

TRIGONOMETRIC RATIOS AND THE UNIT CIRCLE FOR AMC 12: SOLUTIONS

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1. SOLUTIONS

Beginner.

1. (1983 AHSME, Problem #5) Triangle ABC has a right angle at C . If $\sin A = \frac{2}{3}$, then $\tan B$ is

(A) $\frac{3}{5}$ (B) $\frac{\sqrt{5}}{3}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{5}}{2}$ (E) $\frac{5}{3}$

Strategy. Choose a right triangle whose opposite leg and hypotenuse have the ratio $2 : 3$. Then switch to angle B .

Solution. Relative to angle A , the opposite leg is BC , so take $BC = 2$ and $AB = 3$. The remaining leg is $AC = \sqrt{3^2 - 2^2} = \sqrt{5}$. Relative to angle B , the opposite leg is AC and the adjacent leg is BC , so $\tan B = \sqrt{5}/2$.

Answer: (D) $\frac{\sqrt{5}}{2}$. □

2. (1988 AHSME, Problem #13) If $\sin x = 3 \cos x$, then what is $\sin x \cos x$?

(A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{2}{9}$ (D) $\frac{1}{4}$ (E) $\frac{3}{10}$

Strategy. Substitute $\sin x = 3 \cos x$ into the Pythagorean identity to find $\cos^2 x$.

Solution. The identity $\sin^2 x + \cos^2 x = 1$ gives $9 \cos^2 x + \cos^2 x = 1$, hence $\cos^2 x = 1/10$. Thus

$$\sin x \cos x = (3 \cos x) \cos x = 3 \cos^2 x = \frac{3}{10}.$$

Answer: (E) $\frac{3}{10}$. □

3. (1998 AHSME, Problem #19) How many triangles have area 10 and vertices at $(-5, 0)$, $(5, 0)$, and $(5 \cos \theta, 5 \sin \theta)$ for some angle θ ?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Strategy. The fixed base has length 10. Find the required height, then intersect two horizontal lines with the circle.

Solution. The points $(5 \cos \theta, 5 \sin \theta)$ form the circle $x^2 + y^2 = 25$ of radius 5. The base from $(-5, 0)$ to $(5, 0)$ lies on the x -axis and has length 10. A triangle of area 10 on this base has height 2, because $10 = \frac{1}{2} \cdot 10 \cdot 2$. Thus the third vertex has $y = 2$ or $y = -2$. Each of these lines meets the circle at two points, with $x = \pm\sqrt{25 - 4} = \pm\sqrt{21}$. These four vertices give four different triangles.

Answer: (C) 4. □

4. (1987 AHSME, Problem #20) Evaluate

$$\log_{10}(\tan 1^\circ) + \log_{10}(\tan 2^\circ) + \log_{10}(\tan 3^\circ) + \cdots + \log_{10}(\tan 88^\circ) + \log_{10}(\tan 89^\circ).$$

(A) 0 (B) $\frac{1}{2} \log_{10} \left(\frac{\sqrt{3}}{2} \right)$ (C) $\frac{1}{2} \log_{10} 2$ (D) 1 (E) none of these

Strategy. Pair the tangents of complementary angles, and use $\log_{10} u + \log_{10} v = \log_{10}(uv)$ for positive u and v .

Solution. For $1 \leq k \leq 44$, the angles k° and $(90 - k)^\circ$ are complementary, so

$$\tan k^\circ \tan(90 - k)^\circ = 1,$$

and the two logarithms add to $\log_{10} 1 = 0$. These 44 pairs use every angle except 45° , whose term is $\log_{10}(\tan 45^\circ) = \log_{10} 1 = 0$. Hence the whole sum is 0.

Answer: (A) 0. □

5. (1972 AHSME, Problem #27) If the area of $\triangle ABC$ is 64 square units and the geometric mean (mean proportional) between sides AB and AC is 12 inches, then $\sin A$ is equal to:

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{3}{5}$ (C) $\frac{4}{5}$ (D) $\frac{8}{9}$ (E) $\frac{15}{17}$

Strategy. The geometric mean gives the product $AB \cdot AC$. Express the altitude to AB using AC and angle A .

