BINOMIAL COEFFICIENTS AND COMBINATORIAL IDENTITIES FOR AMC 12: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (1965 AHSME, Problem #14) The sum of the numerical coefficients in the complete expansion of $(x^2 - 2xy + y^2)^7$ is:

(A) 0 **(B)** 7 **(C)** 14 **(D)** 128 **(E)** 128^2

Strategy. The sum of the coefficients of a polynomial is its value when every variable equals 1.

Solution. After like terms are collected, the expansion is a sum of terms $c\,x^a y^b$. Putting $x = y = 1$ turns each term into its coefficient c, so the sum of the coefficients is the value of the expression at $x = y = 1$:

$$\left(1^2 - 2 \cdot 1 \cdot 1 + 1^2\right)^7 = 0^7 = 0.$$

Answer: (A) 0. □

2. (2004 AMC 10A, Problem #10) Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same?

(A) $\frac{19}{128}$ **(B)** $\frac{23}{128}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{35}{128}$ **(E)** $\frac{1}{2}$

Strategy. Count the outcomes in which both coins show k heads, and add over k. Rewriting the count for coin B by symmetry makes the lower indices add up to 4, so Vandermonde's identity gives the sum.

Solution. The seven flips have $2^7 = 128$ equally likely outcomes. Coin A shows exactly k heads in $\binom{3}{k}$ of the results of its three flips, and coin B in $\binom{4}{k}$ of the results of its four flips, so both coins show exactly k heads in $\binom{3}{k}\binom{4}{k}$ outcomes, for $k = 0, 1, 2, 3$. By symmetry, $\binom{4}{k} = \binom{4}{4-k}$. The term with $k = 4$, namely $\binom{3}{4}\binom{4}{0}$, is 0, so Vandermonde's identity with $m = 3$, $n = 4$, and $r = 4$ gives the number of outcomes with equal numbers of heads:

$$\sum_{k=0}^{4} \binom{3}{k}\binom{4}{4-k} = \binom{7}{4} = 35.$$

Directly, $1 \cdot 1 + 3 \cdot 4 + 3 \cdot 6 + 1 \cdot 4 = 35$. The probability is $\frac{35}{128}$.

Answer: (D) $\frac{35}{128}$. □

3. (1977 AHSME, Problem #25) Determine the largest positive integer n such that 1005! is divisible by 10^n.

(A) 102 **(B)** 112 **(C)** 249 **(D)** 502 **(E)** none of these

Strategy. Since $10^n = 2^n 5^n$, the answer is the smaller of the exponents of 2 and 5 in 1005!. Find the exponent of 5 with Legendre's formula.

Solution. The power $10^n = 2^n 5^n$ divides 1005! exactly when n is at most the exponent of 2 and at most the exponent of 5 in the prime factorization of 1005!. Since $5^4 = 625 \leq 1005 < 5^5$, Legendre's formula gives the exponent of 5 as

$$\left\lfloor \frac{1005}{5} \right\rfloor + \left\lfloor \frac{1005}{25} \right\rfloor + \left\lfloor \frac{1005}{125} \right\rfloor + \left\lfloor \frac{1005}{625} \right\rfloor = 201 + 40 + 8 + 1 = 250.$$

The exponent of 2 is at least the first term of its formula, $\lfloor 1005/2 \rfloor = 502$, which is larger than 250. So the largest n is 250, which is not among choices (A) to (D).

Answer: (E) none of these. □

4. (1999 AHSME, Problem #14) Four girls — Mary, Alina, Tina, and Hanna — sang songs in a concert as trios, with one girl sitting out each time. Hanna sang 7 songs, which was more than any other girl, and Mary sang 4 songs, which was fewer than any other girl. How many songs did these trios sing?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Strategy. Count the pairs consisting of a song and a girl who sang it in two ways: song by song, and girl by girl.

