THE LAWS OF SINES AND COSINES FOR AMC 12: SOLUTIONS

ANDY JO

Contents

1. Solutions 2
2. Answer Key 11

1. Solutions

Beginner.

1. (1953 AHSME, Problem #34) If one side of a triangle is 12 inches and the opposite angle is 30 degrees, then the diameter of the circumscribed circle is:

(A) 18 inches **(B)** 30 inches **(C)** 24 inches **(D)** 20 inches **(E)** none of these

Strategy. The diameter of the circumscribed circle is $2R = \frac{a}{\sin A}$, by the Extended Law of Sines.

Solution. Let the given side be $a = 12$, opposite the angle $A = 30°$. By the Extended Law of Sines, the diameter of the circumscribed circle, in inches, is

$$2R = \frac{a}{\sin A} = \frac{12}{\sin 30°} = \frac{12}{1/2} = 24.$$

Answer: (C) 24 inches. □

2. (2021 Fall AMC 12B, Problem #8) The product of the lengths of the two congruent sides of an obtuse isosceles triangle is equal to the product of the base and twice the triangle's height to the base. What is the measure, in degrees, of the vertex angle of this triangle?

(A) 105 **(B)** 120 **(C)** 135 **(D)** 150 **(E)** 165

Strategy. Both products in the condition are tied to the area: the base times its height is twice the area, and by the Area Formula so is the square of a congruent side times the sine of the vertex angle.

Solution. Call the congruent sides s, the base t, the height h to the base, the vertex angle θ, and the area K. Then $th = 2K$, and the Area Formula with the two congruent sides gives $s^2 \sin\theta = 2K$. The condition says that

$$s^2 = t \cdot 2h = 4K = 2s^2 \sin\theta,$$

so $\sin\theta = \frac{1}{2}$, and θ is 30° or 150°. The two base angles are equal, so neither of them is obtuse, because a triangle has at most one angle that is not acute. The obtuse angle of the triangle is therefore the vertex angle, and $\theta = 150°$.

Answer: (D) 150. □

3. (2003 AMC 12A, Problem #11) A square and an equilateral triangle have the same perimeter. Let A be the area of the circle circumscribed about the square and B be the area of the circle circumscribed about the triangle. Find $\frac{A}{B}$.

(A) $\frac{9}{16}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{27}{32}$ **(D)** $\frac{3\sqrt{6}}{8}$ **(E)** 1

Strategy. The ratio does not depend on the common perimeter, and the perimeter 12 gives whole-number sides. A diagonal of the square is the common hypotenuse of two right triangles, which locates the circle about the square. The Extended Law of Sines gives the radius of the circle about the triangle.

Solution. Changing the common perimeter multiplies both radii by the same factor and does not change the ratio, so take the perimeter to be 12. Then the square has side $\frac{12}{4} = 3$ and each side of the triangle is $\frac{12}{3} = 4$. Name the square $PQRS$. Its diagonal $PR = 3\sqrt{2}$ is the hypotenuse of the right triangles PQR and PSR, so by the corollary to the Extended Law of Sines, the circumcircles of both triangles are the circle with diameter PR. This

circle passes through all four vertices, so it is the circle circumscribed about the square. Its radius is $r_1 = \frac{3\sqrt{2}}{2}$, so $A = \pi r_1^2 = \frac{9\pi}{2}$.

Each angle of the equilateral triangle is 60°, so by the Extended Law of Sines its circumradius r_2 satisfies

$$2r_2 = \frac{4}{\sin 60^\circ} = \frac{8}{\sqrt{3}}.$$

Thus $r_2 = \frac{4}{\sqrt{3}}$ and $B = \pi r_2^2 = \frac{16\pi}{3}$. Therefore

$$\frac{A}{B} = \frac{9\pi/2}{16\pi/3} = \frac{27}{32}.$$

Answer: (C) $\dfrac{27}{32}$. □

4. (2023 AMC 12B, Problem #17) Triangle ABC has side lengths in arithmetic progression, and the smallest side has length 6. If the triangle has an angle of 120°, what is the area of ABC?

(A) $12\sqrt{3}$ **(B)** $8\sqrt{6}$ **(C)** $14\sqrt{2}$ **(D)** $20\sqrt{2}$ **(E)** $15\sqrt{3}$

Strategy. With common difference d, all three sides depend on one unknown. An obtuse angle is the largest angle of its triangle and lies opposite the longest side, so the Law of Cosines for 120° applies to the side $6 + 2d$.

