INCLUSION-EXCLUSION AND DERANGEMENTS FOR AMC 12: SOLUTIONS

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1. Solutions

Beginner.

1. (1966 AHSME, Problem #29) The number of positive integers less than 1000 divisible by neither 5 nor 7 is:

(A) 688 **(B)** 686 **(C)** 684 **(D)** 658 **(E)** 630

Strategy. Count the multiples of 5 or 7 below 1000 with inclusion-exclusion, and subtract.

Solution. Among $1, 2, \ldots, 999$ there are $\lfloor 999/5 \rfloor = 199$ multiples of 5 and $\lfloor 999/7 \rfloor = 142$ multiples of 7. A number is a multiple of both exactly when it is a multiple of 35, and there are $\lfloor 999/35 \rfloor = 28$ of these. By inclusion-exclusion, $199 + 142 - 28 = 313$ of the numbers are divisible by 5 or 7, so $999 - 313 = 686$ are divisible by neither.

Answer: (B) 686. □

2. (1990 AHSME, Problem #10) An $11 \times 11 \times 11$ wooden cube is formed by gluing together 11^3 unit cubes. What is the greatest number of unit cubes that can be seen from a single point?

(A) 328 **(B)** 329 **(C)** 330 **(D)** 331 **(E)** 332

Strategy. At most three faces of the cube can be seen at once, and they meet at a corner. Count the unit cubes on three such faces with inclusion-exclusion.

Solution. A face of the large cube can be seen from a point only if the point lies on the outer side of the plane of that face. Opposite faces have parallel planes with the cube between them, so no point lies on the outer side of both. Hence at most one face from each pair of opposite faces, at most three faces in all, can be seen, and three such faces meet at a corner. Only the surface of the large cube can be seen, so a visible unit cube has a face on a visible face of the large cube. From a point far out along the diagonal of the cube through that corner, the three faces at the corner are seen completely. So the greatest number is the number of unit cubes on three faces that meet at a corner.

Let A, B, C be the sets of unit cubes on these faces. Each face has $11^2 = 121$ cubes, any two of the faces share an edge of 11 cubes, and all three share the corner cube. By inclusion-exclusion,

$$|A \cup B \cup C| = 3 \cdot 121 - 3 \cdot 11 + 1 = 331.$$

Answer: (D) 331. □

3. (1990 AIME, Problem #1) The increasing sequence 2, 3, 5, 6, 7, 10, 11, ... consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500th term of this sequence.

Strategy. Count the terms up to N by removing the squares and the cubes with inclusion-exclusion, and choose N so that the count reaches 500.

Solution. A positive integer is both a square and a cube exactly when it is a sixth power. For $512 \le N \le 528$, the squares up to N are $1^2, \ldots, 22^2$ (since $22^2 = 484$ and $23^2 = 529$), the cubes up to N are $1^3, \ldots, 8^3$ (since $8^3 = 512$ and $9^3 = 729$), and the sixth powers up to N are 1 and 64. By inclusion-exclusion, exactly $22 + 8 - 2 = 28$ of the numbers $1, \ldots, N$ are squares or cubes, so $N - 28$ of them are terms of the sequence. For $N = 528$ this is 500, and 528 is itself a term, since it is neither a square nor a cube. So the 500th term is 528.

Answer: 528. □

4. Five married couples pose for a photograph. The five husbands stand in a row, and the five wives stand in a second row directly behind them, one wife behind each husband. In how many ways can the ten people be arranged so that no wife stands directly behind her own husband?

Strategy. Place the husbands first. For each order of the husbands, the wives must form a derangement.

Solution. The husbands can stand in $5! = 120$ orders. Fix their order, number the husbands 1 to 5 from left to right, and give each wife her husband's number. Let a_i be the number of the wife behind husband i. Then $(a_1, \ldots, a_5)$ is a permutation of $1, \ldots, 5$, and the condition says that it has no fixed points. So there are $D_5 = 44$ ways to place the wives for each order of the husbands, and $120 \cdot 44 = 5280$ arrangements in all.

Answer: 5280. □

5. (2008 AMC 12A, Problem #19) In the expansion of

$$\left(1 + x + x^2 + \cdots + x^{27}\right)\left(1 + x + x^2 + \cdots + x^{14}\right)^2,$$

what is the coefficient of x^{28}?

(A) 195 **(B)** 196 **(C)** 224 **(D)** 378 **(E)** 405

Strategy. The coefficient of x^{28} is the number of solutions of $a+b+c = 28$ with $0 \le a \le 27$ and $0 \le b, c \le 14$.

