCA, and OAB. Let their base angles be

$$x = \angle OBC = \angle OCB, \qquad y = \angle OCA = \angle OAC, \qquad z = \angle OAB = \angle OBA.$$

Each angle of triangle ABC is the sum of two of these: $A = y + z$, $B = z + x$, and $C = x + y$. Adding the three equations gives $2(x + y + z) = 180°$, so $x = 90° - (y + z) = 90° - A$. Let M be the midpoint of BC. In the isosceles triangle OBC the segment OM is perpendicular to BC, so the right triangle OMB gives

$$\frac{a}{2} = BM = OB\cos x = R\cos(90° - A) = R\sin A.$$

Hence $a = 2R\sin A$, and in the same way $b = 2R\sin B$ and $c = 2R\sin C$.

The other cases are similar. If $A = 90°$, the triangle is half of a rectangle, whose diagonals are equal and bisect each other, so the midpoint of BC is at distance $\frac{a}{2}$ from A, B, and C. It is O, so $a = 2R = 2R\sin A$; and the isosceles triangles OCA and OAB have base angles $C = 90° - B$ and $B = 90° - C$, which give $b = 2R\sin B$ and $c = 2R\sin C$ as before. If the triangle has an obtuse angle, say A, then O lies outside it, on the other side of BC from A, and the angle chase reads $A = y + z$, $B = z - x$, and $C = y - x$. Adding the three equations gives $2(y + z - x) = 180°$, so $x = A - 90°$, $y = 90° - B$, and $z = 90° - C$. That is, the base angle at each side is $|90° - \theta|$, where θ is the angle opposite that side. Since $\cos|90° - \theta| = \sin\theta$, each side is again $2R$ times the sine of the opposite angle. □

Corollary 1.3.3. *In a right triangle, the hypotenuse is a diameter of the circumcircle, and the circumcenter is the midpoint of the hypotenuse.*

Proof. If $C = 90°$, the theorem gives $c = 2R\sin 90° = 2R$. The only chords of length $2R$ are diameters, so the hypotenuse AB is a diameter, and its midpoint is the center. □

Example.

Problem 1.3.4 (2021 Fall AMC 12B, Problem #9)**.** Triangle ABC is equilateral with side length 6. Suppose that O is the center of the inscribed circle of this triangle. What is the area of the circle passing through A, O, and C?

(A) 9π **(B)** 12π **(C)** 18π **(D)** 24π **(E)** 27π

Strategy. The circle through A, O, and C is the circumcircle of triangle AOC. Find the angle of that triangle opposite the known side AC.

Solution. The center of the inscribed circle lies on the bisectors of the angles of the triangle. Each angle of triangle ABC is 60°, so $\angle OAC = \angle OCA = 30°$ and $\angle AOC = 180° - 30° - 30° = 120°$.

The circle through A, O, and C is the circumcircle of triangle AOC, so by the Extended Law of Sines its radius R satisfies

$$2R = \frac{AC}{\sin \angle AOC} = \frac{6}{\sqrt{3}/2} = 4\sqrt{3}.$$

Hence $R = 2\sqrt{3}$, and the area of the circle is $\pi R^2 = 12\pi$.

Answer: (B) 12π. □

2. The Law of Cosines

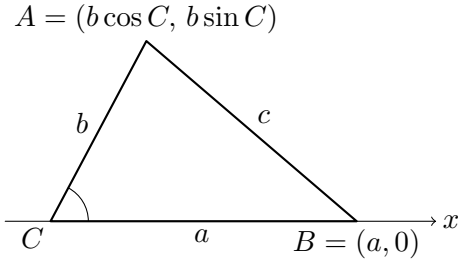
The Law of Cosines extends the Pythagorean theorem to every triangle. It finds the third side from two sides and the angle between them, and it finds every angle from the three sides.

2.1. **Finding a Side.**

Theorem 2.1.1 (Law of Cosines)**.** *In triangle ABC,*

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

and in the same way $a^2 = b^2 + c^2 - 2bc\cos A$ and $b^2 = c^2 + a^2 - 2ca\cos B$.



