BINOMIAL COEFFICIENTS AND COMBINATORIAL IDENTITIES FOR AMC 12

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1. Binomial Coefficients

This section collects the basic facts about binomial coefficients and Pascal's triangle.

1.1. **Pascal's Triangle.** For integers $n \geq 0$ and k, the **binomial coefficient** $\binom{n}{k}$ is the number of k-element subsets of a set of n objects. For $0 \leq k \leq n$,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

and $\binom{n}{k} = 0$ when $k < 0$ or $k > n$.

Definition 1.1.1. Pascal's triangle is the array of rows $0, 1, 2, \ldots$ in which row n lists $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, each row centered under the one before. The entries of a row are counted from 0, so entry k of row n is $\binom{n}{k}$. Rows 0 to 6 are shown below.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Theorem 1.1.2 (Symmetry)**.** *For integers $n \geq 0$ and k,*

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof. Taking the complement sends each k-element subset of a set of n objects to an $(n-k)$-element subset. Taking the complement again returns the original subset, so this rule is a bijection between the two collections of subsets. □

So each row of Pascal's triangle reads the same from both ends.

Theorem 1.1.3 (Pascal's Identity)**.** *For $n \geq 1$ and every integer k,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. Fix one of the n objects. A k-element subset that contains it consists of the fixed object and a $(k-1)$-element subset of the other $n-1$ objects, so there are $\binom{n-1}{k-1}$ such subsets. A k-element subset that does not contain it is a k-element subset of the other $n-1$ objects, so there are $\binom{n-1}{k}$ such subsets. □

In Pascal's triangle, each entry other than the 1's at the ends of a row is therefore the sum of the two entries just above it. The same rule, applied upward from any row of numbers, gives the following.

Proposition 1.1.4 (Sums in a Triangle)**.** Write numbers $b_0, b_1, \ldots, b_n$ in a row. Above each pair of neighbors write their sum, which gives a row of n numbers, and repeat until a single number remains. That number is

$$\sum_{k=0}^{n} \binom{n}{k} b_k = \binom{n}{0} b_0 + \binom{n}{1} b_1 + \cdots + \binom{n}{n} b_n.$$

If each new number is instead the average of the two numbers below it, the last number is

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}b_k.$$

Proof. Number the rows $0, 1, \ldots, n$ from the bottom, and the positions in each row $0, 1, 2, \ldots$ from the left. In the sums below, j runs over all integers; the terms with $j < 0$ or $j > r$ are 0. The claim is that position i of row r holds

$$\sum_j \binom{r}{j} b_{i+j}.$$

For $r = 0$ this is b_i. Suppose that the claim holds for row $r-1$, where $r \geq 1$. Position i of row r holds the sum of positions i and $i+1$ of row $r-1$. Replacing j by $j-1$ in the second of these two sums and applying Pascal's identity gives

$$\sum_j \binom{r-1}{j} b_{i+j} + \sum_j \binom{r-1}{j} b_{i+1+j} = \sum_j \left(\binom{r-1}{j} + \binom{r-1}{j-1}\right) b_{i+j} = \sum_j \binom{r}{j} b_{i+j}.$$

Row n has only position 0, which gives the formula. With averages, each row divides by 2 once more than the row below it, so the top number is the same sum divided by 2^n. □

Example.

Problem 1.1.5. The bottom row of a triangle of numbers is $1, 2, 3, 4, 5$. Each number in a higher row is the sum of the two numbers just below it, so the rows above the bottom row contain 4, 3, 2, and 1 numbers. What number is at the top?

Strategy. Apply the proposition on sums in a triangle with $n = 4$. The weights are the entries of row 4 of Pascal's triangle.

Solution. By the proposition on sums in a triangle, with $n = 4$ and b_0, b_1, b_2, b_3, b_4 equal to $1, 2, 3, 4, 5$, the top number is

$$\binom{4}{0}\cdot 1 + \binom{4}{1}\cdot 2 + \binom{4}{2}\cdot 3 + \binom{4}{3}\cdot 4 + \binom{4}{4}\cdot 5 = 1 + 8 + 18 + 16 + 5 = 48.$$

Building the triangle row by row gives the same number:

$$\begin{array}{ccccccccc}
 & & & & 48 & & & & \\
 & & & 20 & & 28 & & & \\
 & & 8 & & 12 & & 16 & & \\
 & 3 & & 5 & & 7 & & 9 & \\
1 & & 2 & & 3 & & 4 & & 5
\end{array}$$

