BINOMIAL COEFFICIENTS AND COMBINATORIAL IDENTITIES FOR AMC 12: SOLUTIONS

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1. Solutions

Beginner.

1. (1982 AHSME, Problem #8) By definition $r! = r(r-1)\cdots 1$ and $\binom{j}{k} = \frac{j!}{k!(j-k)!}$, where r, j, k are positive integers and $k < j$. If $\binom{n}{1}$, $\binom{n}{2}$, $\binom{n}{3}$ form an arithmetic progression with $n > 3$, then n equals

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 12

Strategy. Three numbers form an arithmetic progression exactly when the middle one is the average of the other two. Write this condition with the factorial formula and solve for n.

Solution. The condition is $2\binom{n}{2} = \binom{n}{1} + \binom{n}{3}$, that is,

$$n(n-1) = n + \frac{n(n-1)(n-2)}{6}.$$

Dividing by n and multiplying by 6 gives $6(n-1) = 6 + (n-1)(n-2)$, which simplifies to

$$n^2 - 9n + 14 = 0, \qquad \text{that is,} \qquad (n-2)(n-7) = 0.$$

Since $n > 3$, the solution is $n = 7$. As a check, $\binom{7}{1}$, $\binom{7}{2}$, $\binom{7}{3}$ are 7, 21, 35, with common difference 14.

Answer: (B) 7. □

2. (1988 AIME, Problem #1) One commercially available ten-button lock may be opened by depressing—in any order—the correct five buttons. One sample has $\{1, 2, 3, 6, 9\}$ as its combination. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow?

Strategy. Count all sets of 1 to 9 buttons with the row sum, then subtract the original combinations.

Solution. A combination of a redesigned lock is a set of between 1 and 9 of the 10 buttons. By the row sum for row 10, the number of such sets is

$$2^{10} - \binom{10}{0} - \binom{10}{10} = 1024 - 1 - 1 = 1022.$$

The original locks use the $\binom{10}{5} = 252$ sets of five buttons, which are among these. So the redesign allows $1022 - 252 = 770$ additional combinations.

Answer: 770. □

3. (1996 AIME, Problem #3) Find the smallest positive integer n for which the expansion of $(xy - 3x + 7y - 21)^n$, after like terms have been collected, has at least 1996 terms.

Strategy. Factor $xy - 3x + 7y - 21$, expand each of the two resulting powers with the binomial theorem, and count the monomials of the product.

Solution. By grouping, $xy - 3x + 7y - 21 = x(y-3) + 7(y-3) = (x+7)(y-3)$, so the expression is $(x+7)^n(y-3)^n$. By the binomial theorem,

$$(x+7)^n = \sum_{i=0}^{n} \binom{n}{i} 7^{n-i} x^i \qquad \text{and} \qquad (y-3)^n = \sum_{j=0}^{n} \binom{n}{j} (-3)^{n-j} y^j,$$

and none of these coefficients is 0. Hence

$$(x+7)^n(y-3)^n = \sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}7^{n-i}(-3)^{n-j}x^iy^j.$$

Different pairs (i,j) give different monomials x^iy^j, and every coefficient is nonzero, so after like terms are collected there are exactly $(n+1)^2$ terms. Since $44^2 = 1936 < 1996 \le 2025 = 45^2$, the condition $(n+1)^2 \ge 1996$ holds exactly when $n+1 \ge 45$. The smallest such n is 44.

Answer: 44. □

4. (2000 AIME I, Problem #3) In the expansion of $(ax+b)^{2000}$, where a and b are relatively prime positive integers, the coefficients of x^2 and x^3 are equal. Find $a+b$.

Strategy. Set the two coefficients equal, and compare $\binom{2000}{3}$ with $\binom{2000}{2}$ by the proposition on neighboring entries.