Solution. The geometric mean is $\sqrt{AB \cdot AC} = 12$, so $AB \cdot AC = 144$. The altitude from C to line AB has length $AC \sin A$. When A is acute, this is the right-triangle definition of sine; when A is obtuse, the right triangle has acute angle $180^\circ - A$, which has the same sine. Therefore

$$64 = \frac{1}{2} \cdot AB \cdot AC \sin A = 72 \sin A,$$

so $\sin A = 8/9$.

Answer: (D) $\frac{8}{9}$. □

Intermediate.

6. (1999 AHSME, Problem #15) Let x be a real number such that $\sec x - \tan x = 2$. What is $\sec x + \tan x$?

(A) 0.1 (B) 0.2 (C) 0.3 (D) 0.4 (E) 0.5

Strategy. Multiply the given expression by the expression that is asked for.

Solution. By the identity $1 + \tan^2 x = \sec^2 x$,

$$(\sec x - \tan x)(\sec x + \tan x) = \sec^2 x - \tan^2 x = 1.$$

Since the first factor is 2, the second is $1/2 = 0.5$.

Answer: (E) 0.5. □

7. (1972 AHSME, Problem #20) If $\tan x = \frac{2ab}{a^2 - b^2}$, where $a > b > 0$ and $0^\circ < x < 90^\circ$, then $\sin x$ is equal to:

(A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{\sqrt{a^2 - b^2}}{2a}$ (D) $\frac{\sqrt{a^2 - b^2}}{2ab}$ (E) $\frac{2ab}{a^2 + b^2}$

Strategy. Use the numerator and denominator of the tangent as the opposite and adjacent legs. Compute the hypotenuse.

Solution. Because $a > b > 0$, both $2ab$ and $a^2 - b^2$ are positive. Since x is acute, x is an angle of a right triangle with opposite leg $2ab$ and adjacent leg $a^2 - b^2$. Its hypotenuse is

$$\sqrt{4a^2b^2 + (a^2 - b^2)^2} = \sqrt{a^4 + 2a^2b^2 + b^4} = a^2 + b^2.$$

Hence $\sin x = \frac{2ab}{a^2 + b^2}$.

Answer: (E) $\frac{2ab}{a^2 + b^2}$. □

8. (1978 AHSME, Problem #15) If $\sin x + \cos x = \frac{1}{5}$ and $0 \leq x < \pi$, then $\tan x$ is
- (A) $-\frac{4}{3}$ (B) $-\frac{3}{4}$ (C) $\frac{3}{4}$ (D) $\frac{4}{3}$ (E) not completely determined by the given information

Strategy. Square the given sum and use the Pythagorean identity to find the product $\sin x \cos x$. Then use the interval to choose the signs.

Solution. Write $s = \sin x$ and $c = \cos x$. Squaring $s + c = 1/5$ and using $s^2 + c^2 = 1$ gives $1 + 2sc = 1/25$, so $sc = -12/25$. Since s and c have sum $\frac{1}{5}$ and product $-\frac{12}{25}$, they are the two roots of

$$u^2 - \frac{1}{5}u - \frac{12}{25} = 0, \quad \text{that is,} \quad (5u - 4)(5u + 3) = 0.$$

The roots are $4/5$ and $-3/5$. On $0 \leq x < \pi$, sine is nonnegative. Therefore $s = 4/5$ and $c = -3/5$, and $\tan x = s/c = -4/3$.

Answer: (A) $-\frac{4}{3}$. □

9. (2012 AMC 12A, Problem #10) A triangle has area 30, one side of length 10, and the median to that side of length 9. Let θ be the acute angle formed by that side and the median. What is $\sin \theta$?

(A) $\frac{3}{10}$ (B) $\frac{1}{3}$ (C) $\frac{9}{20}$ (D) $\frac{2}{3}$ (E) $\frac{9}{10}$

Strategy. Find the altitude to the side of length 10. The median and this altitude form a right triangle that contains θ .