Solution. Let s be the number of songs, and let a and t be the numbers of songs that Alina and Tina sang. Each song was sung by exactly 3 girls, so there are $3s$ pairs consisting of a song and a girl who sang it. Counting the same pairs girl by girl gives

$$3s = 7 + 4 + a + t.$$

Hanna sang more songs than any other girl, and Mary sang fewer than any other girl, so $4 < a < 7$ and $4 < t < 7$. Hence each of a and t is 5 or 6, so $a + t$ is 10, 11, or 12, and $3s$ is 21, 22, or 23. Of these three numbers, only 21 is divisible by 3, which gives $s = 7$, with $a = t = 5$. Such a concert exists: Hanna sits out no song, Mary sits out 3, and Alina and Tina sit out 2 each, one girl for each of the 7 songs.

Answer: (A) 7. □

5. (1980 AHSME, Problem #17) Given that $i^2 = -1$, for how many integers n is $(n + i)^4$ an integer?

(A) none **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Strategy. Expand $(n + i)^4$ with the binomial theorem, carrying the powers of i along. The result is an integer exactly when the terms with odd powers of i add up to 0.

Solution. By the binomial theorem, using $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$,

$$(n + i)^4 = n^4 + 4n^3 i + 6n^2 i^2 + 4ni^3 + i^4 = \left(n^4 - 6n^2 + 1\right) + \left(4n^3 - 4n\right) i.$$

For an integer n, both $n^4 - 6n^2 + 1$ and $4n^3 - 4n$ are integers, and a number $u + vi$ with real u and v is real only when $v = 0$. So $(n + i)^4$ is an integer exactly when

$$4n^3 - 4n = 4n(n - 1)(n + 1) = 0,$$

that is, when n is -1, 0, or 1. For example, $(1 + i)^4 = \left((1 + i)^2\right)^2 = (2i)^2 = -4$. There are 3 such integers.

Answer: (D) 3. □

Intermediate.

6. The number 101 is prime. What is the remainder when

$$1 \cdot \binom{100}{1} + 2 \cdot \binom{100}{2} + 3 \cdot \binom{100}{3} + \cdots + 100 \cdot \binom{100}{100}$$

is divided by 101?

Strategy. Since $100 = 101 - 1$, each coefficient $\binom{100}{k}$ is congruent to 1 or -1 modulo the prime 101. Replace every coefficient by the 1 or -1 it is congruent to, and add.

Solution. Let $0 \le k \le 100$. Reducing each factor modulo 101,

$$k!\binom{100}{k} = 100 \cdot 99 \cdots (101-k) \equiv (-1)(-2)\cdots(-k) = (-1)^k k! \pmod{101}.$$

Since $k < 101$, the prime 101 does not divide $k!$, so $k!$ can be canceled: $\binom{100}{k} \equiv (-1)^k \pmod{101}$. Call the given sum S. Then

$$S \equiv \sum_{k=1}^{100} (-1)^k k = (-1+2) + (-3+4) + \cdots + (-99+100) = 50 \pmod{101},$$

because there are 50 pairs and each pair equals 1. Since $0 \le 50 < 101$, the remainder is 50.

As a check, the corollary of the committee-chair identity gives $S = 100 \cdot 2^{99}$. By the row sum of row 100 and the congruence above,

$$2^{100} = \sum_{k=0}^{100} \binom{100}{k} \equiv \sum_{k=0}^{100} (-1)^k = 1 \pmod{101},$$

so $2S = 100 \cdot 2^{100} \equiv 100 = 2 \cdot 50 \pmod{101}$. The prime 101 does not divide 2, so the factor 2 can be canceled, and again $S \equiv 50 \pmod{101}$.

Answer: 50. □

7. (1989 AIME, Problem #5) When a certain biased coin is flipped five times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let $\frac{i}{j}$, in lowest terms, be the probability that the coin comes up heads in exactly 3 out of 5 flips. Find $i + j$.

Strategy. The probability of exactly k heads is the general term $\binom{5}{k}p^k q^{5-k}$ of $(q+p)^5$, where p is the probability of heads and $q = 1-p$. The condition fixes the ratio of two neighboring terms, which gives a linear equation in p and q.