Solution. By the binomial theorem, the coefficient of x^k in $(ax+b)^{2000}$ is $\binom{2000}{k}a^kb^{2000-k}$. The coefficients of x^2 and x^3 are equal, so

$$\binom{2000}{2}a^2b^{1998} = \binom{2000}{3}a^3b^{1997}.$$

Dividing by a^2b^{1997}, which is positive, gives $\binom{2000}{2}b = \binom{2000}{3}a$. By the proposition on neighboring entries, $\binom{2000}{3} = \frac{1998}{3}\binom{2000}{2} = 666\binom{2000}{2}$, so $b = 666a$. Since a divides b and a and b are relatively prime, $a = 1$. Then $b = 666$ and $a+b = 667$.

Answer: 667. □

5. (2021 Fall AMC 12A, Problem #12) What is the number of terms with rational coefficients among the 1001 terms in the expansion of

$$\left(x\sqrt[3]{2} + y\sqrt{3}\right)^{1000}?$$

(A) 0 **(B)** 166 **(C)** 167 **(D)** 500 **(E)** 501

Strategy. Write the general term and decide when its coefficient is rational.

Solution. By the binomial theorem, for $0 \le k \le 1000$ the term containing x^k is

$$\binom{1000}{k}\left(x\sqrt[3]{2}\right)^k\left(y\sqrt{3}\right)^{1000-k} = \binom{1000}{k}2^{k/3}\,3^{(1000-k)/2}\,x^ky^{1000-k}.$$

Let r be the remainder of k modulo 3, and s the remainder of $1000-k$ modulo 2. The coefficient is a positive integer times $2^{r/3}3^{s/2}$, so it is rational exactly when $2^{r/3}3^{s/2}$ is rational. Suppose that $2^{r/3}3^{s/2} = \frac{u}{v}$ for positive integers u and v. Raising both sides to the sixth power gives $2^{2r}3^{3s}v^6 = u^6$. The exponents of 2 and of 3 in the prime factorizations of the two sides must agree. On the right they are multiples of 6, and on the left they are $2r$ and $3s$ plus multiples of 6. Since $0 \le 2r \le 4$ and $0 \le 3s \le 3$, this forces $r = s = 0$. So the coefficient is rational exactly when k is a multiple of 3 and $1000-k$ is even, that is, when k is a multiple of 6. The values $k = 0, 6, 12, \ldots, 996$ give $\frac{996}{6}+1 = 167$ terms.

Answer: (C) 167. □

Intermediate.

6. (1975 AHSME, Problem #29) What is the smallest integer larger than $(\sqrt{3}+\sqrt{2})^6$?

(A) 972 **(B)** 971 **(C)** 970 **(D)** 969 **(E)** 968

Strategy. Add $(\sqrt{3}-\sqrt{2})^6$, which is small and makes the sum an integer.

Solution. When the expansions of $(\sqrt{3}+\sqrt{2})^6$ and $(\sqrt{3}-\sqrt{2})^6$ are added, the terms with odd powers of $\sqrt{2}$ cancel and the others double:

$$(\sqrt{3}+\sqrt{2})^6+(\sqrt{3}-\sqrt{2})^6 = 2\sum_{k\text{ even}}\binom{6}{k}(\sqrt{3})^{6-k}(\sqrt{2})^k.$$

The terms with $k=0,2,4,6$ are 27, $15\cdot 9\cdot 2=270$, $15\cdot 3\cdot 4=180$, and 8, so the sum is $2\cdot 485=970$. Since $1<\sqrt{2}<\sqrt{3}<2$, the number $\sqrt{3}-\sqrt{2}$ lies between 0 and 1, so $0<(\sqrt{3}-\sqrt{2})^6<1$. Hence

$$969<(\sqrt{3}+\sqrt{2})^6<970,$$

and the smallest integer larger than $(\sqrt{3}+\sqrt{2})^6$ is 970.

Answer: (C) 970. □

7. (1991 AIME, Problem #3) Expanding $(1+0.2)^{1000}$ by the binomial theorem and doing no further manipulation gives

$$\begin{aligned}&\binom{1000}{0}(0.2)^0+\binom{1000}{1}(0.2)^1+\binom{1000}{2}(0.2)^2+\cdots+\binom{1000}{1000}(0.2)^{1000}\\ &\quad = A_0+A_1+\cdots+A_{1000},\end{aligned}$$

where $A_k=\binom{1000}{k}(0.2)^k$ for $k=0$, 1, 2, …, 1000. For which k is A_k the largest?