Solution. Write the sides as 6, $6 + d$, and $6 + 2d$ with $d \geq 0$. If $d = 0$, the triangle is equilateral and has no 120° angle, so $d > 0$. Then $6+2d$ is the longest side, and the obtuse angle of 120° is opposite it. By the Law of Cosines for a 120° angle,

$$(6 + 2d)^2 = 6^2 + (6 + d)^2 + 6(6 + d).$$

Expanding gives $36+24d+4d^2 = 108+18d+d^2$, so $3d^2+6d-72 = 0$, that is, $(d+6)(d-4) = 0$. Hence $d = 4$, so the triangle has sides 6, 10, and 14. The 120° angle lies between the sides 6 and 10, so by the Area Formula

$$[ABC] = \frac{1}{2} \cdot 6 \cdot 10 \sin 120^\circ = 30 \cdot \frac{\sqrt{3}}{2} = 15\sqrt{3}.$$

Answer: (E) $15\sqrt{3}$. □

5. (2014 AMC 12A, Problem #12) Two circles intersect at points A and B. The minor arcs AB measure 30° on one circle and 60° on the other circle. What is the ratio of the area of the larger circle to the area of the smaller circle?

(A) 2 **(B)** $1 + \sqrt{3}$ **(C)** 3 **(D)** $2 + \sqrt{3}$ **(E)** 4

Strategy. The chord AB is common to both circles. Write AB^2 with the Law of Cosines in the isosceles triangle formed by each center with A and B.

Solution. Let O_1 and O_2 be the centers and R_1 and R_2 the radii of the circles on which the minor arc AB measures 30° and 60°, respectively. The measure of an arc is its central angle, so $\angle AO_1B = 30^\circ$ and $\angle AO_2B = 60^\circ$. The Law of Cosines in the isosceles triangles AO_1B and AO_2B gives

$$AB^2 = 2R_1^2(1 - \cos 30^\circ) = 2R_2^2(1 - \cos 60^\circ).$$

Hence

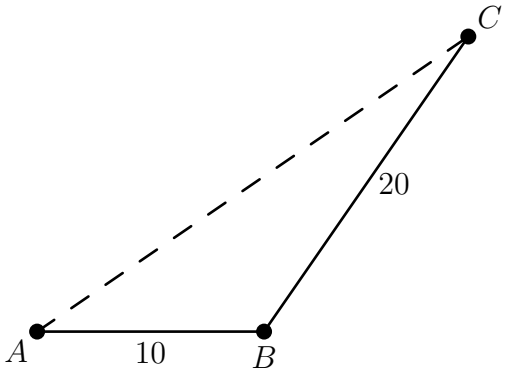
$$\frac{R_1^2}{R_2^2} = \frac{1 - \cos 60^\circ}{1 - \cos 30^\circ} = \frac{1/2}{1 - \sqrt{3}/2} = \frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}.$$

This is greater than 1, so the first circle is the larger one, and the ratio of the areas is $\frac{\pi R_1^2}{\pi R_2^2} = 2 + \sqrt{3}$.

Answer: (D) $2 + \sqrt{3}$. □

Intermediate.

6. (2009 AMC 12A, Problem #13) A ship sails 10 miles in a straight line from A to B, turns through an angle between 45° and 60°, and then sails another 20 miles to C. Let AC be measured in miles. Which of the following intervals contains AC^2?



(A) $[400, 500]$ **(B)** $[500, 600]$ **(C)** $[600, 700]$ **(D)** $[700, 800]$ **(E)** $[800, 900]$

Strategy. The turn at B is the angle between the extension of AB past B and BC, so the angle ABC of the triangle is its supplement. Bound its cosine.

Solution. Let t be the turn, so $45° < t < 60°$. The angle of triangle ABC at B is $\angle ABC = 180° - t$, so $120° < \angle ABC < 135°$. Cosine decreases from 0° to 180°, so

$$-\frac{\sqrt{2}}{2} = \cos 135° < \cos \angle ABC < \cos 120° = -\frac{1}{2}.$$

By the Law of Cosines,

$$AC^2 = 10^2 + 20^2 - 2 \cdot 10 \cdot 20 \cos \angle ABC = 500 - 400 \cos \angle ABC,$$

so $700 < AC^2 < 500 + 200\sqrt{2}$. Since $283^2 = 80089 > 80000 = \left(200\sqrt{2}\right)^2$, the upper bound is less than 783. Every possible value of AC^2 lies in $[700, 800]$, and no other interval among the choices contains all of them.

