INCLUSION-EXCLUSION AND DERANGEMENTS FOR AMC 12

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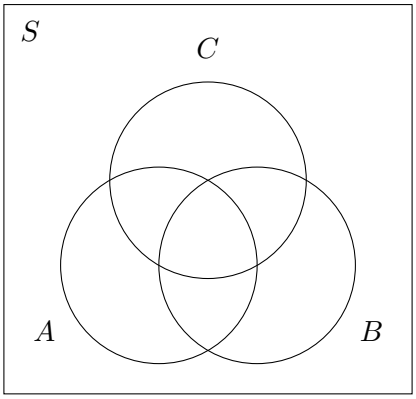
Contents

1. Inclusion-Exclusion 2
1.1. Two and Three Sets 2
1.2. Any Number of Sets 3
2. Counting Integers 5
2.1. Divisibility Conditions 5
2.2. Euler's Function 6
3. Distributions 7
3.1. No Empty Box 7
3.2. Stars and Bars with Upper Bounds 8
4. Derangements 9
4.1. Counting Derangements 9
4.2. The Derangement Recursion 10
4.3. A Given Number of Fixed Points 10
5. Problems 11

1. Inclusion-Exclusion

This section counts the elements of a union of sets, and the elements in none of the sets, from the sizes of the intersections.

1.1. **Two and Three Sets.** For a finite set A, $|A|$ denotes its number of elements. The **union** $A \cup B$ is the set of elements in A or B or both, and the **intersection** $A \cap B$ is the set of elements in both. In a counting problem, all the sets are subsets of a set S of all the objects being counted. The **complement** $\overline{A}$ is the set of elements of S that are not in A, so $|\overline{A}| = |S| - |A|$. An element of S lies in none of the sets A, B, C exactly when it lies outside $A \cup B \cup C$, so the number of such elements is $|S| - |A \cup B \cup C|$. The figure shows three sets A, B, C inside S.



Theorem 1.1.1 (Inclusion-Exclusion for Two and Three Sets)**.** *For finite sets A, B, C,*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Proof. Consider how many times each element is counted on the right side. An element outside the union is not counted at all. In the first formula, an element in exactly one of the sets is counted once, and an element in both is counted $1 + 1 - 1 = 1$ time. In the second formula, an element in exactly one of the sets is counted once, an element in exactly two of them is counted $2 - 1 = 1$ time, and an element in all three is counted $3 - 3 + 1 = 1$ time. □

When the sets are described by properties, the union consists of the objects with at least one of the properties, and its complement consists of the objects with none of them.

Counting in two ways gives a second relation. For sets $A_1, A_2, \ldots, A_n$, let e_m be the number of elements that lie in exactly m of the sets. Count the pairs consisting of a set and an element of that set. Counted set by set, there are $|A_1| + \cdots + |A_n|$ pairs. Counted element by element, there are $e_1 + 2e_2 + \cdots + ne_n$ pairs, since an element in exactly m of the sets belongs to m pairs. Hence

$$|A_1| + |A_2| + \cdots + |A_n| = e_1 + 2e_2 + 3e_3 + \cdots + ne_n.$$

Also $e_1 + e_2 + \cdots + e_n = |A_1 \cup A_2 \cup \cdots \cup A_n|$, since each element of the union lies in exactly m of the sets for one value of $m \geq 1$.

Examples.

Problem 1.1.2 (2017 AIME II, Problem #1)**.** Find the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are subsets of neither $\{1, 2, 3, 4, 5\}$ nor $\{4, 5, 6, 7, 8\}$.

Strategy. Count the subsets of $\{1, \ldots, 5\}$ or of $\{4, \ldots, 8\}$ with inclusion-exclusion, and subtract from the number of all subsets.

Solution. Let S be the set of all $2^8 = 256$ subsets of $\{1, 2, \ldots, 8\}$, A the set of subsets of $\{1, 2, 3, 4, 5\}$ and B the set of subsets of $\{4, 5, 6, 7, 8\}$, so $|A| = |B| = 2^5 = 32$. A subset lies in both A and B exactly when it is a subset of $\{1, \ldots, 5\} \cap \{4, \ldots, 8\} = \{4, 5\}$, so $|A \cap B| = 2^2 = 4$. By inclusion-exclusion,

$$|A \cup B| = 32 + 32 - 4 = 60.$$

The required subsets are the ones in neither A nor B, so there are $256 - 60 = 196$ of them.