Solution. Let V be the vertex opposite the side of length 10, let M be the midpoint of that side, and let H be the foot of the perpendicular from V to the line containing the side. The area gives $30 = \frac{1}{2} \cdot 10 \cdot VH$, so $VH = 6$. Since $VM = 9 \neq 6$, the points H and M are different, and triangle VHM has a right angle at H . Its angle at M is the acute angle θ between the side and the median, so

$$\sin \theta = \frac{VH}{VM} = \frac{6}{9} = \frac{2}{3}.$$

Answer: (D) $\frac{2}{3}$. □

10. (2006 AMC 12A, Problem #15) Suppose $\cos x = 0$ and $\cos(x+z) = \frac{1}{2}$. What is the smallest possible positive value of z ?

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{5\pi}{6}$ (E) $\frac{7\pi}{6}$

Strategy. Locate the points where cosine is 0 and where it is $\frac{1}{2}$ on the unit circle, and compare the counterclockwise turns from one to the other.

Solution. Up to whole turns, the angle x is $\pi/2$ or $3\pi/2$, and the angle $x+z$ is $\pi/3$ or $5\pi/3$. The positive turns of less than 2π from a starting angle to a target angle are

Starting angle	Target $\pi/3$	Target $5\pi/3$
$\pi/2$	$11\pi/6$	$7\pi/6$
$3\pi/2$	$5\pi/6$	$\pi/6$

Every other positive value of z exceeds one of these by a multiple of 2π . The minimum is $\pi/6$, obtained from $x = 3\pi/2$ and $x+z = 5\pi/3$.

Answer: (A) $\frac{\pi}{6}$. □

Advanced.

11. (2024 AMC 12B, Problem #11) Let $x_n = \sin^2(n^\circ)$. What is the mean of $x_1, x_2, x_3, \dots, x_{90}$?

(A) $\frac{11}{45}$ (B) $\frac{22}{45}$ (C) $\frac{89}{180}$ (D) $\frac{1}{2}$ (E) $\frac{91}{180}$

Strategy. Pair n° with $(90 - n)^\circ$. Treat 45° and 90° separately.

Solution. For $n = 1, 2, \dots, 44$,

$$\sin^2 n^\circ + \sin^2(90 - n)^\circ = \sin^2 n^\circ + \cos^2 n^\circ = 1.$$

These 44 pairs account for all terms from x_1 through x_{89} except x_{45} . The remaining terms are $x_{45} = \sin^2 45^\circ = 1/2$ and $x_{90} = \sin^2 90^\circ = 1$. The sum is $44 + \frac{1}{2} + 1 = \frac{91}{2}$, so the mean is $\frac{91}{2} \div 90 = \frac{91}{180}$.

Answer: (E) $\frac{91}{180}$. □

12. (2017 AMC 12B, Problem #7) The functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π . What is the least period of the function $\cos(\sin(x))$?

(A) $\frac{\pi}{2}$ (B) π (C) 2π (D) 4π (E) It's not periodic.

Strategy. Show that π is a period using a half turn. Then find where $\cos(\sin(x))$ equals 1 to rule out smaller periods.

Solution. Let $f(x) = \cos(\sin(x))$. A half turn gives $\sin(x + \pi) = -\sin x$, and cosine satisfies $\cos(-u) = \cos u$, so

$$f(x + \pi) = \cos(-\sin x) = \cos(\sin x) = f(x).$$

Thus π is a period. Suppose $0 < p < \pi$ were a period. Then $f(p) = f(0) = \cos 0 = 1$. Since $-1 \leq \sin p \leq 1$ and the only angle in $[-1, 1]$ with cosine 1 is 0, this forces $\sin p = 0$. But $\sin p > 0$ for $0 < p < \pi$, a contradiction. Hence the least period is π .

Answer: (B) π . □

13. (2016 AMC 12B, Problem #13) Alice and Bob live 10 miles apart. One day Alice looks due north from her house and sees an airplane. At the same time Bob looks due west from his house and sees the same airplane. The angle of elevation of the airplane is 30° from Alice's position and 60° from Bob's position. Which of the following is closest to the airplane's altitude, in miles?

(A) 3.5 (B) 4 (C) 4.5 (D) 5 (E) 5.5

Strategy. Let H be the point on the ground directly below the airplane. The two directions of sight make a right angle at H , so the ground distances form a right triangle with hypotenuse 10.

