THE LAWS OF SINES AND COSINES FOR AMC 12: PRACTICE SET SOLUTIONS

ANDY JO

Contents

1. Solutions 2
2. Answer Key 11

1. Solutions

Beginner.

1. (2007 AMC 12A, Problem #10) A triangle with side lengths in the ratio $3 : 4 : 5$ is inscribed in a circle of radius 3. What is the area of the triangle?

(A) 8.64 **(B)** 12 **(C)** 5π **(D)** 17.28 **(E)** 18

Strategy. Since $3^2 + 4^2 = 5^2$, the triangle is a right triangle. The hypotenuse of a right triangle is a diameter of its circumcircle, and that fixes the size of the triangle.

Solution. Write the sides as $3k$, $4k$, and $5k$ with $k > 0$. Since $(3k)^2 + (4k)^2 = (5k)^2$, the Obtuse Angle Test shows that the angle opposite the side $5k$ is a right angle. By the Extended Law of Sines, the side opposite that angle is $2R \sin 90^\circ = 2R$, where $R = 3$ is the radius of the circle. So the hypotenuse is a diameter: $5k = 6$, and $k = \frac{6}{5}$. The legs are $3k = \frac{18}{5}$ and $4k = \frac{24}{5}$, and the area is

$$\frac{1}{2} \cdot \frac{18}{5} \cdot \frac{24}{5} = \frac{216}{25} = 8.64.$$

Answer: (A) 8.64. □

2. (1953 AHSME, Problem #28) In triangle ABC, sides a, b and c are opposite angles A, B and C respectively. $\overline{AD}$ bisects angle A and meets $\overline{BC}$ at D. Then if $x = CD$ and $y = BD$ the correct proportion is:

(A) $\frac{x}{a} = \frac{a}{b+c}$ **(B)** $\frac{x}{b} = \frac{a}{a+c}$ **(C)** $\frac{y}{c} = \frac{c}{b+c}$ **(D)** $\frac{y}{c} = \frac{a}{b+c}$ **(E)** $\frac{x}{y} = \frac{c}{b}$

Strategy. The Angle Bisector Theorem gives the ratio of y to x, and the two segments add up to a. Together these give x and y; the other four proportions hold only when two sides of the triangle are equal.

Solution. By the Angle Bisector Theorem,

$$\frac{BD}{DC} = \frac{AB}{AC}, \qquad \text{that is,} \qquad \frac{y}{x} = \frac{c}{b}.$$

So $x = bt$ and $y = ct$ for some $t > 0$. The point D lies on BC, so $x + y = a$, which gives $(b + c)t = a$. Hence

$$x = \frac{ab}{b+c}, \qquad y = \frac{ac}{b+c},$$

and dividing y by c gives $\frac{y}{c} = \frac{a}{b+c}$, which is (D). The other proportions do not hold in every triangle. The same formulas give

$$\frac{x}{a} = \frac{b}{b+c}, \qquad \frac{x}{b} = \frac{a}{b+c}, \qquad \frac{y}{c} = \frac{a}{b+c}, \qquad \frac{x}{y} = \frac{b}{c},$$

so (A) and (B) hold only when $a = b$, (C) only when $a = c$, and (E) only when $b = c$.

Answer: (D) $\frac{y}{c} = \frac{a}{b+c}$. □

3. (1958 AHSME, Problem #36) The sides of a triangle are 30, 70, and 80 units. If an altitude is dropped upon the side of length 80, the larger segment cut off on this side is:

(A) 62 **(B)** 63 **(C)** 64 **(D)** 65 **(E)** 66

Strategy. Find the cosines of the two angles at the ends of the side 80 from the three sides. Both are positive, so the foot of the altitude lies between the ends, and each segment is the adjacent side times the cosine of the angle at its end.

