TRIGONOMETRIC RATIOS AND THE UNIT CIRCLE FOR AMC 12: PRACTICE SET SOLUTIONS

ANDY JO

Contents

1. Solutions 2
2. Answer Key 10

1. Solutions

Beginner.

1. (1966 AHSME, Problem #6) AB is a diameter of a circle centered at O. C is a point on the circle such that angle BOC is $60°$. If the diameter of the circle is 5 inches, the length of chord AC, expressed in inches, is:

(A) 3 **(B)** $\frac{5\sqrt{2}}{2}$ **(C)** $\frac{5\sqrt{3}}{2}$ **(D)** $3\sqrt{3}$ **(E)** none of these

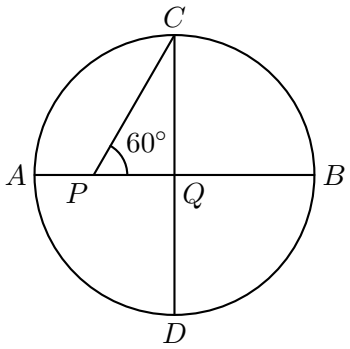
Strategy. Show that triangle OBC is equilateral, so that $\angle ABC = 60°$. Since AB is a diameter, triangle ABC has a right angle at C, and AC is the leg opposite the $60°$ angle.

Solution. The radii OB and OC are equal and $\angle BOC = 60°$, so each base angle of the isosceles triangle OBC is $\frac{1}{2}(180° - 60°) = 60°$. Since O lies on segment AB, this gives $\angle ABC = \angle OBC = 60°$. The angle ACB is inscribed in a semicircle, so it is a right angle. Thus ABC is a $30°$-$60°$-$90°$ triangle with hypotenuse $AB = 5$, and AC is the leg opposite the $60°$ angle:

$$AC = AB \sin 60° = 5 \cdot \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}.$$

Answer: (C) $\frac{5\sqrt{3}}{2}$. □

2. (1980 AHSME, Problem #5) If AB and CD are perpendicular diameters of circle Q, P in AQ, and $\angle QPC = 60°$, then the length of PQ divided by the length of AQ is



(A) $\frac{\sqrt{3}}{2}$ **(B)** $\frac{\sqrt{3}}{3}$ **(C)** $\frac{\sqrt{2}}{2}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

Strategy. The diameters are perpendicular, so triangle PQC has a right angle at Q. Its leg QC is a radius, equal to AQ, and the tangent of the $60°$ angle at P relates the two legs.

Solution. Since $AB \perp CD$, the angle PQC is a right angle. Relative to the $60°$ angle at P, the opposite leg of the right triangle PQC is QC and the adjacent leg is PQ, so

$$\tan 60° = \frac{QC}{PQ}, \qquad \text{that is,} \qquad PQ = \frac{QC}{\sqrt{3}}.$$

Both QC and AQ are radii of the circle, so $QC = AQ$. Therefore

$$\frac{PQ}{AQ} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Answer: (B) $\frac{\sqrt{3}}{3}$. □

3. (1965 AHSME, Problem #2) A regular hexagon is inscribed in a circle. The ratio of the length of a side of the hexagon to the length of the shorter of the arcs intercepted by the side is:

(A) $1:1$ **(B)** $1:6$ **(C)** $1:\pi$ **(D)** $3:\pi$ **(E)** $6:\pi$

Strategy. Each side of the hexagon subtends a central angle of $\frac{\pi}{3}$. Find the side from the triangle it forms with the center, and the arc from $s = r\theta$.

Solution. Let r be the radius. The six equal sides of the hexagon cut the circle into six equal arcs, so each side subtends a central angle of $\frac{2\pi}{6} = \frac{\pi}{3}$ radians, which is $60°$. The center and the endpoints of a side form an isosceles triangle with two sides of length r and apex angle $60°$; its base angles are also $60°$, so it is equilateral and the side of the hexagon has length r. The shorter arc cut off by the side has central angle $\frac{\pi}{3}$, so by $s = r\theta$ its length is $\frac{\pi r}{3}$. The ratio is

$$r : \frac{\pi r}{3} = 3 : \pi.$$

Answer: (D) $3:\pi$. □

4. (2002 AMC 10B, Problem #24) Riders on a Ferris wheel travel in a circle in a vertical plane. A particular wheel has radius 20 feet and revolves at the constant rate of one revolution per minute. How many seconds does it take a rider to travel from the bottom of the wheel to a point 10 vertical feet above the bottom?

