BINOMIAL COEFFICIENTS AND COMBINATORIAL IDENTITIES FOR AMC 12: PRACTICE SET

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1. Problems

Beginner.

1. (1965 AHSME, Problem #14) The sum of the numerical coefficients in the complete expansion of $(x^2 - 2xy + y^2)^7$ is:

(A) 0 **(B)** 7 **(C)** 14 **(D)** 128 **(E)** 128^2

2. (2004 AMC 10A, Problem #10) Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same?

(A) $\frac{19}{128}$ **(B)** $\frac{23}{128}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{35}{128}$ **(E)** $\frac{1}{2}$

3. (1977 AHSME, Problem #25) Determine the largest positive integer n such that 1005! is divisible by 10^n.

(A) 102 **(B)** 112 **(C)** 249 **(D)** 502 **(E)** none of these

4. (1999 AHSME, Problem #14) Four girls — Mary, Alina, Tina, and Hanna — sang songs in a concert as trios, with one girl sitting out each time. Hanna sang 7 songs, which was more than any other girl, and Mary sang 4 songs, which was fewer than any other girl. How many songs did these trios sing?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

5. (1980 AHSME, Problem #17) Given that $i^2 = -1$, for how many integers n is $(n + i)^4$ an integer?

(A) none **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Intermediate.

6. The number 101 is prime. What is the remainder when

$$1 \cdot \binom{100}{1} + 2 \cdot \binom{100}{2} + 3 \cdot \binom{100}{3} + \cdots + 100 \cdot \binom{100}{100}$$

is divided by 101?

7. (1989 AIME, Problem #5) When a certain biased coin is flipped five times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let $\frac{i}{j}$, in lowest terms, be the probability that the coin comes up heads in exactly 3 out of 5 flips. Find $i + j$.

8. (1995 AHSME, Problem #27) Consider the triangular array of numbers with 0, 1, 2, 3, ... along the sides and interior numbers obtained by adding the two adjacent numbers in the previous row. Rows 1 through 6 are shown.

$$\begin{array}{ccccccccccc} & & & & 0 & & & & & & \\ & & & 1 & & 1 & & & & & \\ & & 2 & & 2 & & 2 & & & & \\ & 3 & & 4 & & 4 & & 3 & & & \\ 4 & & 7 & & 8 & & 7 & & 4 & & \\ 5 & & 11 & & 15 & & 15 & & 11 & & 5 \end{array}$$

Let $f(n)$ denote the sum of the numbers in row n. What is the remainder when $f(100)$ is divided by 100?

(A) 12 **(B)** 30 **(C)** 50 **(D)** 62 **(E)** 74

9. (2023 AMC 10B, Problem #21) Each of 2023 balls is randomly placed into one of 3 bins. Which of the following is closest to the probability that each of the bins will contain an odd number of balls?

(A) $\frac{2}{3}$ **(B)** $\frac{3}{10}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{4}$

10. (1982 AHSME, Problem #30) Find the units digit in the decimal expansion of

$$(15+\sqrt{220})^{19}+(15+\sqrt{220})^{82}.$$

(A) 0 **(B)** 2 **(C)** 5 **(D)** 9 **(E)** none of these

Advanced.

11. (2019 AMC 12A, Problem #24) For how many integers n between 1 and 50, inclusive, is

$$\frac{(n^2-1)!}{(n!)^n}$$

an integer? (Recall that $0!=1$.)

(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

12. (1997 AIME, Problem #10) Every card in a deck has a picture of one shape — circle, square, or triangle, which is painted in one of the three colors — red, blue, or green. Furthermore, each color is applied in one of three shades — light, medium, or dark. The deck has 27 cards, with every shape-color-shade combination represented. A set of three cards from the deck is called *complementary* if all of the following statements are true:

- Either each of the three cards has a different shape or all three of the cards have the same shape.
- Either each of the three cards has a different color or all three of the cards have the same color.
- Either each of the three cards has a different shade or all three of the cards have the same shade.

How many different complementary three-card sets are there?

13. (2008 AIME I, Problem #6) A triangular array of numbers has a first row consisting of the odd integers 1, 3, 5, ..., 99 in increasing order. Each row below the first has one fewer entry than the row above it, and the bottom row has a single entry. Each entry in any row after the top row equals the sum of the two entries diagonally above it in the row immediately above it. How many entries in the array are multiples of 67?

$$\begin{array}{ccccccccccc} 1 & & 3 & & 5 & & \cdots & & 97 & & 99 \\ & 4 & & 8 & & 12 & & \cdots & & 196 & \\ & & & & & \vdots & & & & & \end{array}$$

14. (2022 AIME II, Problem #10) Find the remainder when

$$\binom{\binom{3}{2}}{2} + \binom{\binom{4}{2}}{2} + \cdots + \binom{\binom{40}{2}}{2}$$

is divided by 1000.

15. (2003 AIME I, Problem #13) Let N be the number of positive integers that are less than or equal to 2003 and whose base-2 representation has more 1's than 0's. Find the remainder when N is divided by 1000.

2. Answer Key

Problem	Letter	Value
1	A	0
2	D	$\frac{35}{128}$
3	E	none of these
4	A	7
5	D	3
6		50
7		283
8	E	74
9	E	$\frac{1}{4}$
10	D	9
11	D	34
12		117
13		17
14		4
15		155