Answer: 196. □

Problem 1.1.3 (1998 AHSME, Problem #24)**.** Call a 7-digit telephone number $d_1d_2d_3$-$d_4d_5d_6d_7$ *memorable* if the prefix sequence $d_1d_2d_3$ is exactly the same as either of the sequences $d_4d_5d_6$ or $d_5d_6d_7$ (possibly both). Assuming that each d_i can be any of the ten decimal digits 0, 1, 2, ..., 9, the number of different memorable telephone numbers is

(A) 19,810 **(B)** 19,910 **(C)** 19,990 **(D)** 20,000 **(E)** 20,100

Strategy. Count the numbers whose prefix equals $d_4d_5d_6$, those whose prefix equals $d_5d_6d_7$, and those with both properties.

Solution. Let A be the set of telephone numbers with $d_1d_2d_3 = d_4d_5d_6$ and B the set with $d_1d_2d_3 = d_5d_6d_7$. A number in A is determined by the arbitrary digits d_1, d_2, d_3, d_7, so $|A| = 10^4$; a number in B is determined by d_1, d_2, d_3, d_4, so $|B| = 10^4$. A number in both sets has $d_4 = d_1$, $d_5 = d_2$, $d_6 = d_3$ and $d_5 = d_1$, $d_6 = d_2$, $d_7 = d_3$. Then $d_2 = d_5 = d_1$, $d_3 = d_6 = d_2$, $d_4 = d_1$ and $d_7 = d_3$, so all seven digits are equal; conversely, such a number lies in both sets. So $|A \cap B| = 10$, and by inclusion-exclusion

$$|A \cup B| = 10^4 + 10^4 - 10 = 19{,}990.$$

Answer: (C) 19,990. □

Problem 1.1.4 (2016 AMC 12A, Problem #11)**.** Each of the 100 students in a certain summer camp can either sing, dance, or act. Some students have more than one talent, but no student has all three talents. There are 42 students who cannot sing, 65 students who cannot dance, and 29 students who cannot act. How many students have two of these talents?

(A) 16 **(B)** 25 **(C)** 36 **(D)** 49 **(E)** 64

Strategy. Count the talents with multiplicity: the numbers of singers, dancers and actors add up to the number of students with one talent plus twice the number with two talents.

Solution. Every student has at least one of the three talents and none has all three. Let e_1 and e_2 be the numbers of students with exactly one and exactly two talents, so $e_1+e_2 = 100$. The numbers of students who can sing, dance and act are $100 - 42 = 58$, $100 - 65 = 35$ and $100 - 29 = 71$. By the relation $|A_1| + \cdots + |A_n| = e_1 + 2e_2 + \cdots + ne_n$, with $e_3 = 0$,

$$58 + 35 + 71 = e_1 + 2e_2, \qquad \text{that is,} \qquad e_1 + 2e_2 = 164.$$

Subtracting $e_1 + e_2 = 100$ gives $e_2 = 64$.

Answer: (E) 64. □

1.2. **Any Number of Sets.** For sets $A_1, A_2, \ldots, A_n$ inside S and $1 \le k \le n$, write s_k for the sum of the sizes of all the intersections of k of the sets, and put $s_0 = |S|$. The sum s_k has $\binom{n}{k}$ terms. For three sets, $s_1 = |A_1|+|A_2|+|A_3|$, $s_2 = |A_1 \cap A_2|+|A_1 \cap A_3|+|A_2 \cap A_3|$ and $s_3 = |A_1 \cap A_2 \cap A_3|$.

Theorem 1.2.1 (Inclusion-Exclusion)**.** *For finite sets $A_1, A_2, \ldots, A_n$ inside a finite set S,*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = s_1 - s_2 + s_3 - \cdots + (-1)^{n-1}s_n,$$

and the number of elements of S in none of the sets is

$$s_0 - s_1 + s_2 - \cdots + (-1)^n s_n.$$

Proof. An element in none of the sets is counted once in s_0 and in no other s_k, so it contributes 1 to the second expression. Suppose an element lies in exactly $m \geq 1$ of the sets. It lies in exactly $\binom{m}{k}$ of the intersections of k of the sets, namely those formed from k of its m sets, so it contributes

$$\binom{m}{0} - \binom{m}{1} + \binom{m}{2} - \cdots + (-1)^m \binom{m}{m} = (1-1)^m = 0$$

to the second expression, by the binomial theorem. So the second expression counts exactly the elements in none of the sets. Subtracting it from $s_0 = |S|$ gives the first formula. □

To find s_k, sort the intersections of k of the sets into types, so that all the intersections of one type have the same size, and multiply each size by the number of intersections of that type.