Solution. Name the triangle ABC with $BC = 80$, $AB = 70$, and $CA = 30$, and let H be the foot of the altitude from A. By the Law of Cosines,

$$\cos B = \frac{80^2 + 70^2 - 30^2}{2 \cdot 80 \cdot 70} = \frac{10400}{11200} = \frac{13}{14}, \qquad \cos C = \frac{80^2 + 30^2 - 70^2}{2 \cdot 80 \cdot 30} = \frac{2400}{4800} = \frac{1}{2}.$$

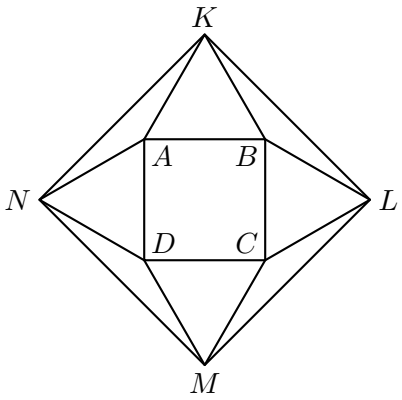
Both cosines are positive, so the angles B and C are acute, and H lies between B and C. In the right triangles ABH and ACH,

$$BH = AB \cos B = 70 \cdot \frac{13}{14} = 65, \qquad CH = AC \cos C = 30 \cdot \frac{1}{2} = 15.$$

As a check, $65 + 15 = 80$. The larger segment is 65.

Answer: (D) 65. □

4. (2003 AMC 12A, Problem #14) Points K, L, M, and N lie in the plane of the square $ABCD$ so that AKB, BLC, CMD, and DNA are equilateral triangles. If $ABCD$ has an area of 16, find the area of $KLMN$.



(A) 32 **(B)** $16 + 16\sqrt{3}$ **(C)** 48 **(D)** $32 + 16\sqrt{3}$ **(E)** 64

Strategy. The angle KBL is what remains of the full turn about B after the two 60° angles of the equilateral triangles and the 90° angle of the square. The Law of Cosines in triangle KBL then gives the side of $KLMN$.

Solution. The square has area 16, so its side is 4, and $BK = BL = 4$ because the triangles AKB and BLC are equilateral. As in the figure, the triangles lie outside the square, so the angles $\angle KBA = 60^\circ$, $\angle ABC = 90^\circ$, and $\angle CBL = 60^\circ$ do not overlap, and

$$\angle KBL = 360^\circ - 60^\circ - 90^\circ - 60^\circ = 150^\circ.$$

By the Law of Cosines in triangle KBL, with $\cos 150^\circ = -\frac{\sqrt{3}}{2}$,

$$KL^2 = 4^2 + 4^2 - 2 \cdot 4 \cdot 4 \cos 150^\circ = 32 + 16\sqrt{3}.$$

A quarter turn about the center of the square takes A, B, C, D to B, C, D, A. It therefore takes each equilateral triangle to the next one, and K, L, M, N to L, M, N, K. Hence $KLMN$ is a square, and its area is $KL^2 = 32 + 16\sqrt{3}$.

Answer: (D) $32 + 16\sqrt{3}$. □

5. (2020 AMC 12A, Problem #14) Regular octagon $ABCDEFGH$ has area n. Let m be the area of quadrilateral $ACEG$. What is $\frac{m}{n}$?

(A) $\frac{\sqrt{2}}{4}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{3\sqrt{2}}{5}$ **(E)** $\frac{2\sqrt{2}}{3}$

Strategy. Draw the radii from the center of the octagon to its vertices. They cut the octagon, and also the quadrilateral $ACEG$, into isosceles triangles whose areas come from the Area Formula.

Solution. Let O be the center of the octagon and R the radius of the circle through its vertices. The radii to the 8 vertices cut the octagon into 8 congruent isosceles triangles, each with two sides R and the angle $\frac{360^\circ}{8} = 45^\circ$ between them. By the Area Formula,

$$n = 8 \cdot \frac{1}{2} R^2 \sin 45^\circ = 2\sqrt{2}\, R^2.$$

The vertices of $ACEG$ lie on the same circle, and the angle between the radii to two consecutive ones, such as OA and OC, is $2 \cdot 45^\circ = 90^\circ$. So the radii OA, OC, OE, and OG cut $ACEG$ into 4 isosceles triangles, each with two sides R and the angle 90° between them, and

$$m = 4 \cdot \frac{1}{2} R^2 \sin 90^\circ = 2R^2.$$

Therefore

$$\frac{m}{n} = \frac{2R^2}{2\sqrt{2}\, R^2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Answer: (B) $\dfrac{\sqrt{2}}{2}$. □

Intermediate.