Solution. Each term of the expansion comes from choosing x^a from the first factor and x^b, x^c from the other two, so the coefficient of x^{28} is the number of solutions of $a + b + c = 28$ in integers with $0 \le a \le 27$, $0 \le b \le 14$ and $0 \le c \le 14$. By stars and bars there are $\binom{30}{2} = 435$ solutions in nonnegative integers. Those with $a \ge 28$ become solutions with sum 0 after subtracting 28 from a, so there is just 1 of them. Those with $b \ge 15$ become solutions with sum 13 after subtracting 15 from b, so there are $\binom{15}{2} = 105$ of them; the same holds for $c \ge 15$. No two of the bounds can fail together, since $28 + 15 > 28$ and $15 + 15 > 28$. By inclusion-exclusion, the coefficient is $435 - 1 - 105 - 105 = 224$.

Answer: (C) 224. □

Intermediate.

6. (2012 AMC 12B, Problem #12) How many sequences of zeros and/or ones of length 20 have all the zeros consecutive, or all the ones consecutive, or both?

(A) 190 **(B)** 192 **(C)** 211 **(D)** 380 **(E)** 382

Strategy. Count the sequences with all zeros consecutive, those with all ones consecutive, and those with both.

Solution. Let Z be the set of sequences whose zeros are consecutive, including the sequence with no zeros, and O the set of sequences whose ones are consecutive. If a sequence in Z has $k \ge 1$ zeros, they form one block starting at one of $21 - k$ positions, so

$$|Z| = 1 + \sum_{k=1}^{20}(21 - k) = 1 + (20 + 19 + \cdots + 1) = 211,$$

and $|O| = 211$ by exchanging zeros and ones. A sequence in both sets consists of at most one block of each digit. If both digits occur, it is $0\cdots01\cdots1$ or $1\cdots10\cdots0$ with 1 to 19

zeros, which gives $2 \cdot 19 = 38$ sequences; the two constant sequences are also in both sets. So $|Z \cap O| = 40$, and by inclusion-exclusion

$$|Z \cup O| = 211 + 211 - 40 = 382.$$

Answer: (E) 382. □

7. (2024 AIME II, Problem #1) Among the 900 residents of Aimeville, there are 195 who own a diamond ring, 367 who own a set of golf clubs, and 562 who own a garden spade. In addition, each of the 900 residents owns a bag of candy hearts. There are 437 residents who own exactly two of these things, and 234 residents who own exactly three of these things. Find the number of residents of Aimeville who own all four of these things.

Strategy. Count the things owned with multiplicity. The bags of candy hearts make every resident own at least one of the four things.

Solution. Let e_m be the number of residents who own exactly m of the four things. Every resident owns a bag of candy hearts, so $e_1 + e_2 + e_3 + e_4 = 900$; with $e_2 = 437$ and $e_3 = 234$ this gives $e_1 + e_4 = 229$. Counting each resident once for each thing owned,

$$195 + 367 + 562 + 900 = e_1 + 2e_2 + 3e_3 + 4e_4,$$

that is, $2024 = e_1 + 874 + 702 + 4e_4$, so $e_1 + 4e_4 = 448$. Subtracting $e_1 + e_4 = 229$ gives $3e_4 = 219$, so $e_4 = 73$.

Answer: 73. □

8. (1992 AIME, Problem #1) Find the sum of all positive rational numbers that are less than 10 and that have denominator 30 when written in lowest terms.

Strategy. The numbers are $\frac{a}{30}$ with $0 < a < 300$ and a relatively prime to 30. Count them with Euler's function, and pair a with $300 - a$.

Solution. A positive rational number less than 10 with denominator 30 in lowest terms is $\frac{a}{30}$, where $0 < a < 300$ and $\gcd(a, 30) = 1$. The integers from 1 to 300 form ten blocks of 30 consecutive integers, and each block contains $\varphi(30) = 30 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 8$ integers relatively prime to 30; since 300 is not one of them, there are 80 numerators. Since $\gcd(300 - a, 30) = \gcd(a, 30)$, the numerators pair up as a and $300 - a$, and no numerator equals 150, which is not relatively prime to 30. Each of the 40 pairs has sum 300, so the numerators add up to $40 \cdot 300 = 12000$, and the required sum is $\frac{12000}{30} = 400$.