Solution. Let p be the probability of heads on one flip and $q = 1 - p$. The flips are independent, so each sequence of five flips with exactly k heads has probability $p^k q^{5-k}$, and there are $\binom{5}{k}$ such sequences. So the probability of exactly k heads is

$$P_k = \binom{5}{k} p^k q^{5-k}.$$

Since $P_1 = 5pq^4$ is not 0, neither p nor q is 0. By the proposition on neighboring entries, $\binom{5}{2} = \frac{5-1}{1+1}\binom{5}{1} = 2\binom{5}{1}$, so

$$\frac{P_2}{P_1} = \frac{\binom{5}{2}p^2q^3}{\binom{5}{1}pq^4} = \frac{2p}{q}.$$

The condition $P_1 = P_2$ gives $q = 2p$. With $p + q = 1$, this gives $p = \frac{1}{3}$ and $q = \frac{2}{3}$, so

$$P_3 = \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = 10 \cdot \frac{1}{27} \cdot \frac{4}{9} = \frac{40}{243}.$$

Since $243 = 3^5$ and 3 does not divide 40, the fraction is in lowest terms, and $i + j = 40 + 243 = 283$.

Answer: 283. □

8. (1995 AHSME, Problem #27) Consider the triangular array of numbers with 0, 1, 2, 3, ... along the sides and interior numbers obtained by adding the two adjacent numbers in the previous row. Rows 1 through 6 are shown.

$$\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & 1 & & 1 & & & & & \\
 & & 2 & & 2 & & 2 & & & & \\
 & 3 & & 4 & & 4 & & 3 & & & \\
4 & & 7 & & 8 & & 7 & & 4 & & \\
5 & & 11 & & 15 & & 15 & & 11 & & 5
\end{array}$$

Let $f(n)$ denote the sum of the numbers in row n. What is the remainder when $f(100)$ is divided by 100?

(A) 12 **(B)** 30 **(C)** 50 **(D)** 62 **(E)** 74

Strategy. Each interior entry is the sum of two neighbors in the row above, as in Pascal's triangle, so every entry of the row above is used twice except the two ends. Compare $f(n)$ with $2f(n-1)$.

Solution. Let $n \geq 3$, and let row $n-1$ be $b_0, b_1, \ldots, b_{n-2}$, so that $b_0 = b_{n-2} = n-2$ and $b_0 + b_1 + \cdots + b_{n-2} = f(n-1)$. Row n consists of the two ends, each equal to $n-1$, and the interior entries b_0+b_1, b_1+b_2, ..., $b_{n-3}+b_{n-2}$. In these sums every entry of row $n-1$ appears twice, except b_0 and b_{n-2}, which appear once. Hence $f(n) = 2(n-1)+2f(n-1)-b_0-b_{n-2}$, that is,

$$f(n) = 2(n-1) + 2f(n-1) - 2(n-2) = 2f(n-1) + 2.$$

This also holds for $n = 2$, since $f(1) = 0$ and $f(2) = 2$. In other words, $f(n) + 2 = 2\,(f(n-1) + 2)$: the numbers $f(n)+2$ double from each row to the next. Since $f(1)+2 = 2$, it follows that $f(n) + 2 = 2^n$, so

$$f(n) = 2^n - 2,$$

which gives 0, 2, 6, 14, 30, 62 for the six rows shown.

Now find 2^{100} modulo 100. Since $2^{10} = 1024 = 25 \cdot 41 - 1$, the binomial theorem gives

$$2^{100} = (25 \cdot 41 - 1)^{10} \equiv (-1)^{10} = 1 \pmod{25},$$

because every term of the expansion except $(-1)^{10}$ contains the factor $25 \cdot 41$. Also, 2^{100} is a multiple of 4. The numbers from 0 to 99 that leave the remainder 1 on division by 25 are 1, 26, 51, and 76, and only 76 is a multiple of 4. So $2^{100} \equiv 76 \pmod{100}$, and $f(100) = 2^{100} - 2 \equiv 74 \pmod{100}$.

Answer: (E) 74. □

9. (2023 AMC 10B, Problem #21) Each of 2023 balls is randomly placed into one of 3 bins. Which of the following is closest to the probability that each of the bins will contain an odd number of balls?