Solution. Let A and B be the two houses and let the altitude be h . The point H is due north of A and due west of B , so the ground segments AH and BH are perpendicular. The angles of elevation give

$$AH = \frac{h}{\tan 30^\circ} = h\sqrt{3}, \quad BH = \frac{h}{\tan 60^\circ} = \frac{h}{\sqrt{3}}.$$

By the Pythagorean theorem in triangle AHB ,

$$100 = AH^2 + BH^2 = 3h^2 + \frac{h^2}{3} = \frac{10h^2}{3}.$$

Thus $h = \sqrt{30}$. Since $5.4^2 < 30 < 5.5^2$, the altitude satisfies $5.4 < h < 5.5$, and the closest listed value is 5.5.

Answer: (E) 5.5. □

14. (2021 Fall AMC 12B, Problem #10) What is the sum of all possible values of t between 0 and 360 such that the triangle in the coordinate plane whose vertices are $(\cos 40^\circ, \sin 40^\circ)$, $(\cos 60^\circ, \sin 60^\circ)$, and $(\cos t^\circ, \sin t^\circ)$ is isosceles?

(A) 100 (B) 150 (C) 330 (D) 360 (E) 380

Strategy. The two equal sides can meet at any of the three vertices. All three vertices lie on the unit circle, so compare central angles of chords.

Solution. Call the fixed vertices A and B , at angles 40° and 60° , and let P be the vertex at angle t° . Chords of the unit circle are equal exactly when their central angles are equal, and chord AB has central angle 20° .

- If $AP = AB$, then P is 20° from A , so $t = 20$ or $t = 60$. The value 60 puts P at B and gives no triangle.
- If $BP = AB$, then P is 20° from B , so $t = 40$ or $t = 80$. The value 40 puts P at A .
- If $PA = PB$, then P lies on the perpendicular bisector of AB . The center is also equidistant from A and B , so this line passes through the center and bisects the 20° central angle. It meets the unit circle at the angles 50° and 230° .

The possible values are 20, 50, 80, and 230. Each gives three distinct points of a circle, which are never collinear, so each gives a triangle. The sum is $20 + 50 + 80 + 230 = 380$.

Answer: (E) 380. □

15. (2021 Fall AMC 12B, Problem #13) Let $c = \frac{2\pi}{11}$. What is the value of

$$\frac{\sin 3c \cdot \sin 6c \cdot \sin 9c \cdot \sin 12c \cdot \sin 15c}{\sin c \cdot \sin 2c \cdot \sin 3c \cdot \sin 4c \cdot \sin 5c}?$$

(A) -1 (B) $-\frac{\sqrt{11}}{5}$ (C) $\frac{\sqrt{11}}{5}$ (D) $\frac{10}{11}$ (E) 1

Strategy. Subtract full turns $2\pi = 11c$ from the angles in the numerator, then use $\sin(-\theta) = -\sin \theta$ to match the factors of the denominator.

Solution. Since $11c = 2\pi$, subtracting a full turn gives $\sin 12c = \sin c$ and $\sin 15c = \sin 4c$. Also,

$$\sin 6c = \sin(6c - 2\pi) = \sin(-5c) = -\sin 5c,$$

$$\sin 9c = \sin(9c - 2\pi) = \sin(-2c) = -\sin 2c.$$

The numerator therefore equals

$$\sin 3c \cdot (-\sin 5c) \cdot (-\sin 2c) \cdot \sin c \cdot \sin 4c = \sin c \cdot \sin 2c \cdot \sin 3c \cdot \sin 4c \cdot \sin 5c,$$

which is the denominator. None of these factors is zero, since $c, 2c, \dots, 5c$ all lie strictly between 0 and π . The ratio is 1.

Answer: (E) 1. □

2. ANSWER KEY

Problem	Letter	Value
1	D	$\frac{\sqrt{5}}{2}$
2	E	$\frac{3}{10}$
3	C	4
4	A	0
5	D	$\frac{8}{9}$
6	E	0.5
7	E	$\frac{2ab}{a^2+b^2}$
8	A	$-\frac{4}{3}$
9	D	$\frac{2}{3}$
10	A	$\frac{\pi}{6}$
11	E	$\frac{91}{180}$
12	B	π
13	E	5.5
14	E	380
15	E	1