In the **symmetric case**, all the intersections of k of the sets have the same size N_k, for each k. Then $s_k = \binom{n}{k} N_k$, and the number of elements in none of the sets is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} N_k = N_0 - \binom{n}{1} N_1 + \binom{n}{2} N_2 - \cdots + (-1)^n N_n,$$

where $N_0 = |S|$. Sections 3 and 4 use this form.

Examples.

Problem 1.2.2 (2001 AIME II, Problem #9)**.** Each unit square of a 3-by-3 unit-square grid is to be colored either blue or red. For each square, either color is equally likely to be used. The probability of obtaining a grid that does not have a 2-by-2 red square is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Strategy. Count the colorings that have a red 2-by-2 square by inclusion-exclusion over the four 2-by-2 squares of the grid. Sort the intersections by how many unit squares they force to be red.

Solution. There are $2^9 = 512$ colorings, all equally likely, so the probability is the number of colorings without a red 2-by-2 square divided by 512. The grid has four 2-by-2 squares: upper left, upper right, lower left and lower right. For each of them, consider the colorings in which it is red. If some of these squares must be red, the c unit squares they cover are red and the other $9-c$ are free, so the intersection has 2^{9-c} colorings.

- One 2-by-2 square covers 4 unit squares: $s_1 = 4 \cdot 2^5 = 128$.
- Two squares side by side (upper left and upper right, for example) cover 6 unit squares, and there are 4 such pairs. Two diagonally opposite squares share only the center and cover 7, and there are 2 such pairs. So $s_2 = 4 \cdot 2^3 + 2 \cdot 2^2 = 40$.
- Any three of the squares cover every unit square except one corner, 8 in all: $s_3 = 4 \cdot 2 = 8$.
- All four cover the grid: $s_4 = 1$.

By inclusion-exclusion, $128 - 40 + 8 - 1 = 95$ colorings have a red 2-by-2 square, so $512 - 95 = 417$ do not. Since $417 = 3 \cdot 139$ is odd, $\frac{417}{512}$ is in lowest terms, and $m + n = 417 + 512 = 929$.

Answer: 929. □

Problem 1.2.3 (2020 AIME II, Problem #9)**.** While watching a show, Ayako, Billy, Carlos, Dahlia, Ehuang, and Frank sat in that order in a row of six chairs. During the break, they went to the kitchen for a snack. When they came back, they sat on those six chairs in such a way that if two of them sat next to each other before the break, then they did not sit next to each other after the break. Find the number of possible seating orders they could have chosen after the break.

Strategy. Apply inclusion-exclusion to the five pairs of people who sat next to each other before the break. People who must sit together form chains, and each chain is treated as one unit.

Solution. Write the six people as A, B, C, D, E, F in their order before the break. The pairs of neighbors before the break are AB, BC, CD, DE, EF; number them 1 to 5 in this order. Let S be the set of $6! = 720$ orders after the break. For each pair, consider the orders in which the two people of the pair sit next to each other. The required orders are those in none of these five sets.

Take a set T of j of the pairs. The pairs in T join the people into chains of consecutive letters; for example, $T = \{1, 2, 5\}$, made of the pairs AB, BC, EF, gives the chains ABC and EF. Suppose every pair in T sits together. Each person inside a chain sits next to both of his or her neighbors in the chain, and a row of chairs gives each person at most two neighbors, so the people of a chain fill consecutive chairs, in alphabetical order or in the reverse order. Treating each chain as one unit leaves $6 - j$ units. So the intersection has $(6 - j)!\, 2^r$ orders, where r is the number of chains with at least two people, that is, the number of runs of consecutive numbers in T. Sorting the sets T by j and by r gives the sums s_j.

- $j = 1$: 5 sets T, each with $r = 1$, so $s_1 = 5 \cdot 2 \cdot 5! = 1200$.
- $j = 2$: 4 sets are two consecutive pairs ($r = 1$) and the other 6 have $r = 2$, so $s_2 = (4 \cdot 2 + 6 \cdot 4) \cdot 4! = 768$.
- $j = 3$: 3 sets are three consecutive pairs ($r = 1$), the set $\{1, 3, 5\}$ has $r = 3$, and the other 6 have $r = 2$, so $s_3 = (3 \cdot 2 + 6 \cdot 4 + 1 \cdot 8) \cdot 3! = 228$.
- $j = 4$: leaving out pair 1 or pair 5 gives $r = 1$, and leaving out pair 2, 3 or 4 gives $r = 2$, so $s_4 = (2 \cdot 2 + 3 \cdot 4) \cdot 2! = 32$.
- $j = 5$: one chain of all six people, so $s_5 = 2 \cdot 1! = 2$.