Answer: (D) $[700, 800]$. □

7. (2019 AMC 12A, Problem #19) In $\triangle ABC$ with integer side lengths,

$$\cos A = \frac{11}{16},$$
$$\cos B = \frac{7}{8},$$
$$\cos C = -\frac{1}{4}.$$

What is the least possible perimeter for $\triangle ABC$?

(A) 9 **(B)** 12 **(C)** 23 **(D)** 27 **(E)** 44

Strategy. Since the sine of an angle of a triangle is positive, each cosine determines its sine. The Law of Sines then fixes the ratio of the sides, and integer sides in a fixed ratio are smallest when the ratio is in lowest terms.

Solution. The sines of the angles of a triangle are positive, so

$$\sin A = \sqrt{1 - \frac{121}{256}} = \frac{3\sqrt{15}}{16},$$
$$\sin B = \sqrt{1 - \frac{49}{64}} = \frac{2\sqrt{15}}{16},$$
$$\sin C = \sqrt{1 - \frac{1}{16}} = \frac{4\sqrt{15}}{16}.$$

Hence $a : b : c = \sin A : \sin B : \sin C = 3 : 2 : 4$ by the Law of Sines, so $(a, b, c) = (3t, 2t, 4t)$ for some $t > 0$. Then $t = a - b$ is a positive integer, and the perimeter $9t$ is at least 9. The triangle with sides 3, 2, 4 has exactly the given cosines; for example, the Law of Cosines gives

$$\cos A = \frac{2^2 + 4^2 - 3^2}{2 \cdot 2 \cdot 4} = \frac{11}{16},$$

and $\cos B$ and $\cos C$ check in the same way. So the least possible perimeter is 9.

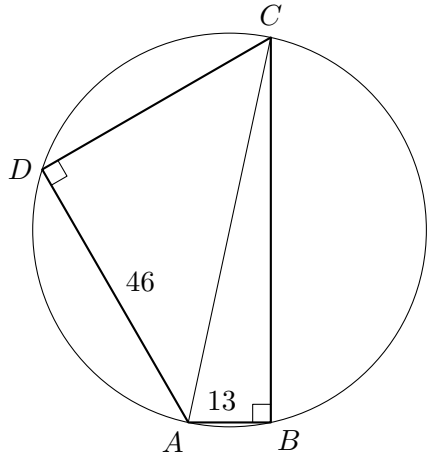
Answer: (A) 9. □

8. (1998 AHSME, Problem #26) In quadrilateral $ABCD$, it is given that $\angle A = 120°$, angles B and D are right angles, $AB = 13$, and $AD = 46$. Then $AC =$

(A) 60 **(B)** 62 **(C)** 64 **(D)** 65 **(E)** 72

Strategy. Triangles ABC and ADC are right triangles with the common hypotenuse AC. Their circumcircles are the same circle, which is also the circumcircle of triangle ABD.

Solution. The triangles ABC and ADC have right angles at B and D and share the hypotenuse AC. By the corollary to the Extended Law of Sines, the circumcircle of each is the circle with diameter AC, so this one circle passes through A, B, C, and D.



This circle is therefore the circumcircle of triangle ABD, and its diameter is AC. In triangle ABD, the Law of Cosines for the 120° angle at A gives

$$BD^2 = 13^2 + 46^2 + 13 \cdot 46 = 169 + 2116 + 598 = 2883 = 3 \cdot 31^2,$$

so $BD = 31\sqrt{3}$. By the Extended Law of Sines in triangle ABD,

$$AC = 2R = \frac{BD}{\sin 120°} = \frac{31\sqrt{3}}{\sqrt{3}/2} = 62.$$

Answer: (B) 62. □

9. (1995 AHSME, Problem #23) The sides of a triangle have lengths 11, 15, and k, where k is an integer. For how many values of k is the triangle obtuse?