6. (1988 AIME, Problem #7) In triangle ABC, $\tan \angle CAB = \frac{22}{7}$, and the altitude from A divides BC into segments of length 3 and 17. What is the area of triangle ABC?

Strategy. Let h be the altitude. Twice the area gives $AB \cdot AC \sin A$ in terms of h, and the Law of Cosines at A gives $AB \cdot AC \cos A$. Their quotient is $\tan A$, which is known.

Solution. Let H be the foot of the altitude from A and $h = AH$. Name the vertices so that $BH = 3$ and $HC = 17$; the other order gives the mirror image, with the same area. Then $BC = 20$, and the Pythagorean theorem in the right triangles ABH and ACH gives $AB^2 = h^2 + 9$ and $AC^2 = h^2 + 289$.

The area of the triangle is $\frac{1}{2} \cdot 20 \cdot h = 10h$, so by the Area Formula

$$\frac{1}{2}\, AB \cdot AC \sin A = 10h, \qquad \text{that is,} \qquad AB \cdot AC \sin A = 20h.$$

By the Law of Cosines at A,

$$400 = AB^2 + AC^2 - 2\, AB \cdot AC \cos A = 2h^2 + 298 - 2\, AB \cdot AC \cos A,$$

which gives

$$AB \cdot AC \cos A = h^2 - 51.$$

Since $\tan A = \frac{22}{7} > 0$, the angle A is acute, so $h^2 - 51 > 0$. Dividing the two equations gives

$$\frac{22}{7} = \tan A = \frac{\sin A}{\cos A} = \frac{20h}{h^2 - 51}.$$

Clearing denominators gives $22h^2 - 1122 = 140h$, that is, $11h^2 - 70h - 561 = 0$, which factors as $(h - 11)(11h + 51) = 0$. Since $h > 0$, $h = 11$, and the area is $10h = 110$.

Answer: 110. □

7. (1967 AHSME, Problem #13) A triangle ABC is to be constructed given side a (opposite angle A), angle B, and h_c, the altitude from C. If N is the number of noncongruent solutions, then N

(A) is 1 **(B)** is 2 **(C)** must be zero **(D)** must be infinite
(E) must be zero or infinite

Strategy. As in the ambiguous case, look at the distance from C to the line on which the third vertex must lie. The side a and the angle B already fix this distance, so the altitude h_c either contradicts them or adds nothing.

Solution. Draw the angle B and put C on one of its rays with $BC = a$. The vertex A must be a point of the other ray, other than B. Wherever A is on that ray, the altitude from C is the distance from C to the line BA, which is $BC \sin B = a \sin B$, whether B is acute, right, or obtuse. So the altitude from C does not depend on the position of A.

If $h_c \neq a \sin B$, no triangle has the given parts, and $N = 0$. If $h_c = a \sin B$, every point A of the ray gives a triangle with the given parts. Its area $\frac{1}{2} \cdot BA \cdot a \sin B$ grows with BA, so different positions of A give triangles that are not congruent, and N is infinite.

Both cases occur, depending on the given values. So N is never 1 or 2, and it is neither always zero nor always infinite: it must be zero or infinite.

Answer: (E) must be zero or infinite. □

8. (1973 AHSME, Problem #15) A sector with acute central angle θ is cut from a circle of radius 6. The radius of the circle circumscribed about the sector is

(A) $3\cos\theta$ **(B)** $3\sec\theta$ **(C)** $3\cos\frac{1}{2}\theta$ **(D)** $3\sec\frac{1}{2}\theta$ **(E)** 3

Strategy. The circle circumscribed about the sector passes through the center of the original circle and the two ends of the arc, so it is the circumcircle of an isosceles triangle with two sides 6. Apply the Extended Law of Sines to one of these sides.