Answer: 400. □

9. (2025 AIME II, Problem #7) Let A be the set of positive integer divisors of 2025. Let B be a randomly selected subset of A. The probability that B is a nonempty set with the property that the least common multiple of its elements is 2025 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Strategy. The least common multiple of a set of divisors of $2025 = 3^4 \cdot 5^2$ is 2025 exactly when the set contains a divisor divisible by 3^4 and a divisor divisible by 5^2. Count the subsets that miss one of these two kinds with inclusion-exclusion.

Solution. The divisors of $2025 = 3^4 \cdot 5^2$ are the numbers $3^a 5^b$ with $0 \le a \le 4$ and $0 \le b \le 2$, so A has 15 elements and 2^{15} subsets, all equally likely. The least common multiple of a nonempty set of divisors is $3^{a'} 5^{b'}$, where a' and b' are the largest exponents of 3 and 5 in the set. So it equals 2025 exactly when the set contains an element of $X = \{3^4, 3^4 \cdot 5, 3^4 \cdot 5^2\}$ and an element of $Y = \{5^2, 3 \cdot 5^2, 3^2 \cdot 5^2, 3^3 \cdot 5^2, 3^4 \cdot 5^2\}$; such a set is automatically nonempty. There are 2^{12} subsets with no element of X, 2^{10} with no element of Y, and

2^8 with no element of $X \cup Y$, a set of $3 + 5 - 1 = 7$ divisors. By inclusion-exclusion, the number of subsets that contain an element of X and an element of Y is

$$2^{15} - 2^{12} - 2^{10} + 2^8 = 32768 - 4096 - 1024 + 256 = 27904.$$

Since $27904 = 2^8 \cdot 109$ and $32768 = 2^{15}$, the probability is $\frac{109}{128}$ in lowest terms, and $m + n = 237$.

Answer: 237. □

10. (2005 AIME II, Problem #4) Find the number of positive integers that are divisors of at least one of 10^{10}, 15^7, 18^{11}.

Strategy. Apply inclusion-exclusion to the three sets of divisors. The common divisors of several numbers are the divisors of their greatest common divisor.

Solution. Write $10^{10} = 2^{10} \cdot 5^{10}$, $15^7 = 3^7 \cdot 5^7$ and $18^{11} = 2^{11} \cdot 3^{22}$. A number $p^a q^b$, with p and q different primes, has $(a+1)(b+1)$ divisors, so the three numbers have $11 \cdot 11 = 121$, $8 \cdot 8 = 64$ and $12 \cdot 23 = 276$ divisors. The common divisors of two of them are the divisors of their greatest common divisor: $\gcd(10^{10}, 15^7) = 5^7$ has 8 divisors, $\gcd(10^{10}, 18^{11}) = 2^{10}$ has 11, and $\gcd(15^7, 18^{11}) = 3^7$ has 8. The only common divisor of all three is 1. By inclusion-exclusion, the answer is

$$121 + 64 + 276 - 8 - 11 - 8 + 1 = 435.$$

Answer: 435. □

11. (2002 AIME II, Problem #9) Let S be the set $\{1, 2, 3, \ldots, 10\}$. Let n be the number of sets of two non-empty disjoint subsets of S. (*Disjoint sets* are defined as sets that have no common elements.) Find the remainder obtained when n is divided by 1000.

Strategy. Count ordered pairs of disjoint subsets by placing each element in the first set, in the second set or in neither. Remove the pairs with an empty set by inclusion-exclusion, then divide by 2.

Solution. An ordered pair (A, B) of disjoint subsets of S is the same as a choice, for each of the 10 elements, of A, B or neither, so there are 3^{10} such pairs. There are 2^{10} pairs with A empty, 2^{10} with B empty, and 1 with both empty. By inclusion-exclusion, the number of ordered pairs with both sets nonempty is

$$3^{10} - 2 \cdot 2^{10} + 1 = 59049 - 2048 + 1 = 57002.$$

Two disjoint nonempty sets are different, so each set $\{A, B\}$ comes from exactly 2 ordered pairs. Hence $n = \frac{57002}{2} = 28501$, and the remainder is 501.

Answer: 501. □

Advanced.

12. (2016 AIME II, Problem #11) For positive integers N and k, define N to be *k-nice* if there exists a positive integer a such that a^k has exactly N positive divisors. Find the number of positive integers less than 1000 that are neither 7-nice nor 8-nice.

Strategy. Show that N is k-nice exactly when $N \equiv 1 \pmod{k}$, then count the integers below 1000 in neither congruence class.