Strategy. Compare A_{k+1} with A_k through their ratio.

Solution. By the proposition on neighboring entries, for $0\le k\le 999$,

$$\frac{A_{k+1}}{A_k}=\frac{\binom{1000}{k+1}}{\binom{1000}{k}}\cdot 0.2=\frac{1000-k}{k+1}\cdot\frac{1}{5}=\frac{1000-k}{5(k+1)}.$$

This ratio is greater than 1 exactly when $1000-k>5k+5$, that is, when $k<\frac{995}{6}=165\frac{5}{6}$. It never equals 1, because 995 is not a multiple of 6. So

$$A_0<A_1<\cdots<A_{166}\qquad\text{and}\qquad A_{166}>A_{167}>\cdots>A_{1000}.$$

The largest term is A_{166}.

Answer: 166. □

8. (2016 AIME II, Problem #6) For polynomial $P(x)=1-\frac{1}{3}x+\frac{1}{6}x^2$, define

$$Q(x)=P(x)P(x^3)P(x^5)P(x^7)P(x^9)=\sum_{i=0}^{50}a_ix^i.$$

Then $\sum_{i=0}^{50}|a_i|=\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Strategy. Show that every coefficient a_i has the sign $(-1)^i$, so that $\sum|a_i|=Q(-1)$.

Solution. In $P(x)=1-\frac{1}{3}x+\frac{1}{6}x^2$, the term in x^j has the sign $(-1)^j$. For odd d, the term of $P(x^d)$ in x^{jd} therefore has the sign $(-1)^j=(-1)^{jd}$. When $Q(x)$ is multiplied out, each product of one term from each of the five factors is a nonzero number times x^i, where i is the total exponent, and its sign is $(-1)^i$. So all contributions to a_i have the sign $(-1)^i$, and $|a_i|=(-1)^ia_i$. Substituting $x=-1$, and using $(-1)^d=-1$ for each odd d,

$$\sum_{i=0}^{50}|a_i|=\sum_{i=0}^{50}a_i(-1)^i=Q(-1)=P(-1)^5=\left(1+\frac{1}{3}+\frac{1}{6}\right)^5=\left(\frac{3}{2}\right)^5=\frac{243}{32}.$$

Since $243 = 3^5$ and $32 = 2^5$ are relatively prime, $m + n = 243 + 32 = 275$.

Answer: 275. □

9. (2021 AMC 12A, Problem #15) A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of groups that can be selected. What is the remainder when N is divided by 100?

(A) 47 **(B)** 48 **(C)** 83 **(D)** 95 **(E)** 96

Strategy. Replace the number of chosen basses by the number of basses left out, so that Vandermonde's identity applies.

Solution. Allow the empty group for now. By symmetry, the number of groups with t tenors and b basses is $\binom{6}{t}\binom{8}{b} = \binom{6}{t}\binom{8}{8-b}$. Put $s = t + (8 - b)$. Since $t - b = s - 8$, the difference $t - b$ is a multiple of 4 exactly when s is a multiple of 4. For a fixed s, Vandermonde's identity gives

$$\sum_t \binom{6}{t}\binom{8}{s-t} = \binom{14}{s}.$$

In other words, a group corresponds to the set of its tenors together with the basses not in it, which is an s-element subset of the 14 singers. The empty group has $s = 8$. Hence

$$N + 1 = \binom{14}{0} + \binom{14}{4} + \binom{14}{8} + \binom{14}{12} = 1 + 1001 + 3003 + 91 = 4096,$$

so $N = 4095$, and the remainder when N is divided by 100 is 95.

Answer: (D) 95. □

10. (2006 AMC 12A, Problem #24) The expression

$$(x + y + z)^{2006} + (x - y - z)^{2006}$$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

(A) 6018 **(B)** 671,676 **(C)** 1,007,514 **(D)** 1,008,016 **(E)** 2,015,028

Strategy. Expand both powers with the multinomial theorem. The terms in which the total power of y and z is odd cancel.

