

TRIGONOMETRIC RATIOS AND THE UNIT CIRCLE FOR AMC 12

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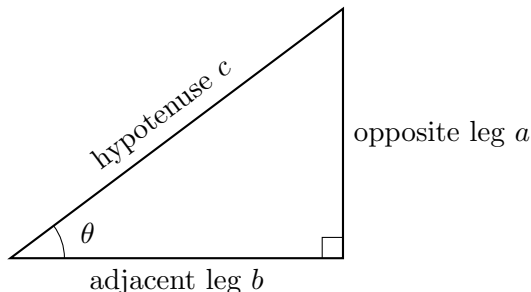
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1. RIGHT-TRIANGLE RATIOS

Right triangles with the same acute angle are similar, so the ratios of their sides depend only on that angle. This section defines these ratios and finds their values at 30° , 45° , and 60° .

1.1. Sine, Cosine, and Tangent. Fix an acute angle θ in a right triangle. The **hypotenuse** is the side opposite the right angle. Of the other two sides, the **opposite leg** is across from θ , and the **adjacent leg** is the leg that forms θ with the hypotenuse.



Definition 1.1.1. The **sine**, **cosine**, and **tangent** of θ are

$$\sin \theta = \frac{a}{c}, \quad \cos \theta = \frac{b}{c}, \quad \tan \theta = \frac{a}{b}.$$

In particular, $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

For the other acute angle, the opposite and adjacent legs exchange roles, and the hypotenuse stays the same. One ratio determines the others. For example, if $\sin \theta = \frac{5}{13}$, every right triangle with angle θ is a scaled copy of the triangle with opposite leg 5 and hypotenuse 13, whose adjacent leg is $\sqrt{13^2 - 5^2} = 12$. Hence $\cos \theta = \frac{12}{13}$ and $\tan \theta = \frac{5}{12}$.

Example.

Problem 1.1.2. Triangle ABC has a right angle at C , $AC = 12$, and $BC = 5$. Find $\sin A$, $\cos A$, and $\tan A$.

Strategy. Find the hypotenuse with the Pythagorean theorem. Then identify the opposite and adjacent legs relative to A .

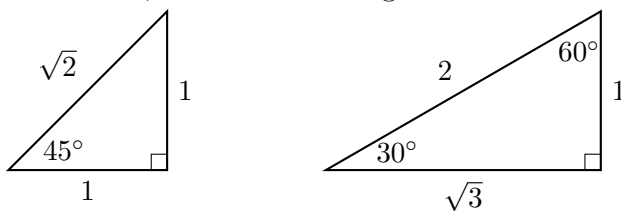
Solution. The hypotenuse is $AB = \sqrt{12^2 + 5^2} = 13$. Relative to angle A , the opposite leg is $BC = 5$ and the adjacent leg is $AC = 12$. Hence

$$\sin A = \frac{5}{13}, \quad \cos A = \frac{12}{13}, \quad \tan A = \frac{5}{12}.$$

Answer: $\sin A = \frac{5}{13}$, $\cos A = \frac{12}{13}$, and $\tan A = \frac{5}{12}$. □

1.2. Special Right Triangles. The diagonal of a square divides it into two 45° - 45° - 90° triangles. If each leg is 1, the Pythagorean theorem gives hypotenuse $\sqrt{2}$.

An altitude of an equilateral triangle of side 2 divides it into two 30° - 60° - 90° triangles. The shorter leg is 1, the hypotenuse is 2, and the other leg is $\sqrt{2^2 - 1^2} = \sqrt{3}$.



Taking ratios in these two triangles gives the exact values below.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Example.

Problem 1.2.1. A ladder of length 8 rests against a vertical wall and makes a 60° angle with level ground. How high above the ground does it reach, and how far is its foot from the wall?

Strategy. The ladder is the hypotenuse. The height is opposite the 60° angle, and the horizontal distance is adjacent to it.

Solution. Let the height be h and the horizontal distance be d . Then

$$\frac{h}{8} = \sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \frac{d}{8} = \cos 60^\circ = \frac{1}{2}.$$

Thus $h = 4\sqrt{3}$ and $d = 4$. These lengths satisfy $h^2 + d^2 = 48 + 16 = 64 = 8^2$, as the Pythagorean theorem requires.