Solution. Let O be the center of the circle of radius 6, and let A and B be the ends of the arc of the sector. The circle circumscribed about the sector passes through O, A, and B, so it is the circumcircle of triangle OAB; let R be its radius. In triangle OAB, $OA = OB = 6$ and $\angle AOB = \theta$, so the base angles are

$$\angle OAB = \angle OBA = \frac{180^\circ - \theta}{2} = 90^\circ - \frac{\theta}{2}.$$

The side OA is opposite the angle at B, so by the Extended Law of Sines and the complementary-angle identity $\sin(90^\circ - t) = \cos t$,

$$2R = \frac{OA}{\sin\angle OBA} = \frac{6}{\sin\left(90^\circ - \frac{\theta}{2}\right)} = \frac{6}{\cos\frac{\theta}{2}}.$$

Hence $R = \dfrac{3}{\cos\frac{1}{2}\theta} = 3\sec\frac{1}{2}\theta$.

This circle does contain the whole sector. Triangle OAB is acute, so the center Q of the circle lies inside it, and since $QA = QB$ and $OA = OB$, both Q and O lie on the perpendicular bisector of AB, which bisects the angle AOB. For a point X of the arc, $\angle QOX \leq \frac{\theta}{2}$, so $\cos\angle QOX \geq \cos\frac{\theta}{2} = \frac{3}{R}$, and the Law of Cosines in triangle QOX gives

$$QX^2 = R^2 + 36 - 12R\cos\angle QOX \leq R^2 + 36 - 36 = R^2.$$

So every point of the arc lies inside or on the circle. The disk bounded by the circle also contains O, so it contains each segment OX, and these segments make up the sector.

Answer: (D) $3\sec\frac{1}{2}\theta$. □

9. (2018 AMC 10B, Problem #22) Real numbers x and y are chosen independently and uniformly at random from the interval $[0, 1]$. Which of the following numbers is closest to the probability that x, y, and 1 are the side lengths of an obtuse triangle?

(A) 0.21 **(B)** 0.25 **(C)** 0.29 **(D)** 0.50 **(E)** 0.79

Strategy. The numbers x, y, 1 form a triangle when $x + y > 1$, and then 1 is a longest side. By the Obtuse Angle Test the triangle is obtuse exactly when $x^2 + y^2 < 1$, so the probability is the area of a region in the unit square.

Solution. The point (x, y) is chosen uniformly from the unit square, which has area 1, so the probability is the area of the set of points that satisfy the conditions. Lines and single points have area 0 and can be ignored. Since $x \le 1$ and $y \le 1$, the inequalities $x + 1 > y$ and $y + 1 > x$ hold except at the corners $(0, 1)$ and $(1, 0)$, so x, y, and 1 are the sides of a triangle exactly when $x + y > 1$. In such a triangle 1 is a longest side, and an obtuse angle can only face a longest side, so by the Obtuse Angle Test the triangle is obtuse exactly when $1 > x^2 + y^2$.

These points form the part of the quarter disk $x^2 + y^2 < 1$, $x \ge 0$, $y \ge 0$, that lies above the line $x + y = 1$. The quarter disk has area $\frac{\pi}{4}$, and its part on or below the line is the right triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$, which has area $\frac{1}{2}$. So the probability is

$$\frac{\pi}{4} - \frac{1}{2}.$$

Since $3.14 < \pi < 3.15$, this number lies between 0.285 and 0.2875, and of the choices, 0.29 is the closest.

Answer: (C) 0.29. □

10. (1980 AHSME, Problem #23) Line segments drawn from the vertex opposite the hypotenuse of a right triangle to the points trisecting the hypotenuse have lengths $\sin x$ and $\cos x$, where x is a real number such that $0 < x < \frac{\pi}{2}$. The length of the hypotenuse is

(A) $\frac{4}{3}$ **(B)** $\frac{3}{2}$ **(C)** $\frac{3\sqrt{5}}{5}$ **(D)** $\frac{2\sqrt{5}}{3}$
(E) not uniquely determined by the given information

Strategy. Stewart's Theorem gives the squares of both segments in terms of the legs and the hypotenuse. Their sum is $\sin^2 x + \cos^2 x = 1$.