Answer: 48. □

1.2. **Neighboring Entries.**

Proposition 1.2.1 (Neighboring Entries)**.** For $0 \leq k \leq n-1$,

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}.$$

Proof. By the factorial formula,

$$\binom{n}{k+1} = \frac{n!}{(k+1)!\,(n-k-1)!} = \frac{n!}{k!\,(n-k)!}\cdot\frac{n-k}{k+1} = \frac{n-k}{k+1}\binom{n}{k}.$$

□

The ratio $\frac{n-k}{k+1}$ is greater than 1 exactly when $n-k > k+1$, that is, when $k < \frac{n-1}{2}$, and it equals 1 exactly when $k = \frac{n-1}{2}$. So the entries of each row increase up to the middle of the row and then decrease, and when n is odd the two middle entries are equal. In general, to find the largest term of a list of positive numbers, compare consecutive terms through their ratio. A condition on the ratio of two neighboring entries becomes an equation in n and k.

Example.

Problem 1.2.2 (1992 AIME, Problem #4)**.** In Pascal's Triangle, each entry is the sum of the two entries above it. The first few rows of the triangle are shown below.

$$\begin{array}{lccccccc} 0: & 1 & & & & & & \\ 1: & 1 & 1 & & & & & \\ 2: & 1 & 2 & 1 & & & & \\ 3: & 1 & 3 & 3 & 1 & & & \\ 4: & 1 & 4 & 6 & 4 & 1 & & \\ 5: & 1 & 5 & 10 & 10 & 5 & 1 & \\ 6: & 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{array}$$

In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3 : 4 : 5$?

Strategy. Write the two ratios of neighboring entries with the proposition on neighboring entries, and solve the resulting equations for the row and the position.

Solution. Let the three entries be $\binom{n}{k}$, $\binom{n}{k+1}$, and $\binom{n}{k+2}$ in row n. By the proposition on neighboring entries,

$$\frac{n-k}{k+1} = \frac{4}{3} \qquad \text{and} \qquad \frac{n-k-1}{k+2} = \frac{5}{4}.$$

Clearing denominators gives $3n - 3k = 4k + 4$ and $4n - 4k - 4 = 5k + 10$, that is,

$$3n = 7k + 4 \qquad \text{and} \qquad 4n = 9k + 14.$$

Multiplying the first equation by 4 and the second by 3 gives $28k + 16 = 27k + 42$, so $k = 26$ and $n = (7 \cdot 26 + 4)/3 = 62$. As a check,

$$\frac{\binom{62}{27}}{\binom{62}{26}} = \frac{36}{27} = \frac{4}{3} \qquad \text{and} \qquad \frac{\binom{62}{28}}{\binom{62}{27}} = \frac{35}{28} = \frac{5}{4}.$$

Answer: 62. □

2. The Binomial Theorem

This section expands powers of a sum and uses the expansions to compute with numbers.

2.1. **The General Term.**

Theorem 2.1.1 (Binomial Theorem)**.** *For all numbers x and y and every integer $n \geq 0$,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + y^n.$$

Proof. Multiplying out the product of n factors $(x + y)$ gives 2^n products, one for each way of choosing x or y from every factor. A product equals $x^{n-k}y^k$ exactly when y is chosen from k of the factors, and there are $\binom{n}{k}$ ways to choose those k factors. □

So the coefficients of $(x+y)^n$ form row n of Pascal's triangle. The term $\binom{n}{k}x^{n-k}y^k$ is the **general term** of the expansion. Constants and signs inside x and y are carried along: in $(2a-b)^n$ the general term is $\binom{n}{k}(2a)^{n-k}(-b)^k$. When x and y are powers of the same variable, possibly with negative or fractional exponents, the general term is a single power of that variable, with an exponent that depends on k. To find the coefficient of a given power, write the exponent of the general term in terms of k and solve for k.

Example.

Problem 2.1.2 (1963 AHSME, Problem #9)**.** In the expansion of $\left(a-\frac{1}{\sqrt{a}}\right)^7$ the coefficient of $a^{-\frac{1}{2}}$ is:

(A) -7 **(B)** 7 **(C)** -21 **(D)** 21 **(E)** 35

Strategy. Write $\frac{1}{\sqrt{a}}$ as $a^{-1/2}$, express the general term as a single power of a, and choose k so that the exponent is $-\frac{1}{2}$.