By inclusion-exclusion, the number of orders is

$$720 - 1200 + 768 - 228 + 32 - 2 = 90.$$

Answer: 90. □

2. Counting Integers

This section counts the integers in a range that satisfy divisibility conditions.

2.1. **Divisibility Conditions.** For positive integers N and d, the multiples of d among $1, 2, \ldots, N$ are $d, 2d, \ldots, \lfloor N/d \rfloor\, d$, so there are $\lfloor N/d \rfloor$ of them. An integer is a multiple of each of several positive integers exactly when it is a multiple of their least common multiple. When the numbers are pairwise relatively prime, their least common multiple is their product; for example, the integers divisible by 2, 3 and 5 are the multiples of 30. To count the integers divisible by none of $d_1, \ldots, d_k$, let A_i be the set of multiples of d_i and apply inclusion-exclusion.

Example.

Problem 2.1.1 (2005 AMC 12A, Problem #18)**.** Call a number "prime-looking" if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?
(A) 100 **(B)** 102 **(C)** 104 **(D)** 106 **(E)** 108

Strategy. Count the numbers less than 1000 divisible by none of 2, 3, 5, then remove 1 and the primes among them.

Solution. For $d = 2, 3, 5$, let A_d be the set of multiples of d among $1, 2, \ldots, 999$. Since 2, 3, 5 are pairwise relatively prime, the intersections consist of the multiples of 6, 10, 15 and 30. With $\lfloor 999/2 \rfloor = 499$, $\lfloor 999/3 \rfloor = 333$, $\lfloor 999/5 \rfloor = 199$, $\lfloor 999/6 \rfloor = 166$, $\lfloor 999/10 \rfloor = 99$, $\lfloor 999/15 \rfloor = 66$ and $\lfloor 999/30 \rfloor = 33$, inclusion-exclusion gives

$$|A_2 \cup A_3 \cup A_5| = (499 + 333 + 199) - (166 + 99 + 66) + 33 = 1031 - 331 + 33 = 733.$$

So $999 - 733 = 266$ numbers less than 1000 are divisible by none of 2, 3, 5. Each of them is 1, a prime or a prime-looking number. There are $168 - 3 = 165$ primes less than 1000 other than 2, 3, 5, and none of them is divisible by 2, 3 or 5. Hence there are $266 - 1 - 165 = 100$ prime-looking numbers.

Answer: (A) 100. □

2.2. Euler's Function.

Definition 2.2.1. For a positive integer n, **Euler's function** $\varphi(n)$ is the number of integers among $1, 2, \ldots, n$ that are relatively prime to n.

Theorem 2.2.2 (Euler's Function)**.** *If $p_1, p_2, \ldots, p_r$ are the different primes that divide n, then*

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_r}\right).$$

Proof. An integer is relatively prime to n exactly when it is divisible by none of $p_1, \ldots, p_r$. Let A_i be the set of multiples of p_i among $1, \ldots, n$. For indices $i_1 < \cdots < i_k$, the product $p_{i_1} \cdots p_{i_k}$ divides n, so $A_{i_1} \cap \cdots \cap A_{i_k}$ consists of the multiples of this product and has $n/(p_{i_1} \cdots p_{i_k})$ elements. By inclusion-exclusion,

$$\varphi(n) = n - \sum_i \frac{n}{p_i} + \sum_{i<j} \frac{n}{p_i p_j} - \cdots + (-1)^r \frac{n}{p_1 p_2 \cdots p_r}.$$

Multiplying out $n\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_r}\right)$, choosing 1 or $-\frac{1}{p_i}$ from each factor, gives exactly these terms. □

For example, $\varphi(30) = 30 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 8$ and $\varphi(1000) = 1000 \cdot \frac{1}{2} \cdot \frac{4}{5} = 400$.

Let $n \geq 2$. Since $\gcd(n - a, n) = \gcd(a, n)$, the integers from 1 to $n - 1$ that are relatively prime to n pair up as a and $n - a$. When $n > 2$, no such integer a satisfies $a = n - a$, since then $\gcd(a, n) = a = \frac{n}{2} > 1$. So each pair has one member less than $\frac{n}{2}$ and one greater, and exactly $\frac{1}{2}\varphi(n)$ of these integers are less than $\frac{n}{2}$. Similarly, $\gcd(a + tn, n) = \gcd(a, n)$ for every integer t, and n consecutive integers leave all n remainders on division by n, so every block of n consecutive integers contains exactly $\varphi(n)$ integers relatively prime to n.

Example.