Proof. Place C at the origin and B on the positive x-axis, so that $B = (a, 0)$, as in the figure. The point A is at distance b from the origin on the ray that makes the angle C with the positive x-axis, so $A = (b\cos C, b\sin C)$. By the distance formula,

$$\begin{aligned} c^2 &= (b\cos C - a)^2 + (b\sin C)^2 \\ &= a^2 - 2ab\cos C + b^2(\cos^2 C + \sin^2 C) \\ &= a^2 + b^2 - 2ab\cos C. \end{aligned}$$

The computation is the same whether C is acute, right, or obtuse. □

When $C = 90°$, $\cos C = 0$ and the law is the Pythagorean theorem. The term $-2ab\cos C$ is negative when C is acute and positive when C is obtuse. The law gives c from a, b, and C alone, so two sides and the angle between them (SAS) determine the third side. Two angles come up often:

$$C = 60° : \quad c^2 = a^2 + b^2 - ab, \qquad\qquad C = 120° : \quad c^2 = a^2 + b^2 + ab.$$

Examples.

Problem 2.1.2. In triangle ABC, $AB = 5$, $AC = 8$, and $\angle A = 60°$. Find BC.

Strategy. The given angle lies between the two given sides, so the Law of Cosines gives the third side directly.

Solution. By the Law of Cosines at A,

$$BC^2 = AB^2 + AC^2 - 2\cdot AB\cdot AC\cos 60° = 25 + 64 - 40 = 49,$$

so $BC = 7$.

Answer: 7. □

Problem 2.1.3 (1959 AHSME, Problem #36)**.** The base of a triangle is 80, and one of the base angles is 60°. The sum of the lengths of the other two sides is 90. The shortest side is:

(A) 45 **(B)** 40 **(C)** 36 **(D)** 17 **(E)** 12

Strategy. One unknown side and the base enclose the 60° angle, and the other unknown side faces it. The two unknown sides have the known sum 90, so one variable describes both, and the Law of Cosines for a 60° angle gives an equation in that variable.

Solution. The 60° angle lies between the base 80 and a side x, and the side opposite it is $90 - x$. By the Law of Cosines for a 60° angle,

$$(90 - x)^2 = 80^2 + x^2 - 80x.$$

Expanding gives $8100 - 180x = 6400 - 80x$, so $100x = 1700$ and $x = 17$. The sides are 80, 17, and 73, which satisfy the triangle inequality, since $17 + 73 > 80$. The shortest side is 17.

Answer: (D) 17. □

2.2. **Finding an Angle.** Solving the Law of Cosines for the cosine gives the angles from the sides.

Corollary 2.2.1. *In triangle ABC,*

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

and similarly for $\cos A$ *and* $\cos B$*.*

As θ increases from 0° to 180°, the point $(\cos\theta, \sin\theta)$ moves along the upper half of the unit circle from $(1, 0)$ to $(-1, 0)$, so its x-coordinate $\cos\theta$ decreases from 1 to -1 and takes each value in between exactly once. An angle of a triangle is therefore determined by its cosine, which is not true of its sine, and three sides (SSS) determine all three angles. The sign of the cosine shows what kind of angle it is.

Corollary 2.2.2 (Obtuse Angle Test)**.** *The angle C of triangle ABC is acute, right, or obtuse according as $c^2 < a^2 + b^2$, $c^2 = a^2 + b^2$, or $c^2 > a^2 + b^2$.*

Proof. By the previous corollary, $\cos C$ has the same sign as $a^2 + b^2 - c^2$. □

A triangle has at most one angle that is not acute, since two such angles would add to at least 180°. An obtuse angle faces the longest side, because $c^2 > a^2 + b^2$ gives $c > a$ and $c > b$. So a triangle is obtuse exactly when the square of its longest side is greater than the sum of the squares of the other two sides.

Examples.

Problem 2.2.3 (1981 AHSME, Problem #21)**.** In a triangle with sides of lengths a, b, and c,

$$(a + b + c)(a + b - c) = 3ab.$$

The measure of the angle opposite the side of length c is

(A) 15° **(B)** 30° **(C)** 45° **(D)** 60° **(E)** 150°

Strategy. Multiplied out, the left side contains $a^2 + b^2 - c^2$, the numerator of the formula for the cosine of the angle opposite c.

Solution. The left side is $(a+b)^2-c^2 = a^2+2ab+b^2-c^2$. Setting it equal to $3ab$ gives $a^2+b^2-c^2 = ab$. Let C be the angle opposite the side c. Then

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{ab}{2ab} = \frac{1}{2},$$

and the only angle of a triangle with cosine $\frac{1}{2}$ is $60°$.