Solution. By the multinomial theorem, with both sums over all triples (a, b, c) of nonnegative integers with $a + b + c = 2006$,

$$(x + y + z)^{2006} = \sum \frac{2006!}{a!\,b!\,c!}\, x^a y^b z^c,$$

$$(x - y - z)^{2006} = \sum \frac{2006!}{a!\,b!\,c!}\, (-1)^{b+c} x^a y^b z^c.$$

In the sum of the two expansions, the monomials with $b + c$ odd cancel, and those with $b + c$ even have the coefficient $2 \cdot \frac{2006!}{a!\,b!\,c!} \neq 0$. Different triples give different monomials, so the number of terms is the number of triples with $a + b + c = 2006$ and $b + c$ even, that is, with a even. For $a = 2006 - 2j$, where $0 \le j \le 1003$, there are $2j + 1$ pairs (b, c) with $b + c = 2j$. The total is

$$1 + 3 + 5 + \cdots + 2007 = 1004^2 = 1{,}008{,}016.$$

Answer: (D) 1,008,016. □

11. (2006 AMC 12A, Problem #23) Given a finite sequence $S = (a_1, a_2, \ldots, a_n)$ of n real numbers, let $A(S)$ be the sequence

$$\left(\frac{a_1 + a_2}{2}, \frac{a_2 + a_3}{2}, \ldots, \frac{a_{n-1} + a_n}{2}\right)$$

of $n-1$ real numbers. Define $A^1(S) = A(S)$ and, for each integer m, $2 \le m \le n-1$, define $A^m(S) = A(A^{m-1}(S))$. Suppose $x > 0$, and let $S = (1, x, x^2, \ldots, x^{100})$. If $A^{100}(S) = (\frac{1}{2^{50}})$, then what is x?

(A) $1 - \dfrac{\sqrt{2}}{2}$ **(B)** $\sqrt{2} - 1$ **(C)** $\dfrac{1}{2}$ **(D)** $2 - \sqrt{2}$ **(E)** $\dfrac{\sqrt{2}}{2}$

Strategy. The one-term sequence $A^{100}(S)$ is the top of a triangle of averages built on the 101 terms of S.

Solution. Write the 101 numbers $1, x, x^2, \ldots, x^{100}$ in a row. Applying A writes above each pair of neighbors their average, and $A^m(S)$ is the row m levels up. So the single term of $A^{100}(S)$ is the number at the top of the triangle of averages built on this row. By the proposition on sums in a triangle, with averages and $n = 100$, and then the binomial theorem, this number is

$$\frac{1}{2^{100}} \sum_{k=0}^{100} \binom{100}{k} x^k = \frac{(1+x)^{100}}{2^{100}} = \left(\frac{1+x}{2}\right)^{100}.$$

Setting it equal to $\frac{1}{2^{50}}$ gives $(1+x)^{100} = 2^{50}$. Since $1 + x > 0$, taking 100th roots gives $1 + x = 2^{1/2} = \sqrt{2}$, so $x = \sqrt{2} - 1$.

Answer: (B) $\sqrt{2} - 1$. □

Advanced.

12. (2017 AIME I, Problem #7) For nonnegative integers a and b with $a + b \le 6$, let $T(a, b) = \binom{6}{a}\binom{6}{b}\binom{6}{a+b}$. Let S denote the sum of all $T(a, b)$, where a and b are nonnegative integers with $a + b \le 6$. Find the remainder when S is divided by 1000.

Strategy. Group the terms by $c = a + b$ and apply Vandermonde's identity twice.

Solution. Group the terms by $c = a + b$, where $0 \le c \le 6$:

$$S = \sum_{c=0}^{6} \binom{6}{c} \sum_{a+b=c} \binom{6}{a}\binom{6}{b}.$$

By Vandermonde's identity, the inner sum is $\binom{12}{c}$. Then, by symmetry and Vandermonde's identity again,

$$S = \sum_{c=0}^{6} \binom{12}{c}\binom{6}{c} = \sum_{c=0}^{6} \binom{12}{c}\binom{6}{6-c} = \binom{18}{6} = 18564.$$

The remainder when S is divided by 1000 is 564.