(A) $\frac{2}{3}$ **(B)** $\frac{3}{10}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{4}$

Strategy. The placements with a, b, and c balls in the three bins are counted by the multinomial coefficient $\frac{2023!}{a!\,b!\,c!}$, the coefficient of $x^a y^b z^c$ in $(x+y+z)^{2023}$. Substituting 1 and -1 for x and y picks out the coefficients with a and b both odd.

Solution. Number the balls. Each ball goes into one of the 3 bins, so there are 3^{2023} equally likely placements. Record a placement as a word of length 2023 in the letters x, y, z, whose jth letter names the bin of ball j. By equal overcounting, the number of words with a

letters x, b letters y, and c letters z is $\frac{2023!}{a!\,b!\,c!}$, which by the multinomial theorem is the coefficient of $x^a y^b z^c$ in

$$F(x,y,z) = (x+y+z)^{2023}.$$

Since $a+b+c = 2023$ is odd, c is odd whenever a and b are both odd. So the number N of placements in which every bin has an odd number of balls is the sum of the coefficients C of the terms $Cx^a y^b z^c$ of F with a and b both odd.

Each term $Cx^a y^b z^c$ of F contributes C, $(-1)^a C$, $(-1)^b C$, and $(-1)^{a+b} C$ to the values $F(1,1,1)$, $F(-1,1,1)$, $F(1,-1,1)$, and $F(-1,-1,1)$. So the number

$$T = F(1,1,1) - F(-1,1,1) - F(1,-1,1) + F(-1,-1,1)$$

is the sum, over all the terms of F, of C times the factor $\left(1-(-1)^a\right)\left(1-(-1)^b\right)$. This factor is 4 when a and b are both odd and 0 otherwise, so $T = 4N$. Evaluating T directly,

$$4N = 3^{2023} - 1 - 1 + (-1)^{2023} = 3^{2023} - 3.$$

Hence the probability is

$$\frac{N}{3^{2023}} = \frac{3^{2023}-3}{4\cdot 3^{2023}} = \frac{1}{4} - \frac{1}{4\cdot 3^{2022}}.$$

It is less than $\frac{1}{4}$ by the tiny amount $\frac{1}{4\cdot 3^{2022}}$, while every other choice differs from $\frac{1}{4}$ by at least $\frac{3}{10} - \frac{1}{4} = \frac{1}{20}$. So $\frac{1}{4}$ is the closest choice.

Answer: (E) $\dfrac{1}{4}$. □

10. (1982 AHSME, Problem #30) Find the units digit in the decimal expansion of

$$(15+\sqrt{220})^{19} + (15+\sqrt{220})^{82}.$$

(A) 0 **(B)** 2 **(C)** 5 **(D)** 9 **(E)** none of these

Strategy. Add the same powers of $15-\sqrt{220}$, a number between 0 and 1. The sums $(15+\sqrt{220})^n + (15-\sqrt{220})^n$ are integers, and their expansions show that they are multiples of 10.

Solution. Let $a = 15+\sqrt{220}$ and $b = 15-\sqrt{220}$. Since $14.5^2 = 210.25 < 220 < 225 = 15^2$, the number b lies between 0 and $\frac{1}{2}$. For $n \geq 1$, adding the expansions of $(15+\sqrt{220})^n$ and $(15-\sqrt{220})^n$ cancels the terms with odd powers of $\sqrt{220}$ and doubles the others:

$$a^n + b^n = 2\sum_{k \text{ even}} \binom{n}{k} 15^{n-k} \left(\sqrt{220}\right)^k = 2\sum_{0\leq j\leq n/2} \binom{n}{2j} 15^{n-2j}\, 220^j.$$

In the last sum, the term with $j = 0$ is 15^n, a multiple of 5, and every term with $j \geq 1$ contains the factor 220, also a multiple of 5. So $a^n + b^n$ is twice a multiple of 5, that is, a multiple of 10, for every $n \geq 1$. Hence

$$a^{19} + a^{82} = M - \left(b^{19} + b^{82}\right), \qquad \text{where } M = \left(a^{19} + b^{19}\right) + \left(a^{82} + b^{82}\right)$$

is a multiple of 10. Since $0 < b < \frac{1}{2}$, the numbers b^{19} and b^{82} lie between 0 and $\frac{1}{2}$, so $0 < b^{19} + b^{82} < 1$ and

$$M - 1 < a^{19} + a^{82} < M.$$

So the integer part of $a^{19} + a^{82}$ is $M-1$, whose units digit is 9.