Answer: (D) $60°$. □

Problem 2.2.4 (1963 AHSME, Problem #34)**.** In triangle ABC, side $a = \sqrt{3}$, side $b = \sqrt{3}$, and side $c > 3$. Let x be the largest number such that the magnitude, in degrees, of the angle opposite side c exceeds x. Then x equals:

(A) 150 **(B)** 120 **(C)** 105 **(D)** 90 **(E)** 60

Strategy. Write $\cos C$ in terms of c, and use the fact that cosine decreases from $0°$ to $180°$.

Solution. By the Law of Cosines,

$$\cos C = \frac{3+3-c^2}{2\cdot 3} = 1 - \frac{c^2}{6} < 1 - \frac{9}{6} = -\frac{1}{2},$$

since $c^2 > 9$. Cosine decreases from $0°$ to $180°$ and $\cos 120° = -\frac{1}{2}$, so $C > 120°$ in every such triangle. No larger number has this property. A number $t \geq 180$ fails, since every angle of a triangle is less than $180°$. If $120 < t < 180$, then $c = \sqrt{6 - 6\cos t°}$ satisfies $3 < c < 2\sqrt{3}$, so the sides $\sqrt{3}$, $\sqrt{3}$, c satisfy the triangle inequality and form a triangle with $\cos C = 1 - \frac{c^2}{6} = \cos t°$. Its angle C measures exactly t degrees, which does not exceed t. Hence $x = 120$.

Answer: (B) 120. □

3. Solving Triangles

To **solve** a triangle is to find its remaining sides and angles from the ones that are given.

3.1. **Choosing a Law.** By the congruence criteria, three sides (SSS), two sides and the angle between them (SAS), and two angles with a side (ASA or AAS) each determine a triangle. The table shows where to start in each case.

Given	First step	Remaining parts
SSS	The Law of Cosines gives the cosine of each angle.	Nothing more: each angle is determined by its cosine.
SAS	The Law of Cosines gives the third side.	The Law of Cosines gives a second angle, and the angle sum gives the third.
ASA, AAS	The angle sum gives the third angle.	The Law of Sines gives the other two sides.

Every angle of a triangle has a positive sine, so $\sin\theta = \sqrt{1-\cos^2\theta}$. A cosine found from the Law of Cosines therefore gives the sine needed by the Area Formula or by the Extended Law of Sines. To find an angle from its sine, use the Law of Sines only for an angle known to be acute, such as an angle opposite a side that is not the longest. Otherwise use the Law of Cosines, which never leaves a choice between θ and $180° - \theta$.

Example.

Problem 3.1.1 (1959 AHSME, Problem #43)**.** The sides of a triangle are 25, 39, and 40. The diameter of the circumscribed circle is:

(A) $\frac{133}{3}$ **(B)** $\frac{125}{3}$ **(C)** 42 **(D)** 41 **(E)** 40

Strategy. Find the cosine of one angle from the three sides, then its sine, and then use $\frac{a}{\sin A} = 2R$.

Solution. Name the triangle ABC with $a = 25$, $b = 39$, and $c = 40$. By the Law of Cosines,

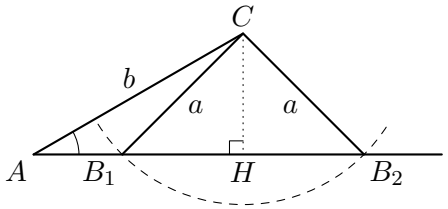
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{1521 + 1600 - 625}{2 \cdot 39 \cdot 40} = \frac{2496}{3120} = \frac{4}{5}.$$

The sine of an angle of a triangle is positive, so $\sin A = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$. By the Extended Law of Sines, the diameter of the circumscribed circle is

$$2R = \frac{a}{\sin A} = \frac{25}{3/5} = \frac{125}{3}.$$

Answer: (B) $\frac{125}{3}$. □

3.2. **The Ambiguous Case.** Two sides and an angle that is not between them (SSA) need not determine a triangle. In the figure, the angle A and the side $b = CA$ are given. The third vertex lies on the other ray of the angle at distance a from C, so it is a point where the circle with center C and radius a meets that ray. Let H be the foot of the perpendicular from C to the ray, so that $CH = b \sin A$ when A is acute. The circle meets the ray at two points B_1 and B_2, at one point, or not at all, depending on how a compares with CH and with b.