(A) 5 **(B)** 6 **(C)** 7.5 **(D)** 10 **(E)** 15

Strategy. Let θ be the angle through which the wheel has turned since the rider left the bottom. Express the rider's height above the bottom as $20 - 20\cos\theta$, using the vertical component of the radius, and find the first θ at which this height is 10.

Solution. The wheel turns 2π radians in 60 seconds, so t seconds after the rider leaves the bottom it has turned through $\theta = \frac{\pi t}{30}$ radians. The center of the wheel is 20 feet above the bottom. For $0 < \theta < \frac{\pi}{2}$, the perpendicular from the rider to the vertical diameter forms a right triangle with hypotenuse 20 and angle θ at the center, so the rider is $20\cos\theta$ feet below the center, and

$$20 - 20\cos\frac{\pi t}{30}$$

feet above the bottom. As θ increases from 0 to $\frac{\pi}{2}$, the cosine decreases from 1 to 0, so this height increases from 0 to 20. It equals 10 exactly when $\cos\theta = \frac{1}{2}$, that is, when $\theta = \frac{\pi}{3}$. Then $\frac{\pi t}{30} = \frac{\pi}{3}$, so $t = 10$.

Answer: (D) 10. □

5. (1980 AHSME, Problem #18) If $b > 1$, $\sin x > 0$, $\cos x > 0$ and $\log_b \sin x = a$, then $\log_b \cos x$ equals

(A) $2\log_b(1 - b^{\frac{a}{2}})$ **(B)** $\sqrt{1 - a^2}$ **(C)** b^{a^2} **(D)** $\frac{1}{2}\log_b(1 - b^{2a})$
(E) none of these

Strategy. Rewrite the logarithm as $\sin x = b^a$. The Pythagorean identity gives $\cos^2 x$, and the condition $\cos x > 0$ selects the positive square root.

Solution. By the definition of the logarithm, $\log_b \sin x = a$ means $\sin x = b^a$. The Pythagorean identity gives

$$\cos^2 x = 1 - \sin^2 x = 1 - b^{2a}.$$

Since $\cos x > 0$, it follows that $\cos x = \sqrt{1-b^{2a}} = (1-b^{2a})^{1/2}$. Taking logarithms to base b,

$$\log_b \cos x = \frac{1}{2}\log_b(1-b^{2a}).$$

Answer: (D) $\frac{1}{2}\log_b(1-b^{2a})$. □

Intermediate.

6. (1991 AHSME, Problem #21) If

$$f\left(\frac{x}{x-1}\right) = \frac{1}{x}$$

for all $x \notin \{0,1\}$, and $0 < \theta < \frac{\pi}{2}$, then $f(\sec^2\theta) =$

(A) $\sin^2\theta$ **(B)** $\cos^2\theta$ **(C)** $\tan^2\theta$ **(D)** $\cot^2\theta$ **(E)** $\csc^2\theta$

Strategy. Find the number x for which $\frac{x}{x-1} = \sec^2\theta$. Then $f(\sec^2\theta) = \frac{1}{x}$, and the identity $1+\tan^2\theta = \sec^2\theta$ simplifies x.

Solution. Solving $\frac{x}{x-1} = y$ for x gives $x = y(x-1)$, so $x(y-1) = y$ and $x = \frac{y}{y-1}$ when $y \neq 1$. Take $y = \sec^2\theta$. By the identity $1+\tan^2\theta = \sec^2\theta$, the denominator is $y-1 = \tan^2\theta$, which is positive for $0 < \theta < \frac{\pi}{2}$. Hence

$$x = \frac{\sec^2\theta}{\tan^2\theta} = \frac{1/\cos^2\theta}{\sin^2\theta/\cos^2\theta} = \frac{1}{\sin^2\theta}.$$

Since $0 < \sin^2\theta < 1$, this x is greater than 1, so it is neither 0 nor 1 and the given equation applies to it:

$$f(\sec^2\theta) = f\left(\frac{x}{x-1}\right) = \frac{1}{x} = \sin^2\theta.$$

Answer: (A) $\sin^2\theta$. □

7. (2017 AMC 12A, Problem #15) Let $f(x) = \sin x + 2\cos x + 3\tan x$, using radian measure for the variable x. In what interval does the smallest positive value of x for which $f(x) = 0$ lie?

(A) $(0,1)$ **(B)** $(1,2)$ **(C)** $(2,3)$ **(D)** $(3,4)$ **(E)** $(4,5)$

Strategy. Use the signs of sine, cosine, and tangent in quadrants I and II to show that f has no zero in $(0,\pi]$. Then compare the values of f at π and at $\frac{5\pi}{4}$.