Solution. With $x=a$ and $y=-a^{-1/2}$, the general term is

$$\binom{7}{k}a^{7-k}\left(-a^{-1/2}\right)^k=(-1)^k\binom{7}{k}a^{7-k-\frac{k}{2}}=(-1)^k\binom{7}{k}a^{7-\frac{3k}{2}}.$$

The exponent $7-\frac{3k}{2}$ is different for different values of k, so each power of a comes from only one term. It equals $-\frac{1}{2}$ exactly when $\frac{3k}{2}=\frac{15}{2}$, that is, $k=5$. The coefficient is $(-1)^5\binom{7}{5}=-21$.

Answer: (C) -21. □

2.2. **Powers of Numbers.** To compute with a power of a number, write the number as a sum of two parts, one of them small or divisible by a given modulus, and expand.

Let a, b, and m be integers, and let n and r be positive integers. If m divides b, every term $\binom{n}{j}a^{n-j}b^j$ of $(a+b)^n$ with $j\geq r$ is divisible by m^r. So modulo m^r only the terms with $j<r$ remain. For example,

$$(a+b)^n\equiv a^n+na^{n-1}b \pmod{b^2}.$$

Adding the expansions of $(x+y)^n$ and $(x-y)^n$ cancels the terms with odd powers of y and doubles the others:

$$(x+y)^n+(x-y)^n=2\sum_{k\text{ even}}\binom{n}{k}x^{n-k}y^k.$$

For example, for nonnegative integers a and d, the number $(a+\sqrt{d})^n+(a-\sqrt{d})^n$ is an integer, because only even powers of $\sqrt{d}$ remain.

Example.

Problem 2.2.1 (1983 AIME, Problem #6)**.** Let $a_n=6^n+8^n$. Determine the remainder on dividing a_{83} by 49.

Strategy. Since $a_{83}=(7-1)^{83}+(7+1)^{83}$, apply the rule above with $m=7$ and $r=2$: modulo 49, each expansion reduces to its terms in 7^0 and 7^1.

Solution. By the binomial theorem,

$$(7+1)^{83}=\sum_{j=0}^{83}\binom{83}{j}7^j \qquad \text{and} \qquad (7-1)^{83}=\sum_{j=0}^{83}\binom{83}{j}7^j(-1)^{83-j}.$$

Since 83 is odd, the sign $(-1)^{83-j}$ is -1 for even j and 1 for odd j. Adding the two expansions, the terms with j even cancel and the terms with j odd double:

$$a_{83} = (7-1)^{83} + (7+1)^{83} = 2\sum_{j \text{ odd}} \binom{83}{j} 7^j.$$

Every term with $j \geq 2$ is divisible by $7^2 = 49$, so only the term with $j = 1$ remains:

$$a_{83} \equiv 2 \cdot 83 \cdot 7 = 1162 = 23 \cdot 49 + 35 \pmod{49}.$$

The remainder is 35.

Answer: 35. □

2.3. **The Multinomial Theorem.**

Theorem 2.3.1 (Multinomial Theorem)**.** *For every integer $n \geq 0$,*

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \frac{n!}{a_1!\, a_2! \cdots a_m!}\, x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m},$$

where the sum is over all lists $(a_1, a_2, \ldots, a_m)$ of nonnegative integers with $a_1 + a_2 + \cdots + a_m = n$. The number $\frac{n!}{a_1!\, a_2! \cdots a_m!}$ is a ***multinomial coefficient****.*

Proof. Choosing one term from each of the n factors gives a word of length n in the letters $x_1, \ldots, x_m$, and the word contributes $x_1^{a_1} \cdots x_m^{a_m}$ when x_i appears a_i times for each i. By equal overcounting, the number of words of length n with a_i copies of x_i for each i is $\frac{n!}{a_1!\, a_2! \cdots a_m!}$. □

For $m = 2$ this is the binomial theorem. When $x_1, \ldots, x_m$ are different variables, different lists of exponents give different monomials and every multinomial coefficient is positive, so the expansion has exactly one term for each list. To find a coefficient, write the general term, collect the exponent of the variable, and list the choices of exponents that give the required power.