Problem 2.2.3 (2014 AIME I, Problem #3)**.** Find the number of rational numbers r, $0 < r < 1$, such that when r is written as a fraction in lowest terms, the numerator and the denominator have a sum of 1000.

Strategy. Write $r = \frac{a}{1000-a}$. The fraction is in lowest terms exactly when a is relatively prime to 1000, and $r < 1$ means $a < 500$.

Solution. If $r = \frac{a}{b}$ in lowest terms with $a + b = 1000$, then $b = 1000 - a$, and $0 < r < 1$ means $0 < a < b$, that is, $a < 500$. Since $\gcd(a, 1000 - a) = \gcd(a, 1000)$, the fraction $\frac{a}{1000-a}$ is in lowest terms exactly when a is relatively prime to 1000. Each such a gives a different r, because a rational number has only one form in lowest terms. So the answer is the number of integers a with $1 \leq a < 500$ relatively prime to 1000. Since $1000 = 2^3 \cdot 5^3$,

$$\varphi(1000) = 1000\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 400,$$

which is also $1000 - 500 - 200 + 100$ by inclusion-exclusion. By the pairing $a \leftrightarrow 1000 - a$, half of these 400 numbers are less than 500. The answer is 200.

Answer: 200. □

3. Distributions

This section counts distributions of objects into boxes in which no box is empty, and in which no box holds too many objects.

3.1. **No Empty Box.** Placing k different objects into n different boxes means choosing a box for each object, so there are n^k distributions. A distribution is the same as a function from the set of objects to the set of boxes. A function f from a set X to a set Y is **onto** if every element of Y equals $f(x)$ for some x in X; a distribution with no empty box is an onto function.

Theorem 3.1.1 (Distributions with No Empty Box)**.** *The number of ways to place k different objects into n different boxes so that no box is empty is*

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-j)^k = n^k - \binom{n}{1}(n-1)^k + \binom{n}{2}(n-2)^k - \cdots .$$

Proof. For each box, consider the distributions in which that box is empty. If all the boxes of a given set of j boxes are empty, every object goes into one of the other $n-j$ boxes, so there are $(n-j)^k$ such distributions. Apply inclusion-exclusion in the symmetric case. □

For $k \geq 1$, the count is $2^k - 2$ for two boxes and $3^k - 3\cdot 2^k + 3$ for three boxes. The count is 0 when $k < n$.

Examples.

Problem 3.1.2. How many functions f from $\{1,2,3,4,5,6\}$ to $\{1,2,3,4\}$ take each of the values 1, 2, 3, and 4 at least once?

Strategy. These are the onto functions from a 6-element set to a 4-element set. Apply the theorem with $k = 6$ objects and $n = 4$ boxes.

Solution. Regard $1,\ldots,6$ as objects and the values $1,2,3,4$ as boxes: f puts object x into box $f(x)$, and f takes every value exactly when no box is empty. By the theorem with $k = 6$ and $n = 4$, the number of such functions is

$$4^6 - \binom{4}{1}3^6 + \binom{4}{2}2^6 - \binom{4}{3}1^6 + \binom{4}{4}0^6 = 4096 - 2916 + 384 - 4 + 0 = 1560.$$

Answer: 1560. □

Problem 3.1.3 (2013 AIME II, Problem #9)**.** A 7×1 board is completely covered by $m\times 1$ tiles without overlap; each tile may cover any number of consecutive squares, and each tile lies completely on the board. Each tile is either red, blue, or green. Let N be the number of tilings of the 7×1 board in which all three colors are used at least once. For example, a 1×1 red tile followed by a 2×1 green tile, a 1×1 green tile, a 2×1 blue tile, and a 1×1 green tile is a valid tiling. Note that if the 2×1 blue tile is replaced by two 1×1 blue tiles, this results in a different tiling. Find the remainder when N is divided by 1000.

Strategy. Sort the tilings by the number k of tiles. The lengths of the tiles are positive integers with sum 7, and the coloring of the tiles must use all three colors, so it is an onto function from the k tiles to the three colors.

Solution. A tiling with k tiles is given by the lengths $\ell_1,\ldots,\ell_k$ of its tiles from left to right, positive integers with $\ell_1+\cdots+\ell_k = 7$, together with a color for each tile. By stars and bars there are $\binom{6}{k-1}$

choices of the lengths. Coloring the k tiles so that all three colors appear is placing k different tiles into 3 boxes with no box empty, in $3^k - 3 \cdot 2^k + 3$ ways; this is 0 for $k = 1, 2$. Hence

$$N = \sum_{k=3}^{7} \binom{6}{k-1} \left(3^k - 3 \cdot 2^k + 3\right) = 15 \cdot 6 + 20 \cdot 36 + 15 \cdot 150 + 6 \cdot 540 + 1 \cdot 1806.$$

The terms are 90, 720, 2250, 3240 and 1806, so $N = 8106$, and the remainder when N is divided by 1000 is 106.