Solution. In quadrant I, that is, for $0 < x < \frac{\pi}{2}$, the sine, cosine, and tangent are all positive, so $f(x) > 0$. The function is not defined at $\frac{\pi}{2}$. In quadrant II, for $\frac{\pi}{2} < x < \pi$, the sine is positive and the cosine is negative, with $-1 < \cos x < 0$. Then $\frac{3}{\cos x} < -3$, so

$$\sin x + 3\tan x = \sin x\left(1 + \frac{3}{\cos x}\right) < 0,$$

and since $2\cos x < 0$ as well, $f(x) < 0$. Finally, $f(\pi) = 0 + 2(-1) + 0 = -2$. So f has no zero in $(0,\pi]$.

The angle $\frac{5\pi}{4}$ lies in quadrant III with reference angle $\frac{\pi}{4}$, so $\sin\frac{5\pi}{4} = \cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$ and $\tan\frac{5\pi}{4} = 1$. Hence

$$f\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \sqrt{2} + 3 = 3 - \frac{3\sqrt{2}}{2} > 0.$$

On $\left[\pi, \frac{5\pi}{4}\right]$ the cosine is never 0, so the graph of f is unbroken there, and f goes from a negative value to a positive one. It therefore has a zero strictly between π and $\frac{5\pi}{4}$. Because f has no zero in $(0, \pi]$, the smallest positive zero also lies strictly between π and $\frac{5\pi}{4}$. Since $3 < \pi$ and $\frac{5\pi}{4} < \frac{5 \cdot 3.2}{4} = 4$, it lies in $(3, 4)$.

Answer: (D) $(3, 4)$. □

8. (2007 AMC 12B, Problem #19) Rhombus $ABCD$, with side length 6, is rolled to form a cylinder of volume 6 by taping $\overline{AB}$ to $\overline{DC}$. What is $\sin(\angle ABC)$?

(A) $\frac{\pi}{9}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{\pi}{6}$ **(D)** $\frac{\pi}{4}$ **(E)** $\frac{\sqrt{3}}{2}$

Strategy. Taping $\overline{AB}$ to $\overline{DC}$ rolls the sides AD and BC into the two circular rims of the cylinder. The cylinder's height is the distance between the parallel sides AD and BC, an altitude of the rhombus.

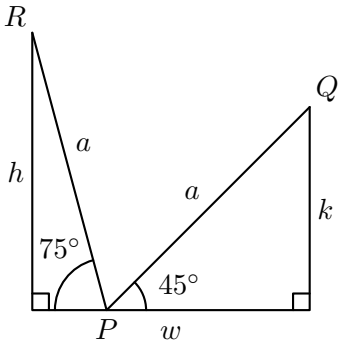
Solution. When $\overline{AB}$ is taped to $\overline{DC}$, the point A meets D and the point B meets C, so the sides AD and BC become the circles at the two ends of the cylinder. Their circumference is 6, so the radius is $r = \frac{6}{2\pi} = \frac{3}{\pi}$. The distance between the parallel lines AD and BC becomes the height of the cylinder, and it equals the length of the perpendicular from A to line BC. In triangle ABC this is the altitude from A, of length $AB \sin(\angle ABC) = 6 \sin(\angle ABC)$; if the angle is obtuse, the perpendicular meets the extension of BC, and the supplementary angle has the same sine. The volume is

$$\pi r^2 \cdot 6 \sin(\angle ABC) = \pi \cdot \frac{9}{\pi^2} \cdot 6 \sin(\angle ABC) = \frac{54 \sin(\angle ABC)}{\pi}.$$

Setting this equal to 6 gives $\sin(\angle ABC) = \frac{6\pi}{54} = \frac{\pi}{9}$.

Answer: (A) $\frac{\pi}{9}$. □

9. (1982 AHSME, Problem #22) In a narrow alley of width w a ladder of length a is placed with its foot at a point P between the walls. Resting against one wall at Q, a distance k above the ground, the ladder makes a 45° angle with the ground. Resting against the other wall at R, a distance h above the ground, the ladder makes a 75° angle with the ground. The width w is equal to



(A) a **(B)** RQ **(C)** k **(D)** $\frac{h+k}{2}$ **(E)** h

Strategy. Show that triangle PQR is equilateral. Then QR crosses the whole alley at 15° to the horizontal, and complementary angles compare its horizontal component with h.