(A) 5 **(B)** 7 **(C)** 12 **(D)** 13 **(E)** 14

Strategy. An obtuse angle faces the longest side. Split into cases according to which side is longest, and use the Obtuse Angle Test.

Solution. By the triangle inequality, $15 - 11 < k < 15 + 11$, so $5 \le k \le 25$.

If $k \ge 16$, the longest side is k. The triangle is obtuse exactly when $k^2 > 11^2 + 15^2$, that is, $k^2 > 346$. Since $18^2 = 324$ and $19^2 = 361$, this means $k \ge 19$, which gives the seven values $19, 20, \ldots, 25$.

If $k \le 15$, a longest side is 15. The triangle is obtuse exactly when $15^2 > 11^2 + k^2$, that is, $k^2 < 104$. Since $10^2 = 100$ and $11^2 = 121$, this means $k \le 10$, which gives the six values $5, 6, \ldots, 10$.

In all, $7 + 6 = 13$ values of k make the triangle obtuse.

Answer: (D) 13. □

10. (2025 AMC 12B, Problem #21) Two non-congruent triangles have the same area. Each triangle has sides of length 8 and 9, and the third side of each triangle has integer length. What is the sum of the lengths of the third sides?

(A) 20 **(B)** 22 **(C)** 24 **(D)** 26 **(E)** 28

Strategy. Equal areas mean equal sines of the angles between the sides 8 and 9. Triangles that are not congruent must have supplementary angles there.

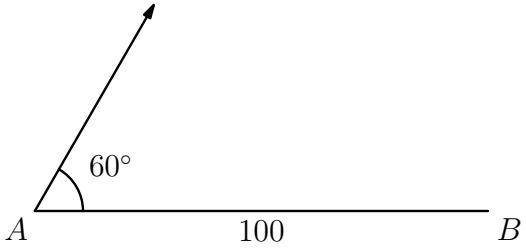
Solution. Let θ_1 and θ_2 be the angles between the sides 8 and 9 in the two triangles. By the Area Formula, the areas are $\frac{1}{2} \cdot 8 \cdot 9 \sin\theta_1 = 36 \sin\theta_1$ and $36 \sin\theta_2$, so $\sin\theta_1 = \sin\theta_2$. If $\theta_1 = \theta_2$, the triangles would be congruent by SAS. Hence $\theta_2 = 180^\circ - \theta_1$ and $\cos\theta_2 = -\cos\theta_1$. Call the third sides x_1 and x_2. The Law of Cosines in the two triangles gives

$$x_1^2 = 145 - 144\cos\theta_1, \qquad x_2^2 = 145 + 144\cos\theta_1,$$

so $x_1^2 + x_2^2 = 290$. By the triangle inequality, each third side is between 1 and 17, so it is one of $2, 3, \ldots, 16$. Of the numbers $290 - x^2$ for these x, only $290 - 11^2 = 169 = 13^2$ and $290 - 13^2 = 121 = 11^2$ are perfect squares. So the third sides are 11 and 13, and indeed $\cos\theta_1 = \frac{145-121}{144} = \frac{1}{6}$ gives two such triangles. The sum of the third sides is 24.

Answer: (C) 24. □

11. (1989 AIME, Problem #6) Two skaters, Allie and Billie, are at points A and B, respectively, on a flat, frozen lake. The distance between A and B is 100 meters. Allie leaves A and skates at a speed of 8 meters per second on a straight line that makes a 60° angle with AB. At the same time Allie leaves A, Billie leaves B at a speed of 7 meters per second and follows the straight path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?



Strategy. Allie and Billie can both reach a point C of Allie's line at time t exactly when $AC = 8t$ and $BC = 7t$. The Law of Cosines at A then becomes a quadratic equation in t, as in the ambiguous case.