Answer: 564. □

13. (2015 AIME I, Problem #12) Consider all 1000-element subsets of the set $\{1, 2, 3, \ldots, 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Strategy. Add up, over j, the number of subsets whose least element is at least j. Then use the hockey stick identity.

Solution. The least element of a 1000-element subset of $\{1, 2, \ldots, 2015\}$ is at most 1016. For $1 \le j \le 1016$, the subsets whose least element is at least j are the 1000-element subsets of $\{j, j+1, \ldots, 2015\}$, a set of $2016 - j$ numbers, so there are $\binom{2016-j}{1000}$ of them.

Count in two ways the pairs (X, j) in which X is one of the subsets and j is a positive integer at most the least element of X. A subset with least element m belongs to exactly m pairs, so the number of pairs is the sum of all the least elements. Counted by j instead, the number of pairs is

$$\sum_{j=1}^{1016} \binom{2016-j}{1000} = \binom{1000}{1000} + \binom{1001}{1000} + \cdots + \binom{2015}{1000} = \binom{2016}{1001}$$

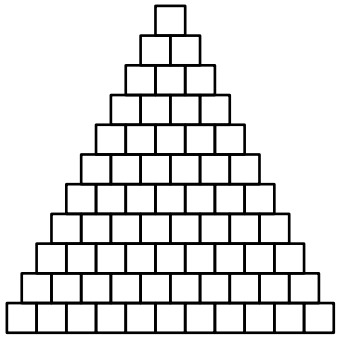
by the hockey stick identity. There are $\binom{2015}{1000}$ subsets, and the committee-chair identity gives $1001\binom{2016}{1001} = 2016\binom{2015}{1000}$. So the mean is

$$\frac{\binom{2016}{1001}}{\binom{2015}{1000}} = \frac{2016}{1001} = \frac{288}{143}.$$

Here $2016 = 7 \cdot 288$ and $1001 = 7 \cdot 143$, and $288 = 2^5 \cdot 3^2$ and $143 = 11 \cdot 13$ have no common prime factor. So $p + q = 288 + 143 = 431$.

Answer: 431. □

14. (2007 AIME II, Problem #13) A triangular array of squares has one square in the first row, two in the second, and, in general, k squares in the kth row for $1 \le k \le 11$. With the exception of the bottom row, each square rests on two squares in the row immediately below, as illustrated in the figure. In each square of the eleventh row, a 0 or a 1 is placed. Numbers are then placed into the other squares, with the entry for each square being the sum of the entries in the two squares below it. For how many initial distributions of 0's and 1's in the bottom row is the number in the top square a multiple of 3?



Strategy. Express the top number in terms of the bottom row with the proposition on sums in a triangle, and reduce the binomial coefficients modulo 3.

Solution. Call the entries of the bottom row $b_0, b_1, \ldots, b_{10}$ from left to right. Each square above the bottom row holds the sum of the two squares below it, so by the proposition on sums in a triangle with $n = 10$, the top square holds

$$\sum_{k=0}^{10} \binom{10}{k} b_k.$$

Row 10 of Pascal's triangle is $1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1$. The entries 45, 120, 210, 252 are multiples of 3, and $10 \equiv 1 \pmod 3$, so the top number is congruent to $b_0 + b_1 + b_9 + b_{10}$ modulo 3. This sum is an integer from 0 to 4, so it is a multiple of 3 exactly when it is 0 or 3. That happens for $\binom{4}{0} + \binom{4}{3} = 1 + 4 = 5$ choices of b_0, b_1, b_9,

b_{10}, and the other seven entries can be chosen freely in $2^7 = 128$ ways. The number of distributions is $5 \cdot 128 = 640$.

Answer: 640. □

15. (2014 AMC 12B, Problem #23) The number 2017 is prime. Let $S = \sum_{k=0}^{62} \binom{2014}{k}$. What is the remainder when S is divided by 2017?

(A) 32 **(B)** 684 **(C)** 1024 **(D)** 1576 **(E)** 2016

Strategy. Show that $\binom{2014}{k} \equiv (-1)^k \binom{k+2}{2} \pmod{2017}$ for $0 \le k \le 62$, then add.