Solution. The ladder has length a in both positions, so $PQ = PR = a$. The two positions make angles of 45° and 75° with the ground on opposite sides of P, so

$$\angle QPR = 180° - 45° - 75° = 60°.$$

The isosceles triangle PQR has apex angle 60°, so it is equilateral, with $QR = a$ and $\angle PRQ = 60^\circ$.

Let M be the foot of the wall through R. In the right triangle PMR, the angle at P is 75°, so $\angle MRP = 15^\circ$ and $h = RM = a\sin 75^\circ$. The point P is on the ground between the walls, so it lies on the same side of the wall RM as Q and below the segment RQ; hence the ray RP lies inside the angle MRQ, and

$$\angle MRQ = \angle MRP + \angle PRQ = 15^\circ + 60^\circ = 75^\circ.$$

Thus QR makes an angle of 75° with the vertical wall, and an angle of $90^\circ - 75^\circ = 15^\circ$ with the horizontal. Its horizontal component runs from one wall to the other, so

$$w = a\cos 15^\circ.$$

The angles 15° and 75° are complementary, so $\cos 15^\circ = \sin 75^\circ$, and therefore $w = a\sin 75^\circ = h$.

Answer: (E) h. □

10. (1979 AHSME, Problem #24) A polygon is called "simple" if it is not self-intersecting. Sides AB, BC, and CD of simple quadrilateral $ABCD$ have lengths 4, 5, and 20, respectively. If vertex angles B and C are obtuse and $\sin C = -\cos B = \frac{3}{5}$, then side AD has length

(A) 24 **(B)** 24.5 **(C)** 24.6 **(D)** 24.8 **(E)** 25

Strategy. Place B at the origin and C on the positive x-axis. The Pythagorean identity and the signs for an obtuse angle give the missing sine and cosine. Then A is the point $(r\cos\theta, r\sin\theta)$ on the ray BA, and D is found the same way on the ray CD, with the coordinates of C added.

Solution. An obtuse angle lies in quadrant II, where the sine is positive and the cosine is negative. By the Pythagorean identity,

$$\sin B = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}, \qquad \cos C = -\sqrt{1 - \frac{9}{25}} = -\frac{4}{5}.$$

Put $B = (0,0)$ and $C = (5,0)$, with A above the x-axis. The ray BA makes the angle B with the positive x-axis, so

$$A = (4\cos B, 4\sin B) = \left(-\frac{12}{5}, \frac{16}{5}\right).$$

The ray CB points along the negative x-axis, and the ray CD makes the angle C with it. If D is above the x-axis, the ray CD is at the angle $180^\circ - C$ in standard position. Since $\cos(180^\circ - C) = -\cos C = \frac{4}{5}$ and $\sin(180^\circ - C) = \sin C = \frac{3}{5}$,

$$D = \left(5 + 20\cdot\frac{4}{5},\ 20\cdot\frac{3}{5}\right) = (21, 12).$$

If instead D were below the x-axis, it would be $(21, -12)$, and side DA would cross side BC at $\left(\frac{48}{19}, 0\right)$, so the quadrilateral would not be simple. Hence $D = (21, 12)$, and

$$AD = \sqrt{\left(21 + \frac{12}{5}\right)^2 + \left(12 - \frac{16}{5}\right)^2} = \frac{1}{5}\sqrt{117^2 + 44^2} = \frac{1}{5}\sqrt{15625} = 25.$$

Answer: (E) 25. □

Advanced.

11. (1977 AHSME, Problem #16) If $i^2 = -1$, then the sum

$$\cos 45^\circ + i\cos 135^\circ + \cdots + i^n \cos(45 + 90n)^\circ + \cdots + i^{40}\cos 3645^\circ$$

equals

(A) $\frac{\sqrt{2}}{2}$ **(B)** $-10i\sqrt{2}$ **(C)** $\frac{21\sqrt{2}}{2}$ **(D)** $\frac{\sqrt{2}}{2}(21 - 20i)$ **(E)** $\frac{\sqrt{2}}{2}(21 + 20i)$

Strategy. Increasing n by 4 adds a full turn to the angle and multiplies i^n by $i^4 = 1$, so the terms repeat in cycles of four. Find one cycle with reference angles and the signs of cosine in each quadrant.