Solution. Let C be the meeting point and t the time, in seconds, from the start to the meeting. Allie has skated $AC = 8t$ along her line, and Billie, skating straight at 7 meters per second, has covered $BC = 7t$. Triangle ABC has the angle 60° at A, so by the Law of Cosines

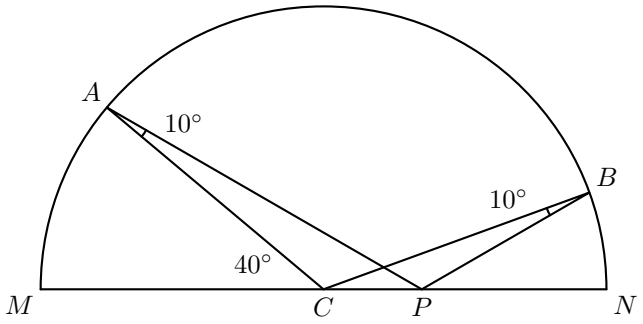
$$(7t)^2 = 100^2 + (8t)^2 - 2 \cdot 100 \cdot 8t \cos 60^\circ,$$

that is, $15t^2 - 800t + 10000 = 0$, or $3t^2 - 160t + 2000 = 0$. This factors as $(t-20)(3t-100) = 0$, so $t = 20$ or $t = \frac{100}{3}$. Conversely, for each of these times the point of Allie's line at distance $8t$ from A is at distance $7t$ from B, so Billie reaches it at the same moment by skating straight toward it. The smaller root, $t = 20$, is the first moment at which Billie can be where Allie is, and by then Allie has covered $8 \cdot 20 = 160$ meters.

Answer: 160. □

Advanced.

12. (1983 AHSME, Problem #30) Distinct points A and B are on a semicircle with diameter MN and center C. The point P is on CN and $\angle CAP = \angle CBP = 10^\circ$. If $\overset{\frown}{MA} = 40^\circ$, then $\overset{\frown}{BN}$ equals



(A) 10° **(B)** 15° **(C)** 20° **(D)** 25° **(E)** 30°

Strategy. In triangle CBP, the side CP faces the 10° angle at B and the radius CB faces the angle at P, so the Law of Sines gives the sine of that angle once CP is known as a multiple of the radius. The same law in triangle CAP gives CP, since its angle at C follows from the arc MA. Of the two angles with that sine, one would make B coincide with A.

Solution. The measure of an arc is its central angle, so $\angle MCA = 40^\circ$, and $\angle ACP = 180^\circ - 40^\circ = 140^\circ$ because P lies on CN. In triangle CAP the angles at A and C are 10° and 140°, so $\angle CPA = 30^\circ$. Let R be the radius. The side CP of triangle CAP is opposite the 10° angle and the side $CA = R$ is opposite the 30° angle, so by the Law of Sines

$$CP = \frac{R \sin 10^\circ}{\sin 30^\circ} = 2R \sin 10^\circ < R,$$

since $\sin 10^\circ < \sin 30^\circ = \frac{1}{2}$. Hence P is inside the circle. In triangle CBP the same side CP is opposite the 10° angle at B, and the side $CB = R$ is opposite the angle at P, so the Law of Sines gives

$$\sin \angle CPB = \frac{R \sin 10^\circ}{CP} = \frac{R \sin 10^\circ}{2R \sin 10^\circ} = \frac{1}{2},$$

and $\angle CPB$ is 30° or 150°.

Suppose $\angle CPB = 30°$. Then the rays PA and PB make the same angle with the ray PC and lie on the same side of the line MN, so they are the same ray. A ray from a point inside a circle meets the circle only once, so $A = B$, which contradicts the statement that A and B are distinct. Hence $\angle CPB = 150°$, and in triangle CBP,

$$\angle BCP = 180° - 10° - 150° = 20°.$$

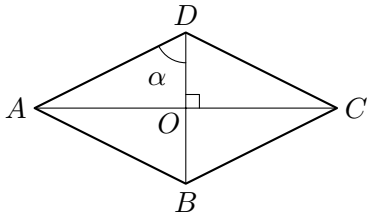
Since P lies on CN, $\angle BCN = 20°$, so the arc BN measures $20°$.

Answer: (C) $20°$. □

13. (2003 AIME II, Problem #7) Find the area of rhombus $ABCD$ given that the radii of the circles circumscribed around triangles ABD and ACD are 12.5 and 25, respectively.

Strategy. Apply $\frac{a}{\sin A} = 2R$ to the side AB in triangle ABD and to the side CD in triangle ACD. The two angles involved are complementary.

Solution. Let s be the side of the rhombus, and let its diagonals meet at O. The diagonals of a rhombus are perpendicular and bisect each other. Let $\alpha = \angle ADB$. In the right triangle AOD, the angle at A is $\angle OAD = 90° - \alpha$, and $\angle OAD = \angle CAD$ because O lies on AC.