Theorem 3.2.1 (SSA)**.** *Let $a, b > 0$ and $0° < A < 180°$. A triangle ABC with $BC = a$, $CA = b$, and the given angle A has $AB = c$ for a positive root c of*

$$c^2 - (2b \cos A)\, c + (b^2 - a^2) = 0,$$

and each positive root gives such a triangle. Hence:

- *if A is acute, there is no triangle when $a < b \sin A$, exactly one (with $B = 90°$) when $a = b \sin A$, two when $b \sin A < a < b$, and exactly one when $a \geq b$;*
- *if A is right or obtuse, there is exactly one triangle when $a > b$ and none when $a \leq b$.*

Proof. In triangle ABC, the Law of Cosines at A reads $a^2 = b^2 + c^2 - 2bc \cos A$, which is the quadratic. Conversely, if $c > 0$ is a root, put B on the other ray of the angle at distance c from A; the Law of Cosines gives $BC^2 = b^2 + c^2 - 2bc \cos A = a^2$. The quadratic has discriminant $4(b^2 \cos^2 A - b^2 + a^2) = 4(a^2 - b^2 \sin^2 A)$, roots $b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}$, sum $2b \cos A$, and product $b^2 - a^2$.

- If $a > b$, the product is negative, so exactly one root is positive.
- If $a = b$, the roots are 0 and $2b \cos A$, so there is one positive root when A is acute and none otherwise.

- If $a < b$, the product is positive, so real roots have the same sign as their sum $2b\cos A$. When A is right or obtuse, no root is positive. When A is acute, there are two positive roots, one double root, or no real roots according as $a > b\sin A$, $a = b\sin A$, or $a < b\sin A$. The double root is $c = b\cos A = AH$, so then the vertex B is H and $\angle ABC = 90°$.

Collecting the cases gives the statement. □

Remark 3.2.2. The Law of Sines shows the same ambiguity. It gives $\sin B = \frac{b\sin A}{a}$. When this number is less than 1, the angle B is either the acute angle θ with this sine or its supplement $180° - \theta$, and the supplement is possible only if $A + (180° - \theta) < 180°$, that is, only if $\theta > A$. In the figure, the two choices are the angles $\angle AB_2C$ and $\angle AB_1C$.

Examples.

Problem 3.2.3. In triangle ABC, $BC = \sqrt{2}$, $CA = 2$, and $\angle A = 30°$. Find all possible values of AB, and the angles B and C in each case.

Strategy. Find AB from the quadratic that the Law of Cosines at A gives. Then find B from its cosine, which leaves no choice.

Solution. Here $a = \sqrt{2}$, $b = 2$, and $b\sin A = 1 < a < b$, so the theorem gives two triangles. With $c = AB$, the Law of Cosines at A gives

$$2 = 4 + c^2 - 2\cdot 2\cdot c\cos 30°, \qquad \text{that is,} \qquad c^2 - 2\sqrt{3}\,c + 2 = 0,$$

so $c = \sqrt{3} \pm 1$. Both roots are positive. By the Law of Cosines at B,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{c^2 - 2}{2\sqrt{2}\,c}.$$

For $c = \sqrt{3} + 1$, $c^2 = 4 + 2\sqrt{3}$, so

$$\cos B = \frac{2 + 2\sqrt{3}}{2\sqrt{2}\,(\sqrt{3} + 1)} = \frac{1}{\sqrt{2}},$$

which gives $B = 45°$ and $C = 180° - 30° - 45° = 105°$. For $c = \sqrt{3} - 1$, $c^2 = 4 - 2\sqrt{3}$, so

$$\cos B = \frac{2 - 2\sqrt{3}}{2\sqrt{2}\,(\sqrt{3} - 1)} = -\frac{1}{\sqrt{2}},$$

which gives $B = 135°$ and $C = 15°$. As a check, the Law of Sines gives $\sin B = \frac{2\sin 30°}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ in both triangles, and the two values of B are supplementary. The figure before the theorem is drawn with these numbers: the first triangle is AB_2C and the second is AB_1C.

Answer: $AB = \sqrt{3} + 1$, with $B = 45°$ and $C = 105°$; or $AB = \sqrt{3} - 1$, with $B = 135°$ and $C = 15°$. □

Problem 3.2.4 (2013 AMC 12A, Problem #12)**.** The angles in a particular triangle are in arithmetic progression, and the side lengths are 4, 5, and x. The sum of the possible values of x equals $a + \sqrt{b} + \sqrt{c}$, where a, b, and c are positive integers. What is $a + b + c$?