Solution. The angle $(45+90n)^\circ$ grows by 360° when n grows by 4. So it is coterminal with 45°, 135°, 225°, or 315° when n leaves remainder 0, 1, 2, or 3, respectively, on division by 4. These four angles lie in quadrants I, II, III, and IV, and each has reference angle 45°, so their cosines are $\frac{\sqrt{2}}{2}$, $-\frac{\sqrt{2}}{2}$, $-\frac{\sqrt{2}}{2}$, and $\frac{\sqrt{2}}{2}$. For the same four remainders, i^n is 1, i, -1, and $-i$. Multiplying, the term $i^n\cos(45 + 90n)^\circ$ is

$$\frac{\sqrt{2}}{2}, \qquad -\frac{\sqrt{2}}{2}\,i, \qquad \frac{\sqrt{2}}{2}, \qquad -\frac{\sqrt{2}}{2}\,i$$

for the four remainders. So every even n contributes $\frac{\sqrt{2}}{2}$ and every odd n contributes $-\frac{\sqrt{2}}{2}\,i$. The last term has $n = 40$, since $45 + 90 \cdot 40 = 3645$, and among $n = 0, 1, \ldots, 40$ there are 21 even values and 20 odd values. The sum is

$$21 \cdot \frac{\sqrt{2}}{2} - 20 \cdot \frac{\sqrt{2}}{2}\,i = \frac{\sqrt{2}}{2}(21 - 20i).$$

Answer: (D) $\frac{\sqrt{2}}{2}(21 - 20i)$. □

12. (2006 AIME II, Problem #1) In convex hexagon $ABCDEF$, all six sides are congruent, $\angle A$ and $\angle D$ are right angles, and $\angle B$, $\angle C$, $\angle E$, and $\angle F$ are congruent. The area of the hexagonal region is $2116(\sqrt{2}+1)$. Find AB.

Strategy. The angle sum gives the four equal angles. Extending pairs of sides then shows that the hexagon is a square with two 45°-45°-90° triangles cut from opposite corners.

Solution. The interior angles of a hexagon add up to 720°, so each of the four equal angles is $\frac{1}{4}(720^\circ - 2 \cdot 90^\circ) = 135^\circ$. Let s be the common side length. Extend AB beyond B and DC beyond C to meet at X. The angles XBC and XCB both equal $180^\circ - 135^\circ = 45^\circ$, so BXC is a 45°-45°-90° triangle with hypotenuse $BC = s$, and $BX = CX = \frac{s}{\sqrt{2}}$. In the same way, AF and DE, extended beyond F and E, meet at a point Y with $FY = EY = \frac{s}{\sqrt{2}}$. The quadrilateral $AXDY$ has right angles at A, X, and D, hence also at Y, and

$$AX = s + \frac{s}{\sqrt{2}} = XD,$$

so it is a square of side $s\left(1 + \frac{1}{\sqrt{2}}\right)$. The hexagon is this square without the triangles BXC and EYF, each of area $\frac{1}{2}\left(\frac{s}{\sqrt{2}}\right)^2 = \frac{s^2}{4}$. Its area is therefore

$$s^2\left(1 + \frac{1}{\sqrt{2}}\right)^2 - \frac{s^2}{2} = s^2\left(\frac{3}{2} + \sqrt{2}\right) - \frac{s^2}{2} = s^2(1 + \sqrt{2}).$$

Setting $s^2(1 + \sqrt{2}) = 2116(\sqrt{2} + 1)$ gives $s^2 = 2116$, so $AB = s = 46$.

Answer: 46. □

13. (2007 AIME I, Problem #4) Three planets revolve about a star in coplanar circular orbits with the star at the center. All planets revolve in the same direction, each at a constant speed, and the periods of their orbits are 60, 84, and 140 years. The positions of the star and all three planets are currently collinear. They will next be collinear after n years. Find n.

Strategy. Describe each planet by the angle of its position around the star. Two planets are collinear with the star exactly when their angles differ by a multiple of 180°, since a half turn takes a ray from the star to the opposite ray of the same line.

Solution. Put the star at the origin and describe each planet by the angle, in standard position, of the ray from the star through the planet. A planet with period T years turns through $\frac{360^\circ}{T}$ each year, so after t years the three angles have grown by

$$\frac{360t}{60} = 6t, \qquad \frac{360t}{84} = \frac{30t}{7}, \qquad \frac{360t}{140} = \frac{18t}{7}$$

degrees. Two planets lie on one line through the star exactly when their rays from the star are the same or opposite, that is, when their angles are coterminal or a half turn apart. So they are collinear with the star exactly when their angles differ by a multiple of 180°. The star and all three planets are collinear exactly when this holds for every pair of planets.