Example.

Problem 2.3.2 (1974 AHSME, Problem #3)**.** The coefficient of x^7 in the polynomial expansion of

$$(1 + 2x - x^2)^4$$

is

(A) -8 **(B)** 12 **(C)** 6 **(D)** -12 **(E)** none of these

Strategy. Apply the multinomial theorem to the three terms 1, $2x$, and $-x^2$, and find the choices of exponents that produce x^7.

Solution. By the multinomial theorem, the general term is

$$\frac{4!}{a!\, b!\, c!}\, 1^a (2x)^b (-x^2)^c = \frac{4!}{a!\, b!\, c!}\, 2^b (-1)^c\, x^{b+2c}, \qquad a + b + c = 4.$$

It contains x^7 when $b + 2c = 7$. Then $c = 7 - (b + c) \geq 7 - 4 = 3$, and $c \leq 3$ because $2c \leq 7$. So $c = 3$, $b = 1$, and $a = 0$ is the only choice, and the coefficient is

$$\frac{4!}{0!\, 1!\, 3!} \cdot 2 \cdot (-1)^3 = 4 \cdot 2 \cdot (-1) = -8.$$

Answer: (A) -8. □

3. Sums of Coefficients

Substituting numbers for the variables in an expansion gives sums of its coefficients.

3.1. **Row Sums.**

Theorem 3.1.1 (Row Sums)**.** *For every integer $n \geq 0$,*

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

Proof. Put $x = y = 1$ in the binomial theorem. Both sides also count the subsets of a set of n objects, the left side by their size. □

When n is odd, symmetry matches the entries $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{(n-1)/2}$ one to one with the other half of the row, $\binom{n}{n}, \binom{n}{n-1}, \ldots, \binom{n}{(n+1)/2}$. So each half of the row sums to 2^{n-1}.

Example.

Problem 3.1.2 (2000 AIME II, Problem #7)**.** Given that

$$\frac{1}{2!17!} + \frac{1}{3!16!} + \frac{1}{4!15!} + \frac{1}{5!14!} + \frac{1}{6!13!} + \frac{1}{7!12!} + \frac{1}{8!11!} + \frac{1}{9!10!} = \frac{N}{1!18!},$$

find the greatest integer that is less than $\frac{N}{100}$.

Strategy. Each fraction on the left equals $\frac{1}{19!}\binom{19}{k}$ for some k, so the left side is part of the first half of row 19, divided by 19!. Compare it with the sum of that half.

Solution. For $2 \leq k \leq 9$, the factorial formula gives $\frac{1}{k!\,(19-k)!} = \frac{1}{19!}\binom{19}{k}$, and the right side is $\frac{N}{1!\,18!} = \frac{19N}{19!}$. Hence

$$\binom{19}{2} + \binom{19}{3} + \cdots + \binom{19}{9} = 19N.$$

Row 19 has the 20 entries $\binom{19}{0}, \ldots, \binom{19}{19}$, and 19 is odd, so its first ten entries sum to $2^{18} = 262144$. Hence

$$19N = 2^{18} - \binom{19}{0} - \binom{19}{1} = 262144 - 1 - 19 = 262124,$$

so $N = 13796$ and $\frac{N}{100} = 137.96$. The greatest integer less than 137.96 is 137.

Answer: 137. □

3.2. **Substituting Values.**

Let $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$ be a polynomial. Then $P(1) = a_0 + a_1 + a_2 + \cdots + a_m$ is the sum of its coefficients, and $P(-1) = a_0 - a_1 + a_2 - a_3 + \cdots$. Adding and subtracting these two equations gives

$$a_0 + a_2 + a_4 + \cdots = \frac{P(1) + P(-1)}{2}, \qquad a_1 + a_3 + a_5 + \cdots = \frac{P(1) - P(-1)}{2}.$$

Example.

Problem 3.2.1 (1966 AHSME, Problem #36)**.** Let

$$(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n}$$

be an identity in x. If we let $s = a_0 + a_2 + a_4 + \cdots + a_{2n}$, then s equals:

(A) 2^n **(B)** $2^n + 1$ **(C)** $\frac{3^n - 1}{2}$ **(D)** $\frac{3^n}{2}$ **(E)** $\frac{3^n + 1}{2}$

Strategy. Here s is the sum of the coefficients with even indices. Find it with the formula above for $a_0 + a_2 + a_4 + \cdots$.