Solution. Call the triangle ABC, with the right angle at C, legs $a = BC$ and $b = CA$, and hypotenuse $c = AB$, so that $a^2 + b^2 = c^2$. Let D and E be the points of AB with $AD = DE = EB = \frac{c}{3}$. Stewart's Theorem for the cevian CD, with $AD = \frac{c}{3}$ and $DB = \frac{2c}{3}$, gives

$$b^2 \cdot \frac{2c}{3} + a^2 \cdot \frac{c}{3} = c\left(CD^2 + \frac{c}{3} \cdot \frac{2c}{3}\right), \qquad \text{so} \qquad CD^2 = \frac{2b^2 + a^2}{3} - \frac{2c^2}{9}.$$

In the same way, with $AE = \frac{2c}{3}$ and $EB = \frac{c}{3}$,

$$CE^2 = \frac{b^2 + 2a^2}{3} - \frac{2c^2}{9}.$$

Adding gives $CD^2 + CE^2 = a^2 + b^2 - \frac{4c^2}{9} = \frac{5c^2}{9}$. The segments CD and CE have lengths $\sin x$ and $\cos x$ in some order, so by the Pythagorean identity $CD^2 + CE^2 = 1$. Hence $c^2 = \frac{9}{5}$, and $c = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$.

So every such triangle has this hypotenuse, and such triangles exist: in the isosceles right triangle with hypotenuse $\frac{3\sqrt{5}}{5}$, both segments have length $\frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} = \cos\frac{\pi}{4}$. The hypotenuse is therefore determined, and it is $\frac{3\sqrt{5}}{5}$.

Answer: (C) $\frac{3\sqrt{5}}{5}$. □

Advanced.

11. (1962 AHSME, Problem #39) The medians AN and BP of a triangle with unequal sides are, respectively, 3 inches and 6 inches long. Its area is $3\sqrt{15}$ square inches. The length of the third median, in inches, is:

(A) 4 **(B)** $3\sqrt{3}$ **(C)** $3\sqrt{6}$ **(D)** $6\sqrt{3}$ **(E)** $6\sqrt{6}$

Strategy. Work in triangle ABG, where G is the centroid. Two of its sides are known, its area is a third of the whole, and its median from G is a third of the third median. The area fixes only the sine of the angle at G, which leaves two cases, and the condition that the sides are unequal rules one of them out.

Solution. Let G be the centroid, where the medians meet, and let M be the midpoint of AB, so that the third median is CM. The centroid divides each median in the ratio $2:1$ from the vertex, so $AG = \frac{2}{3}\cdot 3 = 2$, $BG = \frac{2}{3}\cdot 6 = 4$, and $GM = \frac{1}{3}CM$. The distance from G to the line AB is therefore a third of the distance from C, and $[ABG] = \frac{1}{3}[ABC] = \sqrt{15}$.

Let $\varphi = \angle AGB$. By the Area Formula, $\frac{1}{2}\cdot 2\cdot 4\sin\varphi = \sqrt{15}$, so $\sin\varphi = \frac{\sqrt{15}}{4}$. The angle φ may be acute or obtuse, so by the Pythagorean identity $\cos\varphi = \frac{1}{4}$ or $\cos\varphi = -\frac{1}{4}$. By the Law of Cosines in triangle ABG,

$$AB^2 = 2^2 + 4^2 - 2\cdot 2\cdot 4\cos\varphi = 20 - 16\cos\varphi = 16 \quad \text{or} \quad 24.$$

The segment GM is the median from G in triangle ABG, so by the median length formula

$$GM^2 = \frac{2\cdot AG^2 + 2\cdot BG^2 - AB^2}{4} = \frac{40 - AB^2}{4} = 6 \quad \text{or} \quad 4,$$

and the third median is $CM = 3\,GM = 3\sqrt{6}$ or 6.