Answer: Height $4\sqrt{3}$ and horizontal distance 4. □

2. ANGLE MEASURE

2.1. Degrees and Radians. A full turn is 360° . A **radian** measures an angle by the length of the arc it cuts off on a circle.

Definition 2.1.1. If a central angle cuts off an arc of length s on a circle of radius r , its radian measure is $\theta = s/r$. Equivalently, $s = r\theta$ when θ is measured in radians.

The circumference is $2\pi r$, so a full turn measures 2π radians. Thus

$$180^\circ = \pi \text{ radians}, \quad \theta \text{ degrees} = \frac{\pi\theta}{180} \text{ radians}.$$

To convert radians to degrees, multiply by $180/\pi$. An angle written without a degree symbol is measured in radians unless the problem states otherwise. For example, $\sin 30^\circ = \frac{1}{2}$, but $\sin 30$ does not equal $\frac{1}{2}$.

Degrees	0°	30°	45°	60°	90°	180°	270°	360°
Radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π

Example.

Problem 2.1.2. A central angle of 150° cuts off an arc on a circle of radius 6. Find the angle in radians and the arc length.

Strategy. Convert the angle to radians before using $s = r\theta$.

Solution. The radian measure is $150 \cdot \frac{\pi}{180} = \frac{5\pi}{6}$. Therefore the arc length is

$$s = 6 \cdot \frac{5\pi}{6} = 5\pi.$$

Answer: $\frac{5\pi}{6}$ radians and arc length 5π . □

2.2. Standard Position and Coterminal Angles. An angle in **standard position** has its vertex at the origin and starts on the positive x -axis. A positive angle turns counterclockwise, and a negative angle turns clockwise. The ray where the turn ends is the **terminal ray**. Angles with the same terminal ray are **coterminal**; they differ by a whole number of full turns.

Thus θ and $\theta + 2k\pi$ are coterminal for every integer k . In degrees, use $\theta + 360^\circ k$. To locate an angle, add or subtract full turns until it lies in $[0, 2\pi)$ or $[0^\circ, 360^\circ)$.

Example.

Problem 2.2.1. Find the angle in $[0, 2\pi)$ coterminal with $-\frac{14\pi}{3}$. In which quadrant does its terminal ray lie?

Strategy. Add multiples of $2\pi = \frac{6\pi}{3}$ until the angle lies in the required interval.

Solution. Adding three full turns gives

$$-\frac{14\pi}{3} + 3(2\pi) = \frac{4\pi}{3}.$$

Since $\pi < \frac{4\pi}{3} < \frac{3\pi}{2}$, its terminal ray lies in quadrant III.

Answer: $\frac{4\pi}{3}$, in quadrant III. □

3. THE UNIT CIRCLE

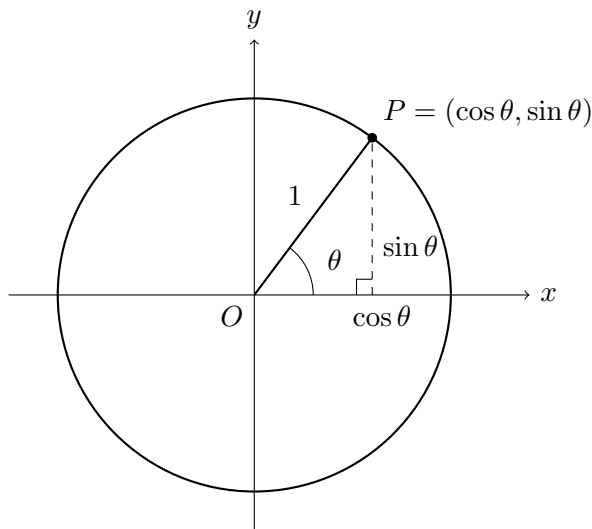
3.1. Coordinates on the Unit Circle. The **unit circle** is the circle $x^2 + y^2 = 1$ centered at the origin. Let the terminal ray of θ meet the unit circle at $P = (x, y)$.

Definition 3.1.1. For any real angle θ , define

$$\cos \theta = x, \quad \sin \theta = y.$$

When $x \neq 0$, define $\tan \theta = y/x$.

For an acute angle, dropping a perpendicular from P to the x -axis gives a right triangle with hypotenuse 1. The coordinate definitions therefore agree with the right-triangle ratios.