Solution. Put $x = 1$ and then $x = -1$ in the identity: $a_0 + a_1 + a_2 + \cdots + a_{2n} = (1+1+1)^n = 3^n$ and $a_0 - a_1 + a_2 - \cdots + a_{2n} = (1-1+1)^n = 1$. Adding the two equations cancels the coefficients with odd indices and doubles those with even indices, as in the formula above, so $2s = 3^n + 1$ and

$$s = \frac{3^n + 1}{2}.$$

Answer: (E) $\dfrac{3^n+1}{2}$. □

4. Combinatorial Identities

Each theorem in this section is proved by counting one collection of objects in two ways.

4.1. The Hockey Stick Identity.

Theorem 4.1.1 (Hockey Stick Identity)**.** *For integers* $0 \le r \le n$,

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

Proof. Count the $(r+1)$-element subsets of $\{1, 2, \ldots, n+1\}$ by their largest element. If the largest element is $i+1$, where $r \le i \le n$, the other r elements form an r-element subset of $\{1, 2, \ldots, i\}$, and there are $\binom{i}{r}$ of these. Adding over i gives the left side. □

The terms on the left run down a diagonal of Pascal's triangle, and the sum is the entry below the last term in the other diagonal direction, so together they have the shape of a hockey stick. For example, $1 + 3 + 6 + 10 = 20$, that is, $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{6}{3}$. Sums of this form appear when subsets of a fixed size are counted by their largest or smallest element.

Example.

Problem 4.1.2 (1986 AIME, Problem #11)**.** The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form

$$a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17},$$

where $y = x + 1$ and the a_i's are constants. Find a_2.

Strategy. Replace $-x$ by $1 - y$ in every term. Then a_2 is a sum of binomial coefficients that lie on one diagonal of Pascal's triangle.

Solution. Since $y = x + 1$, the number $-x$ equals $1 - y$. The terms of the given polynomial are the powers $(-x)^j$ for $0 \le j \le 17$, so the polynomial equals

$$(1-y)^0 + (1-y)^1 + (1-y)^2 + \cdots + (1-y)^{17}.$$

By the binomial theorem, for $j \ge 2$ the power $(1-y)^j$ contains the term $\binom{j}{2}(-y)^2 = \binom{j}{2}y^2$, while $(1-y)^0 = 1$ and $(1-y)^1 = 1 - y$ contain no y^2. Collecting the y^2 terms and applying the hockey stick identity with $r = 2$ and $n = 17$ gives

$$a_2 = \binom{2}{2} + \binom{3}{2} + \cdots + \binom{17}{2} = \binom{18}{3} = 816.$$

Answer: 816. □

4.2. **Vandermonde's Identity.**

Theorem 4.2.1 (Vandermonde's Identity)**.** *For nonnegative integers m, n, and r,*

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

Proof. Count the committees of r people chosen from a group of m men and n women. A committee contains k men and $r-k$ women for exactly one k, and for each k there are $\binom{m}{k}\binom{n}{r-k}$ such committees. Adding over k gives the left side. □

Corollary 4.2.2. *For every integer $n \geq 0$,*

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Proof. By symmetry, $\binom{n}{k}^2 = \binom{n}{k}\binom{n}{n-k}$. Apply Vandermonde's identity with $m = r = n$. □

Vandermonde's identity needs the lower indices of the two factors to have a constant sum. When they do not, rewrite one factor by symmetry, $\binom{n}{j} = \binom{n}{n-j}$, so that they do.

Example.

Problem 4.2.3 (2020 AIME I, Problem #7)**.** A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N.

Strategy. Sort the committees by the number of men. Describing the women by the ones left out makes the lower indices add up to 11, so Vandermonde's identity applies.