The median length formula also shows that two medians are equal only when the sides they go to are equal: the medians m_b and m_c to the sides b and c satisfy

$$m_b^2 - m_c^2 = \frac{2a^2 + 2c^2 - b^2}{4} - \frac{2a^2 + 2b^2 - c^2}{4} = \frac{3}{4}\left(c^2 - b^2\right).$$

If the third median were 6, it would equal the median BP, and two sides of the triangle would be equal. The sides are unequal, so the third median is $3\sqrt{6}$.

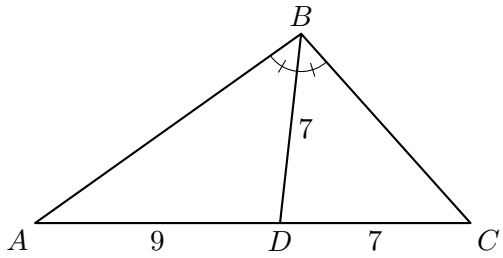
Answer: (C) $3\sqrt{6}$. □

12. (2002 AMC 12A, Problem #23) In triangle ABC, side $\overline{AC}$ and the perpendicular bisector of $\overline{BC}$ meet in point D, and $\overline{BD}$ bisects $\angle ABC$. If $AD = 9$ and $DC = 7$, what is the area of triangle ABD?

(A) 14 **(B)** 21 **(C)** 28 **(D)** $14\sqrt{5}$ **(E)** $28\sqrt{5}$

Strategy. A point of the perpendicular bisector of BC is as far from B as from C, so $DB = 7$. The Angle Bisector Theorem gives the ratio $AB:BC$, and Stewart's Theorem for the cevian BD fixes the scale. After that, all three sides of triangle ABD are known.

Solution. Every point of the perpendicular bisector of BC is equidistant from B and C, and D is such a point, so $DB = DC = 7$. The figure shows the triangle, with the equal angles at B marked.



Since BD bisects the angle ABC and D lies on AC, the Angle Bisector Theorem gives

$$\frac{AB}{BC} = \frac{AD}{DC} = \frac{9}{7},$$

so $AB = 9k$ and $BC = 7k$ for some $k > 0$. Stewart's Theorem for the cevian BD of triangle ABC reads

$$AB^2 \cdot DC + BC^2 \cdot AD = AC\left(BD^2 + AD \cdot DC\right).$$

With $AD = 9$, $DC = 7$, $AC = 16$, and $BD = 7$, this is

$$81k^2 \cdot 7 + 49k^2 \cdot 9 = 16\,(49 + 63),$$

so $1008k^2 = 1792$, $k^2 = \frac{16}{9}$, $k = \frac{4}{3}$, and $AB = 12$.

Triangle ABD has sides $AB = 12$, $AD = 9$, and $BD = 7$. By the Law of Cosines,

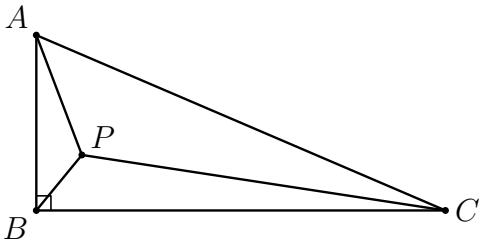
$$\cos A = \frac{12^2 + 9^2 - 7^2}{2 \cdot 12 \cdot 9} = \frac{176}{216} = \frac{22}{27}.$$

The sine of an angle of a triangle is positive, so $\sin A = \sqrt{1 - \frac{484}{729}} = \frac{\sqrt{245}}{27} = \frac{7\sqrt{5}}{27}$. By the Area Formula,

$$[ABD] = \frac{1}{2} \cdot 12 \cdot 9 \cdot \frac{7\sqrt{5}}{27} = 14\sqrt{5}.$$

Answer: (D) $14\sqrt{5}$. □

13. (1987 AIME, Problem #9) Triangle ABC has a right angle at B and contains a point P for which $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.