More generally, the point at distance $r > 0$ from the origin on this ray has coordinates $(r \cos \theta, r \sin \theta)$. Conversely, for a point $(u, v) \neq (0, 0)$ on the ray, $r = \sqrt{u^2 + v^2}$, $\cos \theta = u/r$, and $\sin \theta = v/r$.

Angles whose terminal rays lie on the axes do not come from right triangles. Their values are the coordinates of the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$:

θ	0	$\pi/2$	π	$3\pi/2$
$\cos \theta$	1	0	-1	0
$\sin \theta$	0	1	0	-1
$\tan \theta$	0	undefined	0	undefined

Example.

Problem 3.1.2. The terminal ray of θ passes through $(-8, 6)$. Find $\sin \theta$, $\cos \theta$, and $\tan \theta$.

Strategy. Divide both coordinates by the distance from the point to the origin. Keep the signs of the coordinates.

Solution. The distance to the origin is $\sqrt{(-8)^2 + 6^2} = 10$, so the unit-circle point is $(-\frac{4}{5}, \frac{3}{5})$. Therefore

$$\sin \theta = \frac{3}{5}, \quad \cos \theta = -\frac{4}{5}, \quad \tan \theta = \frac{6}{-8} = -\frac{3}{4}.$$

Answer: $\sin \theta = \frac{3}{5}$, $\cos \theta = -\frac{4}{5}$, and $\tan \theta = -\frac{3}{4}$. □

3.2. Quadrants and Reference Angles. The signs of sine and cosine are the signs of the y - and x -coordinates. Tangent is positive when these signs agree and negative when they differ.

Quadrant	Range	$\sin \theta$	$\cos \theta$	$\tan \theta$
I	$0 < \theta < \pi/2$	+	+	+
II	$\pi/2 < \theta < \pi$	+	-	-
III	$\pi < \theta < 3\pi/2$	-	-	+
IV	$3\pi/2 < \theta < 2\pi$	-	+	-

For a terminal ray off the axes, the **reference angle** is the acute angle between the ray and the x -axis. If $0 \leq \theta < 2\pi$, the reference angle is θ , $\pi - \theta$, $\theta - \pi$, or $2\pi - \theta$ in quadrants I, II, III, or IV, respectively. The point of the unit circle at angle θ is a reflection, across one or both axes, of the point at its reference angle, so the two points have coordinates with the same absolute values. Find the values at the reference angle, then use the quadrant to choose their signs.

Examples.

Problem 3.2.1. Find the exact values of $\sin \frac{29\pi}{6}$, $\cos \frac{29\pi}{6}$, and $\tan \frac{29\pi}{6}$.

Strategy. Remove full turns, find the reference angle, and use the signs in the resulting quadrant.

Solution. Subtracting 4π gives the coterminal angle $\frac{5\pi}{6}$, in quadrant II. Its reference angle is $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$. Sine is positive, cosine is negative, and tangent is negative in quadrant II. Hence

$$\sin \frac{29\pi}{6} = \frac{1}{2}, \quad \cos \frac{29\pi}{6} = -\frac{\sqrt{3}}{2}, \quad \tan \frac{29\pi}{6} = -\frac{\sqrt{3}}{3}.$$

Answer: $\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$, and $-\frac{\sqrt{3}}{3}$, respectively. □

Problem 3.2.2. Find all angles θ in $[0, 2\pi)$ for which $\sin \theta = \frac{1}{2}$.

Strategy. Sine is the y -coordinate. The horizontal line $y = \frac{1}{2}$ meets the unit circle at two points.

Solution. The reference angle is $\pi/6$, since $\sin(\pi/6) = 1/2$. Positive sine places the terminal ray in quadrant I or II, so the angles are $\pi/6$ and $\pi - \pi/6 = 5\pi/6$. The line $y = \frac{1}{2}$ meets the circle at no other points, so there are no other angles in $[0, 2\pi)$.