Solution. If k men serve, then $k+1$ women serve, so $0 \leq k \leq 11$, and these men and women can be chosen in $\binom{11}{k}\binom{12}{k+1}$ ways. By symmetry, $\binom{12}{k+1} = \binom{12}{11-k}$, so in every term the lower indices add up to 11, and Vandermonde's identity with $m = 11$, $n = 12$, and $r = 11$ gives

$$N = \sum_{k=0}^{11} \binom{11}{k}\binom{12}{k+1} = \sum_{k=0}^{11} \binom{11}{k}\binom{12}{11-k} = \binom{23}{11} = \frac{23 \cdot 22 \cdot 21 \cdots 13}{11!}.$$

The numerator is $2^9 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$: the factors 14, 16, 18, 20, 22 give nine 2's, the factors 15, 18, 21 give four 3's, the factors 15 and 20 give two 5's, and the factors 14 and 21 give two 7's. The denominator is $11! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$. After canceling,

$$N = 2 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23,$$

so N has the prime factors 2, 7, 13, 17, 19, and 23, whose sum is 81.

Answer: 81. □

4.3. **Counting in Two Ways.** A collection of pairs (a, b) can be counted by choosing a first and then b, or by choosing b first and then a, and both orders give the same total. In particular, if each of several objects has some of a list of properties, the sum over the objects of the number of properties each one has equals the sum over the properties of the number of objects that have it.

Theorem 4.3.1 (Committee-Chair Identity)**.** *For $1 \leq k \leq n$,*

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Proof. Count the pairs consisting of a committee of k people chosen from n people and a chair who belongs to the committee. Choosing the committee and then the chair gives $\binom{n}{k} \cdot k$ pairs. Choosing the chair first and then the other $k-1$ members from the remaining $n-1$ people gives $n\binom{n-1}{k-1}$ pairs. □

Corollary 4.3.2. *For every integer $n \geq 1$,*

$$\sum_{k=1}^{n} k\binom{n}{k} = n\,2^{n-1}.$$

Proof. By the committee-chair identity and the row sum of row $n-1$,

$$\sum_{k=1}^{n} k\binom{n}{k} = n\sum_{k=1}^{n}\binom{n-1}{k-1} = n\,2^{n-1}.$$ □

Examples.

Problem 4.3.3. Find the value of $\binom{10}{1} + 2\binom{10}{2} + 3\binom{10}{3} + \cdots + 10\binom{10}{10}$.

Strategy. Read $k\binom{10}{k}$ as the number of ways to choose a committee of k people from 10 together with its chair.

Solution. For each k, the term $k\binom{10}{k}$ counts the pairs consisting of a committee of k people chosen from a club of 10 and a chair from that committee. So the sum counts all pairs of a nonempty committee and a chair from that committee. Count the same pairs the other way: choose the chair in 10 ways, and then complete the committee with any subset of the other 9 people, in 2^9 ways. The value is

$$10 \cdot 2^9 = 5120.$$

This agrees with the corollary for $n = 10$.

Answer: 5120. □

Problem 4.3.4 (2023 AIME I, Problem #3)**.** A plane contains 40 lines, no 2 of which are parallel. Suppose that there are 3 points where exactly 3 lines intersect, 4 points where exactly 4 lines intersect, 5 points where exactly 5 lines intersect, 6 points where exactly 6 lines intersect, and no points where more than 6 lines intersect. Find the number of points where exactly 2 lines intersect.

Strategy. Count the pairs of lines in two ways: directly, and by the point where the two lines meet.

Solution. Let x be the number of points that lie on exactly 2 of the lines. Sort the $\binom{40}{2} = 780$ pairs of lines by the point where the two lines of the pair cross. Such a point exists because the problem rules out parallel lines, and it is unique because two distinct lines share at most one point. A point that lies on exactly m of the lines collects the $\binom{m}{2}$ pairs formed from those m lines. Adding these numbers over all the points gives

$$780 = x\binom{2}{2} + 3\binom{3}{2} + 4\binom{4}{2} + 5\binom{5}{2} + 6\binom{6}{2} = x + 9 + 24 + 50 + 90.$$

So $x = 780 - 173 = 607$.

Answer: 607. □

5. Divisibility

This section finds the primes that divide a binomial coefficient and computes binomial coefficients modulo a prime.

5.1. **Prime Factors.**

Theorem 5.1.1 (Legendre's Formula)**.** *For a prime p and a positive integer n, the exponent of p in the prime factorization of $n!$ is*

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots,$$

where only finitely many terms are nonzero.

Proof. Among $1, 2, \ldots, n$, exactly $\lfloor n/p^i \rfloor$ numbers are multiples of p^i. A number divisible by p exactly e times is counted once in each of the first e terms, so it contributes e to the sum, which is its contribution to the exponent of p in $n!$. □

Since $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$, the exponent of p in $\binom{n}{k}$ is

$$\sum_{i \geq 1} \left(\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor \right).$$

When $p^2 > n$, only the term with $i = 1$ can be nonzero.