In triangle ABD, the side AB is opposite the angle $\angle ADB = \alpha$, so by the Extended Law of Sines $\frac{s}{\sin\alpha} = 2 \cdot 12.5 = 25$. In triangle ACD, the side CD is opposite the angle $\angle CAD = 90° - \alpha$, so

$$\frac{s}{\sin(90° - \alpha)} = \frac{s}{\cos\alpha} = 2 \cdot 25 = 50.$$

Hence $\sin\alpha = \frac{s}{25}$ and $\cos\alpha = \frac{s}{50}$, and

$$\frac{s^2}{625} + \frac{s^2}{2500} = \sin^2\alpha + \cos^2\alpha = 1,$$

so $\frac{5s^2}{2500} = 1$ and $s^2 = 500$. In the right triangle AOD, $AO = s\sin\alpha = \frac{s^2}{25} = 20$ and $DO = s\cos\alpha = \frac{s^2}{50} = 10$. The diagonals are $AC = 40$ and $BD = 20$, and the area of the rhombus is $\frac{1}{2} \cdot AC \cdot BD = 400$.

Answer: 400. □

14. (2021 AIME II, Problem #5) For positive real numbers s, let $\tau(s)$ denote the set of all obtuse triangles that have area s and two sides with lengths 4 and 10. The set of all s for which $\tau(s)$ is nonempty, but all triangles in $\tau(s)$ are congruent, is an interval $[a, b)$. Find $a^2 + b^2$.

Strategy. The area fixes the sine of the angle between the sides 4 and 10, so there are at most two candidate triangles, with supplementary angles there. Decide when each of them is obtuse.

Solution. Let θ be the angle between the sides 4 and 10, and let x be the third side. By the Area Formula, $s = \frac{1}{2} \cdot 4 \cdot 10\sin\theta = 20\sin\theta$, and by the Law of Cosines, $x^2 = 116 - 80\cos\theta$. The angle opposite the side 4 is never obtuse, because 4 is not the longest side. So the triangle is obtuse exactly when θ is obtuse or the angle opposite the side 10 is obtuse. By

the Obtuse Angle Test, the second happens exactly when $10^2 > 4^2 + x^2$, that is, when $100 > 132 - 80\cos\theta$, or $\cos\theta > \frac{2}{5}$.

If $s > 20$, there is no such triangle. If $s = 20$, then $\theta = 90°$ and $x^2 = 116$, and the triangle is right, since $100 < 16 + 116$; so $\tau(20)$ is empty. If $0 < s < 20$, then $\sin\theta = \frac{s}{20}$, so θ is an acute angle θ_1 or its supplement $180° - \theta_1$. These give two triangles that are not congruent, since their third sides satisfy $x^2 = 116 \mp 80\cos\theta_1$. The triangle with the obtuse angle $180° - \theta_1$ is always obtuse. The triangle with the angle θ_1 is obtuse exactly when $\cos\theta_1 > \frac{2}{5}$, that is, when $\sin\theta_1 < \sqrt{1 - \frac{4}{25}} = \frac{\sqrt{21}}{5}$, or $s < 4\sqrt{21}$.

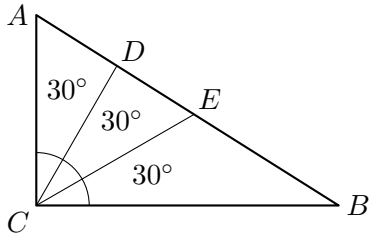
So for $0 < s < 4\sqrt{21}$ the set $\tau(s)$ contains two triangles that are not congruent, and for $4\sqrt{21} \le s < 20$ all of its triangles are congruent to one obtuse triangle. The interval is $[4\sqrt{21}, 20)$, and $a^2 + b^2 = 336 + 400 = 736$.

Answer: 736. □

15. (2012 AIME I, Problem #12) Let $\triangle ABC$ be a right triangle with right angle at C. Let D and E be points on $\overline{AB}$ with D between A and E such that $\overline{CD}$ and $\overline{CE}$ trisect $\angle C$. If $\frac{DE}{BE} = \frac{8}{15}$, then $\tan B$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by the square of any prime. Find $m+n+p$.