Answer: (D) 9. □

Advanced.

11. (2019 AMC 12A, Problem #24) For how many integers n between 1 and 50, inclusive, is

$$\frac{(n^2-1)!}{(n!)^n}$$

an integer? (Recall that $0! = 1$.)

(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

Strategy. For a prime n and for $n = 4$, compare the exponents of one prime in the numerator and in the denominator with Legendre's formula. For every other n, write the quotient as the number of ways to split n^2 people into n unnamed groups of n, times $\frac{(n-1)!}{n}$.

Solution. Write $Q(n) = \frac{(n^2-1)!}{(n!)^n}$.

Let $n = p$ be a prime. Legendre's formula gives the exponent of p in $(p^2-1)!$ as $\lfloor (p^2-1)/p \rfloor = p-1$, since $p^2 > p^2-1$ makes all the later terms 0. The exponent of p in $p!$ is 1, so in $(p!)^p$ it is p. Since $p > p-1$, the quotient $Q(p)$ is not an integer.

For $n = 4$, Legendre's formula gives the exponent of 2 in $15!$ as $7+3+1 = 11$, while $(4!)^4 = (2^3 \cdot 3)^4$ contains 2^{12}. So $Q(4)$ is not an integer.

For every other n, use $n! = n \cdot (n-1)!$ to write

$$Q(n) = \frac{(n^2)!}{n^2\,(n!)^n} = \frac{(n^2)!}{(n!)^n\,n!} \cdot \frac{(n-1)!}{n}.$$

By equal overcounting, the first factor is the number of ways to split n^2 people into n unnamed groups of n, so it is an integer. The second factor is also an integer. For $n = 1$ it is $\frac{0!}{1} = 1$. If n is composite and not the square of a prime, then $n = uv$ with $1 < u < v < n$, and u and v are different factors of $(n-1)!$. If $n = p^2$ for a prime $p \geq 3$, then $p < 2p < p^2 = n$, so p and $2p$ are different factors of $(n-1)!$, and their product $2p^2$ is a multiple of n. In each case n divides $(n-1)!$, so $Q(n)$ is an integer.

Hence $Q(n)$ fails to be an integer exactly when n is a prime or $n = 4$. The primes up to 50 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47, fifteen in all, so the count is $50 - 15 - 1 = 34$.

Answer: (D) 34. □

12. (1997 AIME, Problem #10) Every card in a deck has a picture of one shape — circle, square, or triangle, which is painted in one of the three colors — red, blue, or green. Furthermore, each color is applied in one of three shades — light, medium, or dark. The deck has 27 cards, with every shape-color-shade combination represented. A set of three cards from the deck is called *complementary* if all of the following statements are true:

- Either each of the three cards has a different shape or all three of the cards have the same shape.
- Either each of the three cards has a different color or all three of the cards have the same color.
- Either each of the three cards has a different shade or all three of the cards have the same shade.

How many different complementary three-card sets are there?

Strategy. Show that any two different cards lie in exactly one complementary set. Then count the pairs of cards in two ways: directly, and set by set.

Solution. There are $27 = 3 \cdot 3 \cdot 3$ cards and every shape-color-shade combination appears, so each combination appears on exactly one card. Take two different cards. For each of

the three attributes, the value on a third card that makes a complementary set is forced: if the two cards agree in that attribute, the third card must have the same value, and if they differ, it must have the remaining value, since the three values must then all be different. So exactly one card completes the two cards to a complementary set. This card is different from both of them: the two cards differ in at least one attribute, where the completing card takes the value that neither of them has.