Example.

Problem 5.1.2 (1983 AIME, Problem #8)**.** What is the largest 2-digit prime factor of the integer $n = \binom{200}{100}$?

Strategy. Test the two-digit primes in decreasing order, finding the exponent of each one in $\binom{200}{100}$ with the formula above.

Solution. Let p be a prime with $50 < p < 100$. Then $p^2 > 200$ and $\lfloor 100/p \rfloor = 1$, so the exponent of p in $\binom{200}{100} = \frac{200!}{100!\,100!}$ is

$$\left\lfloor \frac{200}{p} \right\rfloor - 2 \left\lfloor \frac{100}{p} \right\rfloor = \left\lfloor \frac{200}{p} \right\rfloor - 2.$$

For $p \geq 67$, the quotient $\frac{200}{p}$ is less than 3, so the exponent is $2 - 2 = 0$. For $p = 61$, the floor $\lfloor 200/61 \rfloor$ equals 3, so the exponent is 1. Hence 61 divides $\binom{200}{100}$, and none of the larger two-digit primes 67, 71, 73, 79, 83, 89, 97 does.

Answer: 61. □

5.2. **Remainders Modulo a Prime.**

Let p be a prime and $0 \leq k \leq p-1$. Every factor of $k!$ is less than p, so p does not divide $k!$. Therefore $k!\,u \equiv k!\,v \pmod{p}$ implies $u \equiv v \pmod{p}$: the factor $k!$ can be canceled from a congruence modulo p.

Since $k!\binom{n}{k} = n(n-1)\cdots(n-k+1)$, the remainder of $\binom{n}{k}$ modulo p can be found by reducing each of the k factors $n, n-1, \ldots, n-k+1$ modulo p and then canceling $k!$. For example,

$$k!\binom{p-1}{k} = (p-1)(p-2)\cdots(p-k) \equiv (-1)(-2)\cdots(-k) = (-1)^k k! \pmod{p},$$

so $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$ for $0 \leq k \leq p-1$.

Example.

Problem 5.2.1. What is the remainder when $\binom{15}{6}$ is divided by 17?

Strategy. Write $6!\binom{15}{6}$ as a product of six consecutive integers, reduce each factor modulo 17, and cancel 6!.

Solution. Modulo 17,

$$6!\binom{15}{6} = 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \equiv (-2)(-3)(-4)(-5)(-6)(-7) = 5040 = 7 \cdot 6!.$$

Since $6 < 17$, the prime 17 does not divide 6!, so 6! can be canceled: $\binom{15}{6} \equiv 7 \pmod{17}$. As a check, $\binom{15}{6} = 5005 = 17 \cdot 294 + 7$.

Answer: 7. □

6. Problems

Beginner.

1. (1982 AHSME, Problem #8) By definition $r! = r(r-1)\cdots 1$ and $\binom{j}{k} = \frac{j!}{k!(j-k)!}$, where r, j, k are positive integers and $k < j$. If $\binom{n}{1}$, $\binom{n}{2}$, $\binom{n}{3}$ form an arithmetic progression with $n > 3$, then n equals

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 12

2. (1988 AIME, Problem #1) One commercially available ten-button lock may be opened by depressing—in any order—the correct five buttons. One sample has $\{1, 2, 3, 6, 9\}$ as its combination. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow?

3. (1996 AIME, Problem #3) Find the smallest positive integer n for which the expansion of $(xy - 3x + 7y - 21)^n$, after like terms have been collected, has at least 1996 terms.

4. (2000 AIME I, Problem #3) In the expansion of $(ax + b)^{2000}$, where a and b are relatively prime positive integers, the coefficients of x^2 and x^3 are equal. Find $a + b$.

5. (2021 Fall AMC 12A, Problem #12) What is the number of terms with rational coefficients among the 1001 terms in the expansion of

$$\left(x\sqrt[3]{2} + y\sqrt{3}\right)^{1000}?$$

(A) 0 **(B)** 166 **(C)** 167 **(D)** 500 **(E)** 501

Intermediate.