(A) 36 **(B)** 38 **(C)** 40 **(D)** 42 **(E)** 44

Strategy. The middle term of three numbers in arithmetic progression is their average, so one angle of the triangle is 60°. Any of the three sides can face that angle, and each choice gives one case for the Law of Cosines.

Solution. Three angles in arithmetic progression can be written $m-d$, m, and $m+d$. They add to $3m = 180°$, so the middle angle is $60°$. Conversely, every triangle with a $60°$ angle qualifies, since its other two angles add to $120°$. So the condition is that the triangle has a $60°$ angle, and the side opposite that angle is x, 5, or 4. With s the side opposite the $60°$ angle and p and q the other two sides, the Law of Cosines for $60°$ reads $s^2 = p^2 + q^2 - pq$.

- When the side x faces the $60°$ angle, $x^2 = 16 + 25 - 20 = 21$, so $x = \sqrt{21}$.
- When the side 5 faces it, $25 = 16 + x^2 - 4x$, so $x^2 - 4x - 9 = 0$ and $x = 2 \pm \sqrt{13}$, of which only $2 + \sqrt{13}$ is positive.
- When the side 4 faces it, $16 = 25 + x^2 - 5x$, so $x^2 - 5x + 9 = 0$. The discriminant $25 - 36$ is negative, so this case gives no triangle.

These counts agree with the theorem on SSA: in the second case the side opposite the $60°$ angle is longer than the other given side, and in the third case $4 < 5\sin 60°$. Adding the two values of x gives $2+\sqrt{13}+\sqrt{21}$, which has the required form with $a = 2$, $b = 13$, and $c = 21$, so $a+b+c = 36$.

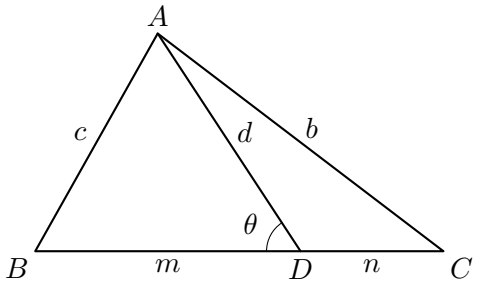
Answer: (A) 36. □

4. Lengths of Cevians

This section finds the lengths of segments from a vertex of a triangle to the opposite side, with the Law of Cosines and the Law of Sines.

4.1. Stewart's Theorem.

Definition 4.1.1. A **cevian** of a triangle is a segment from a vertex to a point of the opposite side. A **median** is the cevian to the midpoint of the opposite side, and an **angle bisector** is the cevian that bisects the angle at its vertex.



Theorem 4.1.2 (Stewart's Theorem)**.** *Let D be a point of side BC of triangle ABC other than B and C, and let $m = BD$, $n = DC$, and $d = AD$. Then*

$$b^2 m + c^2 n = a(d^2 + mn).$$

Proof. Let $\theta = \angle ADB$. The angles $\angle ADB$ and $\angle ADC$ are supplementary, so $\cos\angle ADC = \cos(180° - \theta) = -\cos\theta$. The Law of Cosines in triangles ABD and ADC gives

$$c^2 = m^2 + d^2 - 2md\cos\theta, \qquad b^2 = n^2 + d^2 + 2nd\cos\theta.$$

Multiply the first equation by n and the second by m, and add. The terms with $\cos\theta$ cancel, and since $m + n = a$,

$$c^2 n + b^2 m = mn(m+n) + d^2(m+n) = a(d^2 + mn).$$ □

Corollary 4.1.3 (Median Length)**.** *The median from A has length m_a, where*

$$m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4}.$$

Proof. Take $m = n = \frac{a}{2}$ in Stewart's Theorem: $\frac{a}{2}(b^2 + c^2) = a\left(m_a^2 + \frac{a^2}{4}\right)$, so $m_a^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4}$. □

The idea of the proof of Stewart's Theorem applies whenever two triangles have supplementary angles at a common vertex: the cosines of these angles are negatives of each other, so the two Laws of Cosines can be combined to eliminate the angle.

Examples.

Problem 4.1.4. Triangle ABC has $AB = 8$, $BC = 9$, and $CA = 7$. Point D lies on BC with $BD = 6$. Find AD and the length of the median from C.

Strategy. Apply Stewart's Theorem with $m = BD$ and $n = DC$. For the median, use the median length formula at the vertex C.