Solution. If $a = p_1^{c_1} \cdots p_t^{c_t}$, then $a^k = p_1^{kc_1} \cdots p_t^{kc_t}$ has $(kc_1 + 1) \cdots (kc_t + 1)$ divisors. Each factor is congruent to 1 modulo k, so the product is too; and $a = 1$ gives $N = 1$. So every k-nice number satisfies $N \equiv 1 \pmod{k}$. Conversely, if $N = kq + 1$ with $q \geq 0$, then $a = 2^q$ gives $a^k = 2^{kq}$, which has $kq + 1 = N$ divisors. So N is k-nice exactly when $N \equiv 1 \pmod{k}$.

Among $1, \ldots, 999$, the 7-nice numbers are $7t+1$ with $0 \le t \le 142$, which is 143 numbers, and the 8-nice numbers are $8t+1$ with $0 \le t \le 124$, which is 125 numbers. A number is both exactly when $N-1$ is a multiple of 7 and of 8, that is, of 56: the numbers $56t+1$ with $0 \le t \le 17$, which is 18 numbers. By inclusion-exclusion, $143 + 125 - 18 = 250$ numbers are 7-nice or 8-nice, so $999 - 250 = 749$ are neither.

Answer: 749. □

13. (2012 AIME I, Problem #3) Nine people sit down for dinner where there are three choices of meals. Three people order the beef meal, three order the chicken meal, and three order the fish meal. The waiter serves the nine meals in random order. Find the number of ways in which the waiter could serve the meal types to the nine people so that exactly one person receives the type of meal ordered by that person.

Strategy. Choose the person who receives the meal ordered. Then count the ways to serve the other eight people so that nobody receives the meal ordered.

Solution. There are 9 choices for the person who receives the meal ordered; rename the meals so that this person ordered beef. The other eight people are 2 who ordered beef, 3 who ordered chicken and 3 who ordered fish, and the remaining meals are 2 beef, 3 chicken and 3 fish; none of these eight may receive the meal ordered. Let x be the number of beef meals served to chicken orderers; the other $2-x$ beef meals go to fish orderers. The chicken orderers then get x beef and $3-x$ fish meals. The fish orderers get $2-x$ beef and $3-(2-x) = x+1$ chicken meals. The beef orderers get the remaining $3-(x+1) = 2-x$ chicken and $3-(3-x) = x$ fish meals. So x is 0, 1 or 2, and the servings are determined by choosing which chicken orderers get beef, which fish orderers get beef and which beef orderers get fish: $\binom{3}{x}\binom{3}{2-x}\binom{2}{x}$ ways. This gives $1 \cdot 3 \cdot 1 = 3$, $3 \cdot 3 \cdot 2 = 18$ and $3 \cdot 1 \cdot 1 = 3$ for $x = 0, 1, 2$, so 24 servings, and the answer is $9 \cdot 24 = 216$.

Answer: 216. □

14. (2021 AMC 10B, Problem #22) Ang, Ben, and Jasmin each have 5 blocks, colored red, blue, yellow, white, and green; and there are 5 empty boxes. Each of the people randomly and independently of the other two people places one of their blocks into each box. The probability that at least one box receives 3 blocks all of the same color is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 47 **(B)** 94 **(C)** 227 **(D)** 471 **(E)** 542

Strategy. Count the triples of placements in which some box receives three blocks of one color, with inclusion-exclusion over the boxes.

Solution. Each person places the 5 colors into the 5 boxes, one per box, in $5! = 120$ ways, so there are 120^3 equally likely outcomes. For each box, consider the outcomes in which that box receives three blocks of the same color. If every box in a given set of j boxes does, Ang's placement is arbitrary (120 ways), and Ben and Jasmin must each match Ang's colors in those j boxes and place their other $5-j$ blocks freely, in $(5-j)!$ ways each. So there are $120\left((5-j)!\right)^2$ such outcomes. By inclusion-exclusion, some box receives three blocks of one color in

$$\begin{aligned} &120\left(\binom{5}{1}(4!)^2 - \binom{5}{2}(3!)^2 + \binom{5}{3}(2!)^2 - \binom{5}{4}(1!)^2 + \binom{5}{5}(0!)^2\right) \\ &\quad = 120\,(2880 - 360 + 40 - 5 + 1) = 120 \cdot 2556 \end{aligned}$$

outcomes. The probability is $\frac{120 \cdot 2556}{120^3} = \frac{2556}{14400} = \frac{71}{400}$ (divide by 36), so $m+n = 471$.