6. (1975 AHSME, Problem #29) What is the smallest integer larger than $(\sqrt{3} + \sqrt{2})^6$?

(A) 972 **(B)** 971 **(C)** 970 **(D)** 969 **(E)** 968

7. (1991 AIME, Problem #3) Expanding $(1 + 0.2)^{1000}$ by the binomial theorem and doing no further manipulation gives

$$\begin{aligned}&\binom{1000}{0}(0.2)^0 + \binom{1000}{1}(0.2)^1 + \binom{1000}{2}(0.2)^2 + \cdots + \binom{1000}{1000}(0.2)^{1000}\\ &\quad = A_0 + A_1 + \cdots + A_{1000},\end{aligned}$$

where $A_k = \binom{1000}{k}(0.2)^k$ for $k = 0, 1, 2, \ldots, 1000$. For which k is A_k the largest?

8. (2016 AIME II, Problem #6) For polynomial $P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2$, define

$$Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i.$$

Then $\sum_{i=0}^{50} |a_i| = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

9. (2021 AMC 12A, Problem #15) A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of groups that can be selected. What is the remainder when N is divided by 100?

(A) 47 **(B)** 48 **(C)** 83 **(D)** 95 **(E)** 96

10. (2006 AMC 12A, Problem #24) The expression

$$(x+y+z)^{2006}+(x-y-z)^{2006}$$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

(A) 6018 **(B)** 671,676 **(C)** 1,007,514 **(D)** 1,008,016 **(E)** 2,015,028

11. (2006 AMC 12A, Problem #23) Given a finite sequence $S = (a_1, a_2, \ldots, a_n)$ of n real numbers, let $A(S)$ be the sequence

$$\left(\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}, \ldots, \frac{a_{n-1}+a_n}{2}\right)$$

of $n-1$ real numbers. Define $A^1(S) = A(S)$ and, for each integer m, $2 \le m \le n-1$, define $A^m(S) = A(A^{m-1}(S))$. Suppose $x > 0$, and let $S = (1, x, x^2, \ldots, x^{100})$. If $A^{100}(S) = (\frac{1}{2^{50}})$, then what is x?

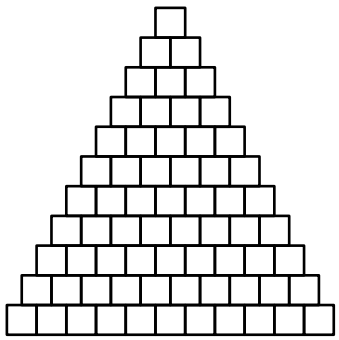
(A) $1 - \dfrac{\sqrt{2}}{2}$ **(B)** $\sqrt{2}-1$ **(C)** $\dfrac{1}{2}$ **(D)** $2-\sqrt{2}$ **(E)** $\dfrac{\sqrt{2}}{2}$

Advanced.

12. (2017 AIME I, Problem #7) For nonnegative integers a and b with $a+b \le 6$, let $T(a,b) = \binom{6}{a}\binom{6}{b}\binom{6}{a+b}$. Let S denote the sum of all $T(a,b)$, where a and b are nonnegative integers with $a+b \le 6$. Find the remainder when S is divided by 1000.

13. (2015 AIME I, Problem #12) Consider all 1000-element subsets of the set $\{1, 2, 3, \ldots, 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

14. (2007 AIME II, Problem #13) A triangular array of squares has one square in the first row, two in the second, and, in general, k squares in the kth row for $1 \le k \le 11$. With the exception of the bottom row, each square rests on two squares in the row immediately below, as illustrated in the figure. In each square of the eleventh row, a 0 or a 1 is placed. Numbers are then placed into the other squares, with the entry for each square being the sum of the entries in the two squares below it. For how many initial distributions of 0's and 1's in the bottom row is the number in the top square a multiple of 3?



15. (2014 AMC 12B, Problem #23) The number 2017 is prime. Let $S = \sum_{k=0}^{62} \binom{2014}{k}$. What is the remainder when S is divided by 2017?

(A) 32 **(B)** 684 **(C)** 1024 **(D)** 1576 **(E)** 2016

16. (2022 AIME I, Problem #12) For any finite set X, let $|X|$ denote the number of elements in X. Define

$$S_n = \sum |A \cap B|,$$

where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \ldots, n\}$ with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\},$$

giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. Let $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.