Now count the pairs consisting of a complementary set and two of its cards. Each complementary set contains $\binom{3}{2} = 3$ pairs of cards, so if there are N complementary sets, there are $3N$ such pairs. On the other hand, each of the $\binom{27}{2} = 351$ pairs of cards lies in exactly one complementary set. Hence $3N = 351$ and $N = 117$.

Answer: 117. □

13. (2008 AIME I, Problem #6) A triangular array of numbers has a first row consisting of the odd integers 1, 3, 5, ..., 99 in increasing order. Each row below the first has one fewer entry than the row above it, and the bottom row has a single entry. Each entry in any row after the top row equals the sum of the two entries diagonally above it in the row immediately above it. How many entries in the array are multiples of 67?

$$\begin{array}{ccccccccccccc} 1 & & 3 & & 5 & & \cdots & & 97 & & 99 \\ & 4 & & 8 & & 12 & & \cdots & & 196 & \\ & & & & & \vdots & & & & & \end{array}$$

Strategy. Each entry comes from a stretch of the first row by repeatedly adding neighbors, so the proposition on sums in a triangle writes it as a sum of binomial coefficients times entries of the first row. Find a closed form for this sum, then decide when 67 divides it.

Solution. Number the rows $0, 1, \ldots, 49$ from the top, so that row r has $50 - r$ entries, and number the positions in each row $0, 1, 2, \ldots$ from the left. The first row holds $b_j = 2j + 1$ in position j, for $0 \le j \le 49$. The entry in position i of row r is obtained from the $r + 1$ entries $b_i, b_{i+1}, \ldots, b_{i+r}$ by r rounds of adding neighbors, so by the proposition on sums in a triangle (with the rows built downward instead of upward) it equals

$$\begin{aligned} \sum_{j=0}^{r} \binom{r}{j} b_{i+j} &= (2i+1) \sum_{j=0}^{r} \binom{r}{j} + 2 \sum_{j=0}^{r} j \binom{r}{j} \\ &= (2i+1)\, 2^r + 2 \cdot r\, 2^{r-1} = 2^r (2i + r + 1). \end{aligned}$$

The first sum is the row sum of row r, and the second is given by the corollary of the committee-chair identity when $r \ge 1$; for $r = 0$ the formula reads $2i+1$, the first row itself. For example, row 1 begins 4, 8, 12 and ends with $2(2 \cdot 48 + 2) = 196$.

Since 67 is an odd prime, it divides $2^r(2i + r + 1)$ exactly when it divides $2i + r + 1$. Position i of row r exists when $0 \le i \le 49 - r$, and then

$$1 \le 2i + r + 1 \le 2(49 - r) + r + 1 = 99 - r \le 99.$$

The only multiple of 67 in this range is 67 itself, so the entries that are multiples of 67 are those with $2i + r = 66$. Then r is even and $i = \frac{66-r}{2}$, and the condition $i \le 49 - r$ becomes $66 - r \le 98 - 2r$, that is, $r \le 32$. Each even r from 0 to 32 gives exactly one such entry, so there are 17 of them.

Answer: 17. □

14. (2022 AIME II, Problem #10) Find the remainder when

$$\binom{\binom{3}{2}}{2} + \binom{\binom{4}{2}}{2} + \cdots + \binom{\binom{40}{2}}{2}$$

is divided by 1000.

Strategy. Count the pairs of different 2-element subsets of $\{1, 2, \ldots, k\}$ by the union of the two subsets, which gives $\binom{\binom{k}{2}}{2} = 3\binom{k}{3} + 3\binom{k}{4}$. Then add each part over k with the hockey stick identity.

Solution. For $k \geq 3$, the number $\binom{\binom{k}{2}}{2}$ counts the unordered pairs of different 2-element subsets of $\{1, 2, \ldots, k\}$. Two different 2-element subsets share at most one element. If they share one, their union has 3 elements, and any two of the three 2-element subsets of a 3-element set share one element, so each 3-element set is the union of $\binom{3}{2} = 3$ such pairs. If they share none, their union has 4 elements, and a 4-element set splits into two 2-element subsets in 3 ways. Counting the pairs by their union,

$$\binom{\binom{k}{2}}{2} = 3\binom{k}{3} + 3\binom{k}{4}.$$

For example, $k = 4$ gives $\binom{6}{2} = 15 = 3 \cdot 4 + 3 \cdot 1$.