Answer: 106. □

3.2. **Stars and Bars with Upper Bounds.** By stars and bars, the equation $x_1+x_2+\cdots+x_k = n$ has $\binom{n+k-1}{k-1}$ solutions in nonnegative integers. When every variable must also be at most m and no two variables can exceed m at the same time, it is enough to subtract the solutions in which one variable exceeds m. Inclusion-exclusion handles the general case, in which several variables can exceed m at the same time.

Theorem 3.2.1 (Upper Bounds)**.** *Let $k \geq 1$, $m \geq 0$ and $n \geq 0$ be integers. The number of solutions of $x_1 + x_2 + \cdots + x_k = n$ in integers with $0 \leq x_i \leq m$ for every i is*

$$\sum_j (-1)^j \binom{k}{j} \binom{n - j(m+1) + k - 1}{k-1},$$

where the sum runs over the integers j with $0 \leq j \leq k$ and $j(m+1) \leq n$.

Proof. For each i, consider the solutions in nonnegative integers with $x_i \geq m+1$. For a given set of j indices, replacing each of those x_i by $x_i - (m+1)$ turns the solutions with all j of these variables at least $m+1$ into the solutions in nonnegative integers of an equation with sum $n - j(m+1)$, and the replacement can be undone. By stars and bars there are $\binom{n-j(m+1)+k-1}{k-1}$ of them when $j(m+1) \leq n$, and none otherwise. Apply inclusion-exclusion in the symmetric case to the $\binom{n+k-1}{k-1}$ solutions in nonnegative integers. □

If the variables have different upper bounds, the same argument works, with the size of each intersection found separately. A lower bound $x_i \geq c_i$ is removed first by putting $y_i = x_i - c_i$; this lowers the sum, and the upper bound of that variable, by c_i.

Bounded variables also describe coefficients. Multiplying out

$$(1 + x + \cdots + x^{m_1})(1 + x + \cdots + x^{m_2}) \cdots (1 + x + \cdots + x^{m_k})$$

gives one term $x^{t_1+t_2+\cdots+t_k}$ for each choice of exponents with $0 \leq t_i \leq m_i$ for every i, so the coefficient of x^n is the number of solutions of $t_1 + t_2 + \cdots + t_k = n$ with these bounds.

Example.

Problem 3.2.2. Four dice, colored red, blue, green, and yellow, are rolled, and each shows one of the numbers 1 through 6. In how many of the 6^4 possible outcomes is the sum of the four numbers equal to 16?

Strategy. Subtract 1 from each number to get variables from 0 to 5 with sum 12, and apply the theorem.

Solution. An outcome is a list (x_1, x_2, x_3, x_4) of integers from 1 to 6, one for each color, and the condition is $x_1+x_2+x_3+x_4 = 16$. Put $y_i = x_i - 1$. Then $0 \leq y_i \leq 5$ and $y_1+y_2+y_3+y_4 = 12$. Apply the theorem with $k = 4$, $n = 12$ and $m = 5$. The term $\binom{15}{3}$ counts all solutions in nonnegative integers; $\binom{9}{3}$ counts those with a given $y_i \geq 6$; $\binom{3}{3} = 1$ counts those with two given variables at

least 6 (both equal to 6, the others 0); three variables cannot all be at least 6, since $3 \cdot 6 > 12$. So the number of outcomes is

$$\binom{15}{3} - \binom{4}{1}\binom{9}{3} + \binom{4}{2}\binom{3}{3} = 455 - 4 \cdot 84 + 6 \cdot 1 = 125.$$

Answer: 125. □

4. Derangements

This section counts the permutations that leave no object in its place, and those that leave a given number of objects in place.

4.1. Counting Derangements.

Definition 4.1.1. A **permutation** of $1, 2, \ldots, n$ is a list $(a_1, a_2, \ldots, a_n)$ that contains each of $1, 2, \ldots, n$ exactly once. The number i is a **fixed point** of the permutation if $a_i = i$. A **derangement** is a permutation with no fixed points. The number of derangements of $1, 2, \ldots, n$ is denoted D_n, and $D_0 = 1$.

In applications, n objects are given back to their n owners, one object to each owner. Number the owners and their objects from 1 to n, and let a_i be the object that owner i receives. A derangement is a way in which no owner receives his or her own object.