Solution. Here $a = 9$, $b = 7$, $c = 8$, $m = 6$, and $n = 3$. Stewart's Theorem gives

$$49 \cdot 6 + 64 \cdot 3 = 9(d^2 + 18), \qquad \text{so} \qquad 486 = 9d^2 + 162,$$

hence $d^2 = 36$ and $AD = 6$. As a check, the Law of Cosines in triangles ABC and ABD gives

$$\cos B = \frac{81 + 64 - 49}{2 \cdot 9 \cdot 8} = \frac{2}{3}, \qquad AD^2 = 64 + 36 - 2 \cdot 8 \cdot 6 \cdot \frac{2}{3} = 36.$$

The median from C goes to the midpoint of AB, the side of length $c = 8$. By the median length formula, with the roles of the vertices exchanged,

$$m_c^2 = \frac{2a^2 + 2b^2 - c^2}{4} = \frac{2 \cdot 81 + 2 \cdot 49 - 64}{4} = \frac{196}{4} = 49,$$

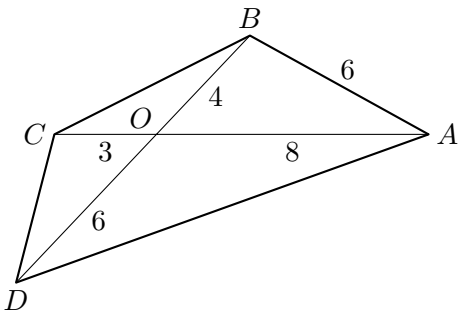
so the median has length 7.

Answer: $AD = 6$, and the median from C has length 7. □

Problem 4.1.5 (1967 AHSME, Problem #32)**.** In quadrilateral $ABCD$ with diagonals AC and BD intersecting at O, $BO = 4$, $OD = 6$, $AO = 8$, $OC = 3$, and $AB = 6$. The length of AD is:
(A) 9 **(B)** 10 **(C)** $6\sqrt{3}$ **(D)** $8\sqrt{2}$ **(E)** $\sqrt{166}$

Strategy. All three sides of triangle AOB are known, so the Law of Cosines gives the cosine of its angle at O. Because O lies on the diagonal BD, between B and D, the angle $\angle AOD$ is the supplement of that angle and has the opposite cosine, as in the proof of Stewart's Theorem.

Solution. The figure shows the given lengths.



In triangle AOB, the Law of Cosines gives

$$\cos \angle AOB = \frac{AO^2 + BO^2 - AB^2}{2 \cdot AO \cdot BO} = \frac{64 + 16 - 36}{2 \cdot 8 \cdot 4} = \frac{11}{16}.$$

The diagonal BD passes through O, so the angles $\angle AOB$ and $\angle AOD$ are supplementary, and $\cos \angle AOD = -\frac{11}{16}$. In triangle AOD,

$$AD^2 = AO^2 + OD^2 - 2 \cdot AO \cdot OD \cos \angle AOD = 64 + 36 + 2 \cdot 8 \cdot 6 \cdot \frac{11}{16} = 100 + 66 = 166.$$

Answer: (E) $\sqrt{166}$. □

4.2. **Angle Bisectors.**

Theorem 4.2.1 (Angle Bisector Theorem)**.** *If D is the point of side BC with $\angle BAD = \angle DAC$, then*

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

Proof. By the Law of Sines in triangles ABD and ADC,

$$\frac{BD}{\sin\angle BAD} = \frac{AB}{\sin\angle ADB}, \qquad \frac{DC}{\sin\angle DAC} = \frac{AC}{\sin\angle ADC}.$$

The angles at A are equal, and the angles at D are supplementary, so they have equal sines. Dividing the first equation by the second gives $\frac{BD}{DC} = \frac{AB}{AC}$. □

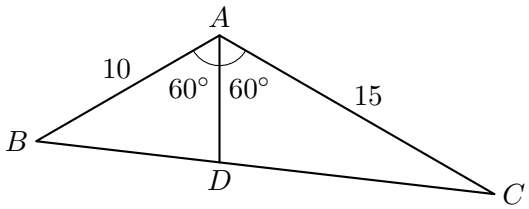
The length of the bisector AD can be found by splitting the area: $[ABD] + [ADC] = [ABC]$, where each area comes from the Area Formula, with the angle $\frac{A}{2}$ at A in the two smaller triangles. This is direct when $\sin\frac{A}{2}$ and $\sin A$ are both known, for example when $A = 120°$, where both equal $\frac{\sqrt{3}}{2}$. Stewart's Theorem gives the same length.