Answer: $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. □

3.3. Symmetry and Periodicity. The coordinate definitions give the following relations directly.

- A full turn returns to the same point:

$$\sin(\theta + 2\pi) = \sin \theta, \quad \cos(\theta + 2\pi) = \cos \theta.$$

- Reflection across the x -axis changes the sign of the y -coordinate:

$$\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta.$$

- Reflection across the y -axis changes the sign of the x -coordinate:

$$\sin(\pi - \theta) = \sin \theta, \quad \cos(\pi - \theta) = -\cos \theta.$$

- A half turn changes the signs of both coordinates:

$$\sin(\theta + \pi) = -\sin \theta, \quad \cos(\theta + \pi) = -\cos \theta.$$

Thus $\tan(\theta + \pi) = \tan \theta$ wherever tangent is defined.

One trigonometric value does not determine the angle. For example, θ and $\pi - \theta$ have the same sine. A given interval or quadrant decides which angle, or which sign, is meant.

The distance between two points of the unit circle depends only on the angle between them. The **central angle** of a chord is the angle, between 0° and 180° , that the chord subtends at the center. Two chords of the same circle are equal exactly when their central angles are equal. For chords shorter than a diameter, the center and the endpoints form an isosceles triangle with two sides equal to the radius; two such triangles are congruent by SAS when their angles at the center are equal, and by SSS when their third sides are equal. A diameter is the only chord with central angle 180° , and it is longer than every other chord.

4. BASIC IDENTITIES

4.1. The Pythagorean Identity.

Theorem 4.1.1. *For every real angle θ ,*

$$\sin^2 \theta + \cos^2 \theta = 1.$$

Proof. The point $(\cos \theta, \sin \theta)$ lies on $x^2 + y^2 = 1$. □

Here $\sin^2 \theta$ means $(\sin \theta)^2$, not $\sin(\theta^2)$. If one of $\sin \theta$ and $\cos \theta$ is known, the identity gives the square of the other. Its sign must still be found from the quadrant.

Example.

Problem 4.1.2. Suppose $\sin \theta = -\frac{5}{13}$ and $\pi < \theta < \frac{3\pi}{2}$. Find $\cos \theta$ and $\tan \theta$.

Strategy. Use the Pythagorean identity to find $\cos^2 \theta$. The quadrant fixes the sign of $\cos \theta$.

Solution. By the Pythagorean identity,

$$\cos^2 \theta = 1 - \frac{25}{169} = \frac{144}{169}.$$

Quadrant III has negative cosine, so $\cos \theta = -12/13$. Consequently $\tan \theta = (-5/13)/(-12/13) = 5/12$.

Answer: $\cos \theta = -\frac{12}{13}$ and $\tan \theta = \frac{5}{12}$. □

4.2. Reciprocal Ratios. The **secant**, **cosecant**, and **cotangent** of θ are

$$\sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}.$$

Each is defined only when its denominator is nonzero. Where both tangent and cotangent are defined, $\cot \theta = 1/\tan \theta$.

Dividing the Pythagorean identity by $\cos^2 \theta$ or by $\sin^2 \theta$ gives

$$1 + \tan^2 \theta = \sec^2 \theta, \quad 1 + \cot^2 \theta = \csc^2 \theta,$$

wherever the ratios are defined. Written as differences of squares, these identities become

$$(\sec \theta - \tan \theta)(\sec \theta + \tan \theta) = 1,$$

$$(\csc \theta - \cot \theta)(\csc \theta + \cot \theta) = 1.$$

Example.

Problem 4.2.1. An acute angle θ satisfies $\csc \theta - \cot \theta = \frac{1}{3}$. Find $\sin \theta$ and $\cos \theta$.

Strategy. Use the difference of squares to find $\csc \theta + \cot \theta$, then add and subtract the two equations.

Solution. Since $(\csc \theta - \cot \theta)(\csc \theta + \cot \theta) = 1$, it follows that $\csc \theta + \cot \theta = 3$. Adding the two equations gives $2 \csc \theta = 10/3$, so $\csc \theta = 5/3$ and $\sin \theta = 3/5$. Subtracting them gives $2 \cot \theta = 8/3$, so $\cot \theta = 4/3$. Finally,

$$\cos \theta = \cot \theta \sin \theta = \frac{4}{3} \cdot \frac{3}{5} = \frac{4}{5}.$$

Answer: $\sin \theta = \frac{3}{5}$ and $\cos \theta = \frac{4}{5}$. □

4.3. Complementary Angles. The two acute angles of a right triangle are complementary. Exchanging them exchanges the opposite and adjacent legs. Therefore, for $0 < \theta < \pi/2$,

$$\sin \left(\frac{\pi}{2} - \theta \right) = \cos \theta, \quad \cos \left(\frac{\pi}{2} - \theta \right) = \sin \theta, \quad \tan \left(\frac{\pi}{2} - \theta \right) = \frac{1}{\tan \theta}.$$

Reflecting the unit circle across the line $y = x$ exchanges the coordinates and takes the point at angle θ to the point at angle $\frac{\pi}{2} - \theta$, so the first two identities hold for all real θ . The tangent identity holds wherever both sides are defined. In degrees, replace $\pi/2$ by 90° .