Solution. Let $p = 2017$, so $2014 = p - 3$. For $0 \le k \le 62$,

$$k!\binom{2014}{k} = 2014 \cdot 2013 \cdots (2015 - k) \equiv (-3)(-4) \cdots (-(k+2))$$

$$= (-1)^k \frac{(k+2)!}{2} = (-1)^k \, k! \binom{k+2}{2} \pmod{p}.$$

Since $k < p$, the factor $k!$ can be canceled, so $\binom{2014}{k} \equiv (-1)^k \binom{k+2}{2} \pmod{p}$. Write $T_j = \binom{j+1}{2} = \frac{j(j+1)}{2}$. With $j = k + 1$,

$$S \equiv T_1 - T_2 + T_3 - T_4 + \cdots - T_{62} + T_{63} \pmod{p}.$$

Since $T_{2i} - T_{2i-1} = \frac{2i(2i+1) - (2i-1)2i}{2} = 2i$, pairing the terms gives

$$S \equiv T_{63} - \sum_{i=1}^{31} (T_{2i} - T_{2i-1}) = 2016 - \sum_{i=1}^{31} 2i = 2016 - 992 = 1024 \pmod{p}.$$

Since $0 \le 1024 < 2017$, the remainder is 1024.

Answer: (C) 1024. □

16. (2022 AIME I, Problem #12) For any finite set X, let $|X|$ denote the number of elements in X. Define

$$S_n = \sum |A \cap B|,$$

where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \ldots, n\}$ with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\},$$

giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. Let $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

Strategy. Add up the contributions to S_n element by element: for each x, find how many pairs (A, B) with $|A| = |B|$ satisfy $x \in A$ and $x \in B$. Then find $\frac{S_{n+1}}{S_n}$ for general n with the factorial formula.

Solution. Each pair (A, B) contributes 1 to S_n for every element of $A \cap B$. Adding these contributions element by element instead of pair by pair gives $S_n = c_1 + c_2 + \cdots + c_n$, where c_x is the number of pairs (A, B) with $|A| = |B|$ in which both sets contain x. A set of size $k \ge 1$ that contains x consists of x and $k - 1$ of the other $n - 1$ elements, so $\binom{n-1}{k-1}^2$ of the pairs counted by c_x have $|A| = |B| = k$. By the corollary of Vandermonde's identity, applied to row $n - 1$,

$$c_x = \sum_{k=1}^{n} \binom{n-1}{k-1}^2 = \binom{2n-2}{n-1}$$

for every x. Hence $S_n = n\binom{2n-2}{n-1}$; for example, $S_2 = 2\binom{2}{1} = 4$.

By the factorial formula,

$$\binom{2n}{n} = \frac{(2n)(2n-1)}{n \cdot n} \cdot \frac{(2n-2)!}{(n-1)!\,(n-1)!} = \frac{2(2n-1)}{n}\binom{2n-2}{n-1},$$

so

$$\frac{S_{n+1}}{S_n} = \frac{(n+1)\binom{2n}{n}}{n\binom{2n-2}{n-1}} = \frac{2(n+1)(2n-1)}{n^2}.$$

A prime dividing n and $n+1$ would divide their difference 1, and a prime dividing n and $2n-1$ would divide $2n-(2n-1) = 1$. For odd n, the prime 2 does not divide n either, so the numerator $2(n+1)(2n-1)$ and the denominator n^2 have no common prime factor, and the fraction is in lowest terms. With $n = 2021$,

$$\frac{S_{2022}}{S_{2021}} = \frac{2 \cdot 2022 \cdot 4041}{2021^2},$$

so

$$p = 2 \cdot 2022 \cdot 4041 = 16{,}341{,}804 \qquad \text{and} \qquad q = 2021^2 = 4{,}084{,}441.$$

Their sum $p + q = 20{,}426{,}245$ leaves the remainder 245 on division by 1000.

Answer: 245. □

2. Answer Key

Problem	Letter	Value
1	B	7
2		770
3		44
4		667
5	C	167
6	C	970
7		166
8		275
9	D	95
10	D	1,008,016
11	B	$\sqrt{2}-1$
12		564
13		431
14		640
15	C	1024
16		245