Strategy. The three equal angles at P make up a full turn, so each is 120°. The Law of Cosines for a 120° angle gives the three sides of triangle ABC in terms of PC, and the right angle at B gives an equation.

Solution. The point P is inside the triangle, so the angles $\angle APB$, $\angle BPC$, and $\angle CPA$ together make up the full turn about P. They are equal, so each is 120°. Let $x = PC$. The Law of Cosines for a 120° angle, $c^2 = a^2 + b^2 + ab$, in the triangles APB, BPC, and CPA gives

$$AB^2 = 10^2 + 6^2 + 10 \cdot 6 = 196, \quad BC^2 = 6^2 + x^2 + 6x, \quad CA^2 = x^2 + 10^2 + 10x.$$

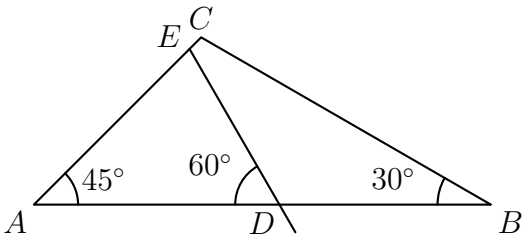
The angle at B is a right angle, so $CA^2 = AB^2 + BC^2$, that is,

$$x^2 + 10x + 100 = 196 + x^2 + 6x + 36.$$

Hence $4x = 132$ and $x = 33$.

Answer: 33. □

14. (1987 AHSME, Problem #30) In the figure, $\triangle ABC$ has $\angle A = 45°$ and $\angle B = 30°$. A line DE, with D on AB and $\angle ADE = 60°$, divides $\triangle ABC$ into two pieces of equal area. (Note: the figure may not be accurate; perhaps E is on CB instead of AC.) The ratio $\frac{AD}{AB}$ is



(A) $\frac{1}{\sqrt{2}}$ **(B)** $\frac{2}{2+\sqrt{2}}$ **(C)** $\frac{1}{\sqrt{3}}$ **(D)** $\frac{1}{\sqrt[3]{6}}$ **(E)** $\frac{1}{\sqrt[4]{12}}$

Strategy. All lines that make the angle 60° with AB in this way are parallel, and one of them passes through C. Comparing the area that this one cuts off with half the triangle shows that E lies on AC. Then write $[ADE]$ and $[ABC]$ with the Area Formula at the common angle A, and AE and AC with the Law of Sines.

Solution. The third angle of triangle ABC is $C = 180° - 45° - 30° = 105°$. Let D_0 be the point of AB with $\angle AD_0C = 60°$. Triangle AD_0C has angles 45° at A, 60° at D_0, and 75° at C. By the Law of Sines in triangles AD_0C and ABC,

$$AD_0 = \frac{AC \sin 75°}{\sin 60°}, \qquad AB = \frac{AC \sin 105°}{\sin 30°}.$$

Since $\sin 105° = \sin 75°$, dividing gives $\frac{AD_0}{AB} = \frac{\sin 30°}{\sin 60°} = \frac{1}{\sqrt{3}}$. The triangles AD_0C and ABC have the same height from C, so $[AD_0C] = \frac{1}{\sqrt{3}}[ABC]$, which is more than half of $[ABC]$. Every line DE with $\angle ADE = 60°$ is parallel to D_0C. If D were on D_0B, the piece at A would contain triangle AD_0C and would have more than half the area. So D lies between A and D_0, the line meets AC at E, and the piece at A is triangle ADE, with angles 45° at A, 60° at D, and 75° at E.