Example.

Problem 4.3.1. Evaluate $\sin^2 17^\circ + \sin^2 73^\circ + \tan 28^\circ \tan 62^\circ$.

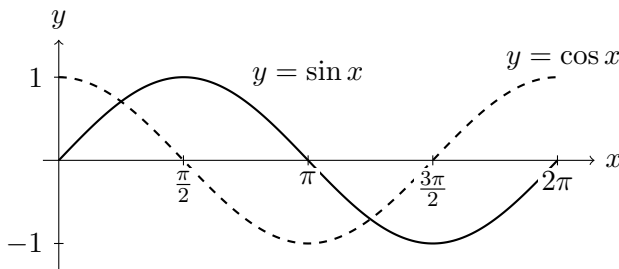
Strategy. Pair complementary angles. The sine pair becomes a Pythagorean identity, and the tangent pair becomes a product of reciprocals.

Solution. Since $73^\circ = 90^\circ - 17^\circ$, the first two terms add to $\sin^2 17^\circ + \cos^2 17^\circ = 1$. Also, $\tan 62^\circ = 1/\tan 28^\circ$, so the product of tangents is 1. The total is 2.

Answer: 2. □

5. GRAPHS OF SINE AND COSINE

5.1. Amplitude, Period, and Phase. With x in radians, the graph of $y = \sin x$ starts at 0 when $x = 0$, rises to 1 at $x = \frac{\pi}{2}$, returns to 0 at $x = \pi$, falls to -1 at $x = \frac{3\pi}{2}$, and returns to 0 at $x = 2\pi$. Then it repeats. Since $\cos x = \sin\left(\frac{\pi}{2} - x\right) = \sin\left(\pi - \left(\frac{\pi}{2} - x\right)\right) = \sin\left(x + \frac{\pi}{2}\right)$, the graph of $y = \cos x$ is the graph of $y = \sin x$ shifted left by $\frac{\pi}{2}$.



Definition 5.1.1. A function f is **periodic** with **period** $p > 0$ if $f(x + p) = f(x)$ for all x . The smallest such p , if one exists, is the **least period**.

Sine and cosine have least period 2π . A full turn returns to the same point, so 2π is a period. If $0 < p < 2\pi$, then $\sin\left(\frac{\pi}{2} + p\right) \neq 1 = \sin \frac{\pi}{2}$, because $\sin x = 1$ only at $x = \frac{\pi}{2} + 2k\pi$; so p is not a period of sine, and a similar argument with $\cos 0 = 1$ applies to cosine. In the same way, tangent has least period π : $\tan(x + \pi) = \tan x$, and $\tan x = 0$ only at multiples of π .

Theorem 5.1.2. Let A, B, C, D be constants with $A \neq 0$ and $B > 0$. The graph of

$$y = A \sin(B(x - C)) + D$$

has **amplitude** $|A|$, least period $\frac{2\pi}{B}$, and **phase shift** C : it is the graph of $y = A \sin Bx$ shifted right by C (left if $C < 0$) and up by D . Its values range from $D - |A|$ to $D + |A|$. The same holds with cosine in place of sine.

Proof. The values of $\sin(B(x - C))$ fill the interval $[-1, 1]$. Multiplying by A and adding D gives the range. When x increases by p , the input $B(x - C)$ increases by Bp , so the graph repeats exactly when Bp is a multiple of 2π ; the least such $p > 0$ is $2\pi/B$. Finally, y takes at $x + C$ the value that $A \sin Bx + D$ takes at x , so the graph is shifted right by C . $\square$

Example.

Problem 5.1.3. Find the amplitude, least period, phase shift, and the largest and smallest values of $y = 3 \sin\left(2x - \frac{\pi}{3}\right) + 1$. What is the smallest positive x at which y is largest?