Answer: (D) 471. □

15. (2011 AIME II, Problem #12) Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Let the probability that each delegate sits next to at least one delegate from another country be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Strategy. A delegate has no neighbor from another country exactly when he or she sits between the other two delegates of the same country. Count the seatings in which the three delegates of some country sit together, with inclusion-exclusion over the countries.

Solution. Number the chairs, so that the 9! seatings are equally likely. A delegate fails the condition exactly when both neighbors come from the delegate's own country, that is, when the three delegates of that country sit in three consecutive chairs with this delegate in the middle. So the condition fails exactly when the three delegates of some country sit in three consecutive chairs. Around the table there are 9 sets of three consecutive chairs. For one country, its block takes one of 9 positions, which gives $9 \cdot 3! \cdot 6! = 38880$ seatings. For two given countries, the first block takes one of 9 positions; the other six chairs are consecutive, and the second block takes one of 4 positions among them. With the orders inside the two blocks and of the last three delegates, this gives $9 \cdot 4 \cdot 3! \cdot 3! \cdot 3! = 7776$ seatings. For all three countries, the first block takes one of 9 positions and the other two blocks fill the remaining six chairs in 2 ways, which gives $9 \cdot 2 \cdot (3!)^3 = 3888$ seatings. By inclusion-exclusion, the condition fails in

$$3 \cdot 38880 - 3 \cdot 7776 + 3888 = 97200$$

seatings, so it holds in $9! - 97200 = 362880 - 97200 = 265680$. The probability is $\frac{265680}{362880} = \frac{41}{56}$, and $m+n = 97$.

Answer: 97. □

16. (2018 AIME II, Problem #11) Find the number of permutations of 1, 2, 3, 4, 5, 6 such that for each k with $1 \le k \le 5$, at least one of the first k terms of the permutation is greater than k.

Strategy. A permutation fails the condition for k exactly when its first k terms are $1, \ldots, k$ in some order. Apply inclusion-exclusion to the five values of k; each intersection cuts the permutation into blocks.

Solution. For $1 \le k \le 5$, the first k terms are all at most k exactly when they are $1, 2, \ldots, k$ in some order. Let A_k be the set of permutations with this property; the required permutations are those in none of $A_1, \ldots, A_5$.

Take $k_1 < k_2 < \cdots < k_t$. A permutation lies in $A_{k_1} \cap \cdots \cap A_{k_t}$ exactly when its first k_1 terms are $1, \ldots, k_1$, the next $k_2 - k_1$ terms are $k_1 + 1, \ldots, k_2$, and so on, and the last $6 - k_t$ terms are $k_t + 1, \ldots, 6$, each block in any order. So the intersection has $k_1!\,(k_2 - k_1)! \cdots (6 - k_t)!$ elements, the product of the factorials of the block lengths. The block lengths are positive integers with sum 6, in order, and every such list of $t+1$ lengths comes from exactly one choice of $k_1 < \cdots < k_t$ (the empty choice, $t = 0$, gives the single

block of length 6). Group the lists of lengths by their number $\ell = t + 1$ of blocks; for each ℓ, the sizes of the intersections add up as follows.

- $\ell = 1$: $6! = 720$.
- $\ell = 2$: $1!\,5! + 2!\,4! + 3!\,3! + 4!\,2! + 5!\,1! = 120 + 48 + 36 + 48 + 120 = 372$.
- $\ell = 3$: $3 \cdot 24 + 6 \cdot 12 + 8 = 152$, from the 3 orders of $1, 1, 4$, the 6 orders of $1, 2, 3$ and the list $2, 2, 2$.
- $\ell = 4$: $4 \cdot 6 + 6 \cdot 4 = 48$, from the 4 orders of $1, 1, 1, 3$ and the 6 orders of $1, 1, 2, 2$.
- $\ell = 5$: $5 \cdot 2 = 10$, from the 5 orders of $1, 1, 1, 1, 2$.
- $\ell = 6$: 1.

The sign of each term is $(-1)^t = (-1)^{\ell-1}$, so by inclusion-exclusion the number of permutations is

$$720 - 372 + 152 - 48 + 10 - 1 = 461.$$

Answer: 461. □

2. Answer Key

Problem	Letter	Value
1	B	686
2	D	331
3		528
4		5280
5	C	224
6	E	382
7		73
8		400
9		237
10		435
11		501
12		749
13		216
14	D	471
15		97
16		461