Theorem 4.1.2 (Derangements)**.** *For every integer $n \geq 0$,*

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)!.$$

Proof. For each i, consider the permutations with $a_i = i$. A permutation with $a_i = i$ for every i in a given set of j positions has its other $n - j$ entries in any order of the remaining $n - j$ numbers, so there are $(n-j)!$ such permutations. Inclusion-exclusion in the symmetric case gives $D_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)!$, and $\binom{n}{j}(n-j)! = \frac{n!}{j!}$. □

The values of D_n for $n \leq 8$ are shown in the table.

n	0	1	2	3	4	5	6	7	8
D_n	1	0	1	2	9	44	265	1854	14833

Example.

Problem 4.1.3. Six guests leave their coats at a party. When the party ends, the six coats are handed back at random, one to each guest. In how many ways can the coats be handed back so that no guest receives his or her own coat?

Strategy. A way of handing back the coats is a permutation; no guest receives his or her own coat exactly when the permutation is a derangement.

Solution. Number the guests and their coats 1 to 6, and let a_i be the coat handed to guest i. The ways of handing back the coats are the permutations $(a_1, \ldots, a_6)$ of $1, \ldots, 6$, and no guest receives his or her own coat exactly when there is no fixed point. By the theorem, with the terms $\binom{6}{j}(6-j)! = \frac{720}{j!}$ for $j = 0, 1, \ldots, 6$,

$$D_6 = 720 - 720 + 360 - 120 + 30 - 6 + 1 = 265.$$

Answer: 265. □

4.2. **The Derangement Recursion.**

Theorem 4.2.1 (Derangement Recursion)**.** *For $n \geq 2$, $D_n = (n-1)(D_{n-1} + D_{n-2})$.*

Proof. In a derangement, $a_1 = c$ for one of the $n-1$ numbers $c \neq 1$. Fix c and count the derangements with $a_1 = c$. If $a_c = 1$, the entries in the $n-2$ positions other than 1 and c are the numbers other than 1 and c, none in its own position; there are D_{n-2} such derangements. If $a_c \neq 1$, the entries in positions $2, 3, \ldots, n$ are the numbers other than c. Rename the number 1 as c. Then the conditions $a_c \neq 1$ and $a_i \neq i$ for $i \neq 1, c$ say that no entry in positions $2, \ldots, n$ equals its position, so these derangements correspond one to one to the derangements of $n-1$ numbers; there are D_{n-1} of them. Each of the $n-1$ values of c gives $D_{n-1} + D_{n-2}$ derangements. □

For example, $D_5 = 4(9+2) = 44$, $D_6 = 5(44+9) = 265$, $D_7 = 6(265+44) = 1854$ and $D_8 = 7(1854+265) = 14833$, in agreement with the table. Starting from $D_0 = 1$ and $D_1 = 0$, the recursion gives every value.

Example.

Problem 4.2.2. How many permutations $(a_1, a_2, \ldots, a_7)$ of $1, 2, \ldots, 7$ satisfy $a_1 = 2$ and $a_i \neq i$ for every i?

Strategy. Split according to whether $a_2 = 1$, as in the proof of the recursion.

Solution. If $a_2 = 1$, then $a_3, \ldots, a_7$ are $3, \ldots, 7$ in some order with no fixed point, so there are $D_5 = 44$ such permutations. If $a_2 \neq 1$, the entries $a_2, \ldots, a_7$ are the numbers $1, 3, 4, 5, 6, 7$. Rename the number 1 as 2; the conditions $a_2 \neq 1$ and $a_i \neq i$ for $3 \leq i \leq 7$ then say that the renamed list $a_2, \ldots, a_7$ is a derangement of $2, 3, \ldots, 7$, so there are $D_6 = 265$ such permutations. The total is $44 + 265 = 309$. This agrees with the recursion: each of the six possible values of a_1 gives $D_6 + D_5 = 309$ derangements, and $6 \cdot 309 = 1854 = D_7$.

Answer: 309. □

4.3. **A Given Number of Fixed Points.**

Theorem 4.3.1 (Fixed Points)**.** *For $0 \leq k \leq n$, the number of permutations of $1, 2, \ldots, n$ with exactly k fixed points is $\binom{n}{k} D_{n-k}$.*

Proof. Choose the set of k fixed points in $\binom{n}{k}$ ways. The other $n-k$ positions must hold the other $n-k$ numbers with none in its own position, which can be done in D_{n-k} ways. A permutation determines its set of fixed points, so each permutation is counted exactly once. □

The same idea counts arrangements in which exactly k objects are in their own places: choose those objects first, then count the arrangements of the rest in which no object is in its own place.