Strategy. Factor the coefficient of x out of the input to read off the phase shift.

Solution. Since $2x - \frac{\pi}{3} = 2\left(x - \frac{\pi}{6}\right)$, the function is $y = 3 \sin\left(2\left(x - \frac{\pi}{6}\right)\right) + 1$. The amplitude is 3, the least period is $\frac{2\pi}{2} = \pi$, and the phase shift is $\frac{\pi}{6}$ to the right. The values range from $1 - 3 = -2$ to $1 + 3 = 4$. The value 4 occurs when $\sin\left(2x - \frac{\pi}{3}\right) = 1$, that is, when

$$2x - \frac{\pi}{3} = \frac{\pi}{2} + 2k\pi \quad \text{for an integer } k, \quad \text{so} \quad x = \frac{5\pi}{12} + k\pi.$$

The smallest positive such x is $\frac{5\pi}{12}$.

Answer: Amplitude 3, least period π , phase shift $\frac{\pi}{6}$ to the right, largest value 4, smallest value -2 , and $x = \frac{5\pi}{12}$. $\square$

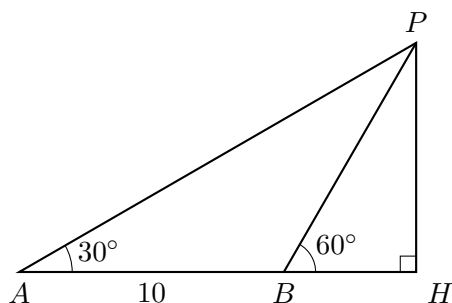
6. GEOMETRIC APPLICATIONS

When a figure contains no right triangle, drop a perpendicular. If a segment of length r makes an acute angle θ with a horizontal line, its horizontal and vertical components have lengths $r \cos \theta$ and $r \sin \theta$. An **angle of elevation** is the angle above the horizontal from an observer to an object.

The same idea gives the height of a triangle. If sides AB and AC form angle A , the altitude from C to line AB has length $AC \sin A$. When A is obtuse, the altitude meets the extension of AB , and the acute angle in the right triangle is $\pi - A$, which has the same sine.

Example.

Problem 6.1. A point P is above a horizontal line. From a point A on the line, the angle of elevation to P is 30° . From a point B on the line, 10 units closer to the foot of the perpendicular from P , the angle of elevation is 60° . The points A and B are on the same side of the foot. Find the height of P above the line.



Strategy. Write the height in two ways using tangent in the two right triangles.

Solution. Let H be the foot of the perpendicular, and let $BH = x$ and $PH = h$. Then $AH = x+10$, and

$$h = x \tan 60^\circ = x\sqrt{3}, \quad h = (x+10) \tan 30^\circ = \frac{x+10}{\sqrt{3}}.$$

Equating and multiplying by $\sqrt{3}$ gives $3x = x+10$, so $x = 5$ and $h = 5\sqrt{3}$.

Answer: $5\sqrt{3}$.

□

7. PROBLEMS

Beginner.

- (1983 AHSME, Problem #5) Triangle ABC has a right angle at C . If $\sin A = \frac{2}{3}$, then $\tan B$ is
 (A) $\frac{3}{5}$ (B) $\frac{\sqrt{5}}{3}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{5}}{2}$ (E) $\frac{5}{3}$
- (1988 AHSME, Problem #13) If $\sin x = 3 \cos x$, then what is $\sin x \cos x$?
 (A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{2}{9}$ (D) $\frac{1}{4}$ (E) $\frac{3}{10}$
- (1998 AHSME, Problem #19) How many triangles have area 10 and vertices at $(-5, 0)$, $(5, 0)$, and $(5 \cos \theta, 5 \sin \theta)$ for some angle θ ?
 (A) 0 (B) 2 (C) 4 (D) 6 (E) 8
- (1987 AHSME, Problem #20) Evaluate
 $\log_{10}(\tan 1^\circ) + \log_{10}(\tan 2^\circ) + \log_{10}(\tan 3^\circ) + \cdots + \log_{10}(\tan 88^\circ) + \log_{10}(\tan 89^\circ)$.
 (A) 0 (B) $\frac{1}{2} \log_{10} \left(\frac{\sqrt{3}}{2} \right)$ (C) $\frac{1}{2} \log_{10} 2$ (D) 1 (E) none of these
- (1972 AHSME, Problem #27) If the area of $\triangle ABC$ is 64 square units and the geometric mean (mean proportional) between sides AB and AC is 12 inches, then $\sin A$ is equal to:
 (A) $\frac{\sqrt{3}}{2}$ (B) $\frac{3}{5}$ (C) $\frac{4}{5}$ (D) $\frac{8}{9}$ (E) $\frac{15}{17}$

Intermediate.