By the Area Formula with the common angle A,

$$\frac{[ADE]}{[ABC]} = \frac{\frac{1}{2} AD \cdot AE \sin 45°}{\frac{1}{2} AB \cdot AC \sin 45°} = \frac{AD \cdot AE}{AB \cdot AC}.$$

By the Law of Sines in triangles ADE and ABC,

$$AE = \frac{AD \sin 60°}{\sin 75°}, \qquad AC = \frac{AB \sin 30°}{\sin 105°} = \frac{AB \sin 30°}{\sin 75°}.$$

Hence

$$\frac{1}{2} = \frac{[ADE]}{[ABC]} = \frac{AD^2 \sin 60°}{AB^2 \sin 30°} = \sqrt{3} \left(\frac{AD}{AB}\right)^2,$$

so $\left(\frac{AD}{AB}\right)^2 = \frac{1}{2\sqrt{3}} = \frac{1}{\sqrt{12}}$, and $\frac{AD}{AB} = \frac{1}{\sqrt[4]{12}}$.

Answer: (E) $\frac{1}{\sqrt[4]{12}}$. □

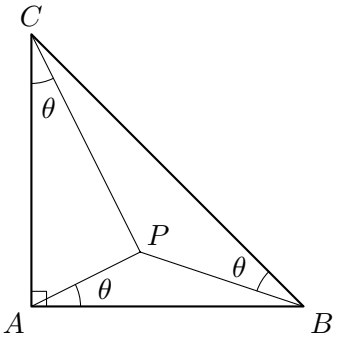
15. (2023 AIME II, Problem #3) Let $\triangle ABC$ be an isosceles triangle with $\angle A = 90°$. There exists a point P inside $\triangle ABC$ such that $\angle PAB = \angle PBC = \angle PCA$ and $AP = 10$. Find the area of $\triangle ABC$.

Strategy. Let θ be the common angle and chase the angles: triangles PAB and PBC both have angles θ, $45° - \theta$, and $135°$. The Law of Sines in these two triangles relates PA, PB, and PC to the sides AB and BC, whose ratio is $\sqrt{2}$.

Solution. Let θ be the common angle. The triangle has a right angle at A and $AB = AC$, so $\angle ABC = \angle ACB = 45°$ and $BC = \sqrt{2}\,AB$. Since P is inside the triangle,

$$\angle PBA = 45° - \theta, \qquad \angle PCB = 45° - \theta, \qquad \angle PAC = 90° - \theta,$$

with $0° < \theta < 45°$. The figure is drawn to scale.



Triangle PAB has angles θ at A and $45° - \theta$ at B, so $\angle APB = 135°$. Triangle PBC has angles θ at B and $45° - \theta$ at C, so $\angle BPC = 135°$. Triangle PCA has angles θ at C and $90° - \theta$ at A, so $\angle CPA = 90°$.

By the Law of Sines in triangles PAB and PBC,

$$\frac{PA}{\sin(45° - \theta)} = \frac{PB}{\sin\theta} = \frac{AB}{\sin 135°}, \qquad \frac{PB}{\sin(45° - \theta)} = \frac{PC}{\sin\theta} = \frac{BC}{\sin 135°}.$$

The angles in the two lists are the same, so each side of triangle PBC is $\frac{BC}{AB} = \sqrt{2}$ times the side of triangle PAB opposite the same angle. Hence $PB = \sqrt{2}\,PA = 10\sqrt{2}$ and $PC = \sqrt{2}\,PB = 20$. The angle of triangle PCA at P is a right angle, so

$$AC^2 = PA^2 + PC^2 = 100 + 400 = 500,$$

and the area of triangle ABC is $\frac{1}{2}\,AB \cdot AC = \frac{1}{2}\,AC^2 = 250$.

Answer: 250. □

2. Answer Key

Problem	Letter	Value
1	A	8.64
2	D	$\frac{y}{c} = \frac{a}{b+c}$
3	D	65
4	D	$32 + 16\sqrt{3}$
5	B	$\frac{\sqrt{2}}{2}$
6		110
7	E	must be zero or infinite
8	D	$3 \sec \frac{1}{2}\theta$
9	C	0.29
10	C	$\frac{3\sqrt{5}}{5}$
11	C	$3\sqrt{6}$
12	D	$14\sqrt{5}$
13		33
14	E	$\frac{1}{\sqrt[4]{12}}$
15		250