Example.

Problem 4.3.2. How many permutations of $1, 2, \ldots, 8$ have exactly three fixed points?

Strategy. Choose the three fixed points, then derange the other five numbers.

Solution. The three fixed points can be chosen in $\binom{8}{3} = 56$ ways. The other five positions hold the other five numbers with none in its own position, in $D_5 = 44$ ways. By the theorem, the number of permutations is $\binom{8}{3} D_5 = 56 \cdot 44 = 2464$.

Answer: 2464. □

5. Problems

Beginner.

1. (1966 AHSME, Problem #29) The number of positive integers less than 1000 divisible by neither 5 nor 7 is:

(A) 688 **(B)** 686 **(C)** 684 **(D)** 658 **(E)** 630

2. (1990 AHSME, Problem #10) An $11 \times 11 \times 11$ wooden cube is formed by gluing together 11^3 unit cubes. What is the greatest number of unit cubes that can be seen from a single point?

(A) 328 **(B)** 329 **(C)** 330 **(D)** 331 **(E)** 332

3. (1990 AIME, Problem #1) The increasing sequence 2, 3, 5, 6, 7, 10, 11, ... consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500th term of this sequence.

4. Five married couples pose for a photograph. The five husbands stand in a row, and the five wives stand in a second row directly behind them, one wife behind each husband. In how many ways can the ten people be arranged so that no wife stands directly behind her own husband?

5. (2008 AMC 12A, Problem #19) In the expansion of

$$\left(1 + x + x^2 + \cdots + x^{27}\right)\left(1 + x + x^2 + \cdots + x^{14}\right)^2,$$

what is the coefficient of x^{28}?

(A) 195 **(B)** 196 **(C)** 224 **(D)** 378 **(E)** 405

Intermediate.

6. (2012 AMC 12B, Problem #12) How many sequences of zeros and/or ones of length 20 have all the zeros consecutive, or all the ones consecutive, or both?

(A) 190 **(B)** 192 **(C)** 211 **(D)** 380 **(E)** 382

7. (2024 AIME II, Problem #1) Among the 900 residents of Aimeville, there are 195 who own a diamond ring, 367 who own a set of golf clubs, and 562 who own a garden spade. In addition, each of the 900 residents owns a bag of candy hearts. There are 437 residents who own exactly two of these things, and 234 residents who own exactly three of these things. Find the number of residents of Aimeville who own all four of these things.

8. (1992 AIME, Problem #1) Find the sum of all positive rational numbers that are less than 10 and that have denominator 30 when written in lowest terms.

9. (2025 AIME II, Problem #7) Let A be the set of positive integer divisors of 2025. Let B be a randomly selected subset of A. The probability that B is a nonempty set with the property that the least common multiple of its elements is 2025 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

10. (2005 AIME II, Problem #4) Find the number of positive integers that are divisors of at least one of 10^{10}, 15^7, 18^{11}.

11. (2002 AIME II, Problem #9) Let S be the set $\{1, 2, 3, \ldots, 10\}$. Let n be the number of sets of two non-empty disjoint subsets of S. (*Disjoint sets* are defined as sets that have no common elements.) Find the remainder obtained when n is divided by 1000.

Advanced.

12. (2016 AIME II, Problem #11) For positive integers N and k, define N to be *k-nice* if there exists a positive integer a such that a^k has exactly N positive divisors. Find the number of positive integers less than 1000 that are neither 7-nice nor 8-nice.

13. (2012 AIME I, Problem #3) Nine people sit down for dinner where there are three choices of meals. Three people order the beef meal, three order the chicken meal, and three order the fish meal. The waiter serves the nine meals in random order. Find the number of ways in which the waiter could serve the meal types to the nine people so that exactly one person receives the type of meal ordered by that person.

14. (2021 AMC 10B, Problem #22) Ang, Ben, and Jasmin each have 5 blocks, colored red, blue, yellow, white, and green; and there are 5 empty boxes. Each of the people randomly and independently of the other two people places one of their blocks into each box. The probability that at least one box receives 3 blocks all of the same color is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 47 **(B)** 94 **(C)** 227 **(D)** 471 **(E)** 542

15. (2011 AIME II, Problem #12) Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Let the probability that each delegate sits next to at least one delegate from another country be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

16. (2018 AIME II, Problem #11) Find the number of permutations of 1, 2, 3, 4, 5, 6 such that for each k with $1 \leq k \leq 5$, at least one of the first k terms of the permutation is greater than k.