Example.

Problem 4.2.2. In triangle ABC, $AB = 10$, $AC = 15$, and $\angle BAC = 120°$. The bisector of angle BAC meets BC at D. Find BD and AD.

Strategy. Find BC with the Law of Cosines and divide it in the ratio given by the Angle Bisector Theorem. Find AD by splitting the area of the triangle.

Solution. The figure shows the triangle and its bisector.



By the Law of Cosines for a 120° angle,

$$BC^2 = 10^2 + 15^2 + 10 \cdot 15 = 475, \qquad \text{so} \qquad BC = 5\sqrt{19}.$$

By the Angle Bisector Theorem, $\frac{BD}{DC} = \frac{10}{15} = \frac{2}{3}$, so $BD = \frac{2}{5}BC = 2\sqrt{19}$ and $DC = 3\sqrt{19}$.

The bisector makes angles of 60° with AB and with AC. Since $[ABD] + [ADC] = [ABC]$, the Area Formula gives

$$\frac{1}{2} \cdot 10 \cdot AD \sin 60° + \frac{1}{2} \cdot 15 \cdot AD \sin 60° = \frac{1}{2} \cdot 10 \cdot 15 \sin 120°.$$

As $\sin 120° = \sin 60°$, this simplifies to $25\,AD = 150$, so $AD = 6$. Stewart's Theorem gives the same value: $15^2 \cdot 2\sqrt{19} + 10^2 \cdot 3\sqrt{19} = 5\sqrt{19}\,(AD^2 + 114)$, so $AD^2 = 36$.

Answer: $BD = 2\sqrt{19}$ and $AD = 6$. □

5. Problems

Beginner.

1. (1953 AHSME, Problem #34) If one side of a triangle is 12 inches and the opposite angle is 30 degrees, then the diameter of the circumscribed circle is:

(A) 18 inches **(B)** 30 inches **(C)** 24 inches **(D)** 20 inches
(E) none of these

2. (2021 Fall AMC 12B, Problem #8) The product of the lengths of the two congruent sides of an obtuse isosceles triangle is equal to the product of the base and twice the triangle's height to the base. What is the measure, in degrees, of the vertex angle of this triangle?

(A) 105 **(B)** 120 **(C)** 135 **(D)** 150 **(E)** 165

3. (2003 AMC 12A, Problem #11) A square and an equilateral triangle have the same perimeter. Let A be the area of the circle circumscribed about the square and B be the area of the circle circumscribed about the triangle. Find $\frac{A}{B}$.

(A) $\frac{9}{16}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{27}{32}$ **(D)** $\frac{3\sqrt{6}}{8}$ **(E)** 1

4. (2023 AMC 12B, Problem #17) Triangle ABC has side lengths in arithmetic progression, and the smallest side has length 6. If the triangle has an angle of 120°, what is the area of ABC?

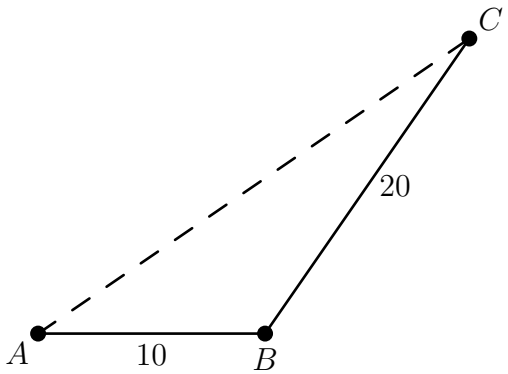
(A) $12\sqrt{3}$ **(B)** $8\sqrt{6}$ **(C)** $14\sqrt{2}$ **(D)** $20\sqrt{2}$ **(E)** $15\sqrt{3}$

5. (2014 AMC 12A, Problem #12) Two circles intersect at points A and B. The minor arcs AB measure 30° on one circle and 60° on the other circle. What is the ratio of the area of the larger circle to the area of the smaller circle?

(A) 2 **(B)** $1+\sqrt{3}$ **(C)** 3 **(D)** $2+\sqrt{3}$ **(E)** 4

Intermediate.

6. (2009 AMC 12A, Problem #13) A ship sails 10 miles in a straight line from A to B, turns through an angle between 45° and 60°, and then sails another 20 miles to C. Let AC be measured in miles. Which of the following intervals contains AC^2?