Now add over $k = 3, 4, \ldots, 40$. Since $\binom{3}{4} = 0$, the hockey stick identity with $n = 40$ and $r = 3$, then $r = 4$, followed by Pascal's identity, gives

$$3\sum_{k=3}^{40}\binom{k}{3} + 3\sum_{k=4}^{40}\binom{k}{4} = 3\binom{41}{4} + 3\binom{41}{5} = 3\binom{42}{5}.$$

Since $\binom{42}{5} = \frac{42 \cdot 41 \cdot 40 \cdot 39 \cdot 38}{120} = 850668$, the sum is $3 \cdot 850668 = 2552004$, and its remainder on division by 1000 is 4.

Answer: 4. □

15. (2003 AIME I, Problem #13) Let N be the number of positive integers that are less than or equal to 2003 and whose base-2 representation has more 1's than 0's. Find the remainder when N is divided by 1000.

Strategy. Sort the numbers by their number d of binary digits. After the leading 1, the other $d - 1$ digits must contain enough 1's, and by symmetry the count is half the sum of row $d - 1$ of Pascal's triangle, plus half its middle entry when $d - 1$ is even. Then remove the numbers from 2004 to 2047.

Solution. Since $2^{10} \leq 2003 < 2^{11} = 2048$, the numbers up to 2003 have at most 11 binary digits. First count all the numbers from 1 to 2047 that have more 1's than 0's.

A number with exactly d binary digits starts with 1, and its other $d - 1$ digits can be any string of 0's and 1's. If m of them are 1's, the number has $m + 1$ ones and $d - 1 - m$ zeros, and it has more 1's than 0's exactly when $m + 1 > d - 1 - m$, that is, $2m \geq d - 1$. So the count for d digits is the sum of the entries $\binom{d-1}{m}$ of row $d - 1$ with $2m \geq d - 1$.

- If $d = 2t$, then $d - 1 = 2t - 1$ is odd and the condition is $m \geq t$. These entries form one half of row $2t - 1$, so they sum to $2^{2t-2} = 2^{d-2}$.
- If $d = 2t + 1$, then $d - 1 = 2t$ is even and the condition is $m \geq t$. By symmetry, the entries of row $2t$ with $m > t$ have the same sum as those with $m < t$, so the entries with $m \geq t$ sum to $\frac{1}{2}\left(2^{2t} + \binom{2t}{t}\right)$.

For $d = 2, 4, 6, 8, 10$, the counts add up to $1+4+16+64+256 = 341$. For $d = 1, 3, 5, 7, 9, 11$, that is, $t = 0, 1, \ldots, 5$, the powers 2^{2t} add up to $1 + 4 + 16 + 64 + 256 + 1024 = 1365$ and the middle entries $\binom{2t}{t}$ add up to $1 + 2 + 6 + 20 + 70 + 252 = 351$, so the counts add up to

$$\frac{1}{2}(1365 + 351) = 858.$$

So $341 + 858 = 1199$ numbers from 1 to 2047 have more 1's than 0's.

The numbers from 2004 to 2047 are larger than $1984 = 11111000000_2$ and at most $2047 = 11111111111_2$. So each of them has 11 binary digits, begins with five 1's, and has at least one more 1 among its last six digits. Each therefore has at least six 1's and at most five 0's, and all 44 of these numbers were counted. Hence

$$N = 1199 - 44 = 1155,$$

and the remainder when N is divided by 1000 is 155.

Answer: 155. □

2. Answer Key

Problem	Letter	Value
1	A	0
2	D	$\frac{35}{128}$
3	E	none of these
4	A	7
5	D	3
6		50
7		283
8	E	74
9	E	$\frac{1}{4}$
10	D	9
11	D	34
12		117
13		17
14		4
15		155