- (1999 AHSME, Problem #15) Let x be a real number such that $\sec x - \tan x = 2$. What is $\sec x + \tan x$?
 (A) 0.1 (B) 0.2 (C) 0.3 (D) 0.4 (E) 0.5
- (1972 AHSME, Problem #20) If $\tan x = \frac{2ab}{a^2 - b^2}$, where $a > b > 0$ and $0^\circ < x < 90^\circ$, then $\sin x$ is equal to:
 (A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{\sqrt{a^2 - b^2}}{2a}$ (D) $\frac{\sqrt{a^2 - b^2}}{2ab}$ (E) $\frac{2ab}{a^2 + b^2}$
- (1978 AHSME, Problem #15) If $\sin x + \cos x = \frac{1}{5}$ and $0 \leq x < \pi$, then $\tan x$ is
 (A) $-\frac{4}{3}$ (B) $-\frac{3}{4}$ (C) $\frac{3}{4}$ (D) $\frac{4}{3}$ (E) not completely determined by the given information
- (2012 AMC 12A, Problem #10) A triangle has area 30, one side of length 10, and the median to that side of length 9. Let θ be the acute angle formed by that side and the median. What is $\sin \theta$?
 (A) $\frac{3}{10}$ (B) $\frac{1}{3}$ (C) $\frac{9}{20}$ (D) $\frac{2}{3}$ (E) $\frac{9}{10}$
- (2006 AMC 12A, Problem #15) Suppose $\cos x = 0$ and $\cos(x+z) = \frac{1}{2}$. What is the smallest possible positive value of z ?
 (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{5\pi}{6}$ (E) $\frac{7\pi}{6}$

Advanced.

11. (2024 AMC 12B, Problem #11) Let $x_n = \sin^2(n^\circ)$. What is the mean of $x_1, x_2, x_3, \dots, x_{90}$?
- (A) $\frac{11}{45}$ (B) $\frac{22}{45}$ (C) $\frac{89}{180}$ (D) $\frac{1}{2}$ (E) $\frac{91}{180}$
12. (2017 AMC 12B, Problem #7) The functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π . What is the least period of the function $\cos(\sin(x))$?
- (A) $\frac{\pi}{2}$ (B) π (C) 2π (D) 4π (E) It's not periodic.
13. (2016 AMC 12B, Problem #13) Alice and Bob live 10 miles apart. One day Alice looks due north from her house and sees an airplane. At the same time Bob looks due west from his house and sees the same airplane. The angle of elevation of the airplane is 30° from Alice's position and 60° from Bob's position. Which of the following is closest to the airplane's altitude, in miles?
- (A) 3.5 (B) 4 (C) 4.5 (D) 5 (E) 5.5
14. (2021 Fall AMC 12B, Problem #10) What is the sum of all possible values of t between 0 and 360 such that the triangle in the coordinate plane whose vertices are $(\cos 40^\circ, \sin 40^\circ)$, $(\cos 60^\circ, \sin 60^\circ)$, and $(\cos t^\circ, \sin t^\circ)$ is isosceles?
- (A) 100 (B) 150 (C) 330 (D) 360 (E) 380
15. (2021 Fall AMC 12B, Problem #13) Let $c = \frac{2\pi}{11}$. What is the value of
- $$\frac{\sin 3c \cdot \sin 6c \cdot \sin 9c \cdot \sin 12c \cdot \sin 15c}{\sin c \cdot \sin 2c \cdot \sin 3c \cdot \sin 4c \cdot \sin 5c}?$$
- (A) -1 (B) $-\frac{\sqrt{11}}{5}$ (C) $\frac{\sqrt{11}}{5}$ (D) $\frac{10}{11}$ (E) 1