(A) $[400, 500]$ **(B)** $[500, 600]$ **(C)** $[600, 700]$ **(D)** $[700, 800]$ **(E)** $[800, 900]$

7. (2019 AMC 12A, Problem #19) In $\triangle ABC$ with integer side lengths,

$$\cos A = \frac{11}{16},$$
$$\cos B = \frac{7}{8},$$
$$\cos C = -\frac{1}{4}.$$

What is the least possible perimeter for $\triangle ABC$?

(A) 9 **(B)** 12 **(C)** 23 **(D)** 27 **(E)** 44

8. (1998 AHSME, Problem #26) In quadrilateral $ABCD$, it is given that $\angle A = 120^\circ$, angles B and D are right angles, $AB = 13$, and $AD = 46$. Then $AC =$

(A) 60 **(B)** 62 **(C)** 64 **(D)** 65 **(E)** 72

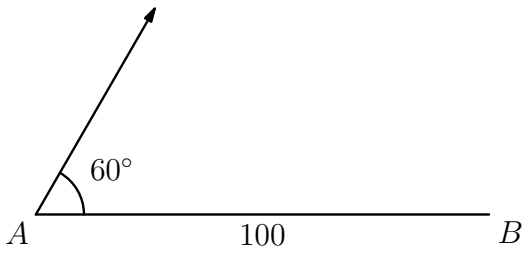
9. (1995 AHSME, Problem #23) The sides of a triangle have lengths 11, 15, and k, where k is an integer. For how many values of k is the triangle obtuse?

(A) 5 **(B)** 7 **(C)** 12 **(D)** 13 **(E)** 14

10. (2025 AMC 12B, Problem #21) Two non-congruent triangles have the same area. Each triangle has sides of length 8 and 9, and the third side of each triangle has integer length. What is the sum of the lengths of the third sides?

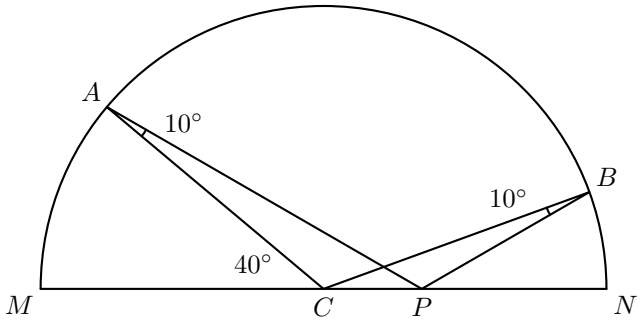
(A) 20 **(B)** 22 **(C)** 24 **(D)** 26 **(E)** 28

11. (1989 AIME, Problem #6) Two skaters, Allie and Billie, are at points A and B, respectively, on a flat, frozen lake. The distance between A and B is 100 meters. Allie leaves A and skates at a speed of 8 meters per second on a straight line that makes a 60° angle with AB. At the same time Allie leaves A, Billie leaves B at a speed of 7 meters per second and follows the straight path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?



Advanced.

12. (1983 AHSME, Problem #30) Distinct points A and B are on a semicircle with diameter MN and center C. The point P is on CN and $\angle CAP = \angle CBP = 10^\circ$. If $\overset{\frown}{MA} = 40^\circ$, then $\overset{\frown}{BN}$ equals



(A) 10° **(B)** 15° **(C)** 20° **(D)** 25° **(E)** 30°

13. (2003 AIME II, Problem #7) Find the area of rhombus $ABCD$ given that the radii of the circles circumscribed around triangles ABD and ACD are 12.5 and 25, respectively.

14. (2021 AIME II, Problem #5) For positive real numbers s, let $\tau(s)$ denote the set of all obtuse triangles that have area s and two sides with lengths 4 and 10. The set of all s for which $\tau(s)$ is nonempty, but all triangles in $\tau(s)$ are congruent, is an interval $[a, b)$. Find $a^2 + b^2$.

15. (2012 AIME I, Problem #12) Let $\triangle ABC$ be a right triangle with right angle at C. Let D and E be points on $\overline{AB}$ with D between A and E such that $\overline{CD}$ and $\overline{CE}$ trisect $\angle C$. If $\frac{DE}{BE} = \frac{8}{15}$, then $\tan B$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by the square of any prime. Find $m+n+p$.

16. (2013 AIME II, Problem #13) In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, the area of $\triangle ABC$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.