Strategy. Only triangle DCB is needed. Its angle at C is two of the three equal parts of the right angle, and CE halves it. The Angle Bisector Theorem then turns $DE : EB$ into $CD : CB$. This ratio, with the 60° angle between CD and CB, determines every angle of the triangle, including the angle at B.

Solution. Each of the angles $\angle ACD$, $\angle DCE$, and $\angle ECB$ is a third of the right angle at C, that is, 30°. So triangle DCB has the angle 60° at C, and CE is the bisector of that angle.



Since CE bisects the angle at C of triangle DCB, the Angle Bisector Theorem makes the sides at C proportional to the segments of DB:

$$\frac{CD}{CB} = \frac{DE}{BE} = \frac{8}{15},$$

so $CD = 8k$ and $CB = 15k$ for some $k > 0$. By the Law of Cosines for the 60° angle at C,

$$DB^2 = 64k^2 + 225k^2 - 120k^2 = 169k^2,$$

so $DB = 13k$. The angle of triangle DCB at B is the angle B of triangle ABC, since D lies on BA. By the Law of Cosines,

$$\cos B = \frac{CB^2 + DB^2 - CD^2}{2 \cdot CB \cdot DB} = \frac{225 + 169 - 64}{2 \cdot 15 \cdot 13} = \frac{330}{390} = \frac{11}{13}.$$

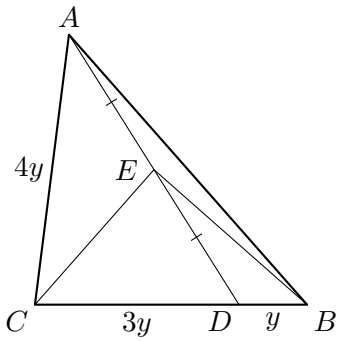
The sine of an angle of a triangle is positive, so $\sin B = \sqrt{1 - \cos^2 B} = \frac{\sqrt{48}}{13} = \frac{4\sqrt{3}}{13}$. Dividing by $\cos B = \frac{11}{13}$ gives $\tan B = \frac{4\sqrt{3}}{11}$, the required form with $m = 4$, $n = 11$, and $p = 3$, so $m + n + p = 18$.

Answer: 18. □

16. (2013 AIME II, Problem #13) In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, the area of $\triangle ABC$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Strategy. Because E is the midpoint of AD, the given lengths CE and BE are lengths of medians, in triangles ACD and ABD. Together with Stewart's Theorem for the cevian AD of triangle ABC, that makes three equations in three unknowns.

Solution. Let $BD = y$, so that $CD = 3y$ and $AC = BC = 4y$, and let $AD = d$ and $AB = c$.



The segment CE is the median from C in triangle ACD, and BE is the median from B in triangle ABD. By the median length formula,

$$7 = \frac{2(4y)^2 + 2(3y)^2 - d^2}{4}, \qquad 9 = \frac{2c^2 + 2y^2 - d^2}{4},$$

so $d^2 = 50y^2 - 28$ and $2c^2 = 36 - 2y^2 + d^2 = 48y^2 + 8$, that is, $c^2 = 24y^2 + 4$. Stewart's Theorem for the cevian AD of triangle ABC, with $m = BD = y$ and $n = DC = 3y$, gives

$$(4y)^2 \cdot y + c^2 \cdot 3y = 4y\,(d^2 + 3y^2), \qquad \text{so} \qquad 3c^2 + 4y^2 = 4d^2.$$

Substituting gives $72y^2 + 12 + 4y^2 = 200y^2 - 112$, so $124y^2 = 124$ and $y = 1$. Then $AC = BC = 4$ and $c^2 = 28$, so $AB = 2\sqrt{7}$. Because $CA = CB$, the altitude from C meets AB at its midpoint, so it has length $\sqrt{16 - 7} = 3$, and

$$[ABC] = \frac{1}{2} \cdot 2\sqrt{7} \cdot 3 = 3\sqrt{7}.$$

Hence $m = 3$, $n = 7$, and $m + n = 10$.

Answer: 10. □

2. Answer Key

Problem	Letter	Value
1	C	24 inches
2	D	150
3	C	$\frac{27}{32}$
4	E	$15\sqrt{3}$
5	D	$2+\sqrt{3}$
6	D	$[700, 800]$
7	A	9
8	B	62
9	D	13
10	C	24
11		160
12	C	20°
13		400
14		736
15		18
16		10