At the start every difference is a multiple of 180°. After t years the three differences have changed by

$$6t - \frac{30t}{7} = \frac{12t}{7}, \qquad \frac{30t}{7} - \frac{18t}{7} = \frac{12t}{7}, \qquad 6t - \frac{18t}{7} = \frac{24t}{7}$$

degrees. So the positions are collinear again exactly when $\frac{12t}{7}$ is a multiple of 180, since $\frac{24t}{7}$ is then one too. This happens exactly when t is a multiple of $\frac{7 \cdot 180}{12} = 105$, so the next time is $n = 105$.

Answer: 105. □

14. (1985 AHSME, Problem #10) An arbitrary circle can intersect the graph of $y = \sin x$ in

(A) at most 2 points **(B)** at most 4 points **(C)** at most 6 points
(D) at most 8 points **(E)** more than 16 points

Strategy. A circle of very large radius tangent to the x-axis has a long, nearly flat arc just above the axis. The graph of $y = \sin x$ rises to 1 and falls to -1 in every period, so it crosses such an arc many times.

Solution. Consider the circle with center $(0, 10^4)$ and radius 10^4, which touches the x-axis at the origin. For $0 \le x \le 60$, the lower half of this circle is the graph of

$$g(x) = 10^4 - \sqrt{10^8 - x^2} = \frac{x^2}{10^4 + \sqrt{10^8 - x^2}},$$

so $0 \le g(x) \le \frac{x^2}{10^4} \le \frac{3600}{10^4} < 1$. The sine equals 1 at $\frac{\pi}{2} + 2k\pi$ and -1 at $\frac{3\pi}{2} + 2k\pi$ for every integer k. Consider the 18 points

$$x_j = \frac{\pi}{2} + j\pi, \qquad j = 0, 1, \ldots, 17,$$

which all lie in $[0, 60]$, since $x_{17} = \frac{35\pi}{2} < 60$. The difference $\sin x - g(x)$ is positive at x_j for even j, where $\sin x_j = 1 > g(x_j)$, and negative for odd j, where $\sin x_j = -1 < 0 \le g(x_j)$. The graphs of $y = \sin x$ and $y = g(x)$ are unbroken, so this difference equals 0 somewhere strictly between x_j and x_{j+1} for each $j = 0, 1, \ldots, 16$. These 17 zeros give 17 different

points where the circle meets the graph of $y = \sin x$. So a circle can meet the graph in more than 16 points, and the statements (A) through (D) are false.

Answer: (E) more than 16 points. □

15. (2001 AIME I, Problem #4) In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

Strategy. Find angle C and show that triangle ATC is isosceles, so $AC = 24$. Then drop the perpendicular from T to AB to find AB, and use the altitude $AC \sin A$ for the area.

Solution. The third angle is $C = 180° - 60° - 45° = 75°$. The bisector makes $\angle TAC = \angle TAB = 30°$, so in triangle ATC,

$$\angle ATC = 180° - 30° - 75° = 75° = \angle ACT.$$

Thus triangle ATC is isosceles with $AC = AT = 24$.

Let H be the foot of the perpendicular from T to AB. It lies between A and B, because the angles of triangle ATB at A and at B are acute. The right triangle AHT has hypotenuse 24 and a 30° angle at A, so

$$TH = 24 \sin 30° = 12, \qquad AH = 24 \cos 30° = 12\sqrt{3}.$$

The right triangle THB has a 45° angle at B, so it is a 45°-45°-90° triangle and $HB = TH = 12$. Hence $AB = 12\sqrt{3} + 12$.

The altitude from C to line AB has length $AC \sin A = 24 \sin 60° = 12\sqrt{3}$, so the area of triangle ABC is

$$\frac{1}{2}\left(12 + 12\sqrt{3}\right) \cdot 12\sqrt{3} = 216 + 72\sqrt{3}.$$

Thus $a = 216$, $b = 72$, and $c = 3$, and $a + b + c = 291$.

Answer: 291. □

2. Answer Key

Problem	Letter	Value
1	C	$\frac{5\sqrt{3}}{2}$
2	B	$\frac{\sqrt{3}}{3}$
3	D	$3 : \pi$
4	D	10
5	D	$\frac{1}{2}\log_b(1 - b^{2a})$
6	A	$\sin^2\theta$
7	D	$(3, 4)$
8	A	$\frac{\pi}{9}$
9	E	h
10	E	25
11	D	$\frac{\sqrt{2}}{2}(21 - 20i)$
12		46
13		105
14	E	more than 16 points
15		291