

ADVANCED COUNTING AND BIJECTIONS FOR AMC 12

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CONTENTS

1. Counting Principles	2
1.1. Choices and Cases	2
1.2. Equal Overcounting	3
2. Bijections	3
2.1. Maps and Inverses	3
2.2. Stars and Bars	4
2.3. Pairing Outcomes	5
3. Gaps and Runs	6
3.1. Separated Selections	6
3.2. Runs in a String	6
4. Lattice Paths	7
4.1. Encoding Steps	7
5. Parking Functions	7
5.1. Counting by Rotation	7
6. Problems	9

1. COUNTING PRINCIPLES

This section reviews the addition and multiplication principles and counting by division.

1.1. Choices and Cases. The **Addition Principle** adds counts of cases that do not overlap and include every outcome. The **Multiplication Principle** multiplies the numbers of options in a sequence of choices when every outcome comes from exactly one sequence and the number of options at each stage is fixed. The options at a stage may depend on earlier choices, but their number may not.

There are $n! = n(n-1) \cdots 1$ orders of n distinct objects, with $0! = 1$. If only k positions are filled, there are $n!/(n-k)!$ orders. In the dictionary-ordered list of the orders of n distinct letters, the words with a given first letter form a block of $(n-1)!$ consecutive words, and the words with given first two letters form a block of $(n-2)!$.

The number of k -element subsets of a set of n distinct objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

A subset is an unordered selection. Each subset appears in $k!$ of the $n!/(n-k)!$ ordered selections, one for each order of its members, which gives the formula. By convention $\binom{n}{k} = 0$ when $k < 0$ or $k > n$, and $\binom{n}{0} = 1$ counts the empty subset.

There are 2^n subsets altogether: each element is either included or omitted. Equivalently, encode the subset by a string of n zeros and ones, with a 1 in exactly the positions of the selected elements. Reading the 1 positions recovers the subset.

Examples.

Problem 1.1.1 (1995 AHSME, Problem #11). How many base 10 four-digit numbers, $N = \underline{a} \underline{b} \underline{c} \underline{d}$, satisfy all three of the following conditions?

- (i) $4,000 \leq N < 6,000$;
- (ii) N is a multiple of 5;
- (iii) $3 \leq b < c \leq 6$.

(A) 10 (B) 18 (C) 24 (D) 36 (E) 48

Strategy. Choose the first digit, the pair of middle digits, and the last digit separately.

Solution. The first digit is 4 or 5, so it has 2 choices. The last digit is 0 or 5, also 2 choices. Choose two different digits from $\{3, 4, 5, 6\}$ for the middle positions. Their required increasing order determines which is b and which is c , so there are $\binom{4}{2} = 6$ choices. Each triple of choices produces one number, and every permitted number is obtained once. Therefore the count is $2 \cdot 6 \cdot 2 = 24$.

Answer: (C) 24. □

Problem 1.1.2 (2013 AMC 12A, Problem #15). Rabbits Peter and Pauline have three offspring—Flopsie, Mopsie, and Cottontail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

(A) 96 (B) 108 (C) 156 (D) 204 (E) 372

Strategy. Split the distributions according to whether the two parents go to the same store.

Solution. If both parents go to the same store, choose that store in 4 ways. Each of the three distinct offspring can go to any of the other 3 stores, giving $4 \cdot 3^3 = 108$ distributions.

If the parents go to different stores, choose Peter's store and then Pauline's in $4 \cdot 3$ ways. Each offspring can go to either of the 2 remaining stores, giving $4 \cdot 3 \cdot 2^3 = 96$ distributions.

Every distribution belongs to exactly one case, so the total is $108 + 96 = 204$.

Answer: (D) 204. □

1.2. Equal Overcounting.

Theorem 1.2.1. *Suppose a collection of descriptions represents a collection of objects, and every object has exactly r descriptions. Then the number of objects is the number of descriptions divided by r .*

Proof. Group the descriptions according to the object they represent. Every group has r members, so the number of descriptions is r times the number of groups. □

For example, the arrangements of n letters in which the distinct letters appear $m_1, \dots, m_t$ times

$$\frac{n!}{m_1! \cdots m_t!}.$$

Temporarily label equal letters to make all letters distinct. Erasing the labels sends exactly $m_1! \cdots m_t!$ labeled arrangements to each arrangement of the original letters.

Likewise, the number of ways to split mk distinct people into k unnamed groups of m people is

$$\frac{(mk)!}{(m!)^k k!}.$$

An ordering of all the people gives consecutive groups of size m . Each grouping arises $(m!)^k$ times from the orders within its groups and $k!$ times from the order of its groups. If the groups have different sizes, divide only by the orders among groups of equal size.

Division requires every object to have the same number of descriptions. For example, the ordered pairs (a, b) with $1 \leq a, b \leq 3$ cannot all be divided by 2 to count unordered pairs: $(1, 2)$ and $(2, 1)$ describe the same pair, but $(1, 1)$ has only one description. Separating equal and unequal entries gives $3 + (9 - 3)/2 = 6$ unordered pairs.

2. BIJECTIONS

2.1. Maps and Inverses.

Definition 2.1.1. A **bijection** between two sets is a rule pairing every object of the first set with exactly one object of the second, so that every object of the second set has exactly one partner. The sets then have the same number of objects.

To prove that a rule is a bijection, give its **inverse**: a rule that reconstructs the original object. Check that both rules produce objects of the required kind and that applying one and then the other returns the starting object.

The subset-to-binary-string rule above is a bijection. Another example takes a k -element subset of $\{1, \dots, n\}$ and lists its members in increasing order. The inverse forgets the order. Thus there are $\binom{n}{k}$ increasing lists of k numbers from $\{1, \dots, n\}$, with no factor of $k!$.

Examples.

Problem 2.1.2 (2005 AMC 12A, Problem #19). A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. For example, after traveling one mile the odometer changed from 000039 to 000050. If the odometer now reads 002005, how many miles has the car actually traveled?

(A) 1404 (B) 1462 (C) 1604 (D) 1605 (E) 1804

Strategy. Relabel the nine allowed odometer digits as the digits of base 9.

Solution. Replace the displayed digits 0, 1, 2, 3, 5, 6, 7, 8, 9 by 0, 1, 2, 3, 4, 5, 6, 7, 8, respectively. This replacement is reversible, position by position. The odometer counts through the resulting strings in base-9 order, starting with 000000. Thus its reading 002005 corresponds to 002004 in base 9.

In base 9, each place is worth nine times the place to its right. The number of miles traveled is therefore

$$2 \cdot 9^3 + 4 = 1458 + 4 = 1462.$$

Answer: (B) 1462. □

Problem 2.1.3 (2006 AMC 12B, Problem #18). An object in the plane moves from one lattice point to another. At each step, the object may move one unit to the right, one unit to the left, one unit up, or one unit down. If the object starts at the origin and takes a ten-step path, how many different points could be the final point?

(A) 120 (B) 121 (C) 221 (D) 230 (E) 231

Strategy. Change coordinates from (x, y) to $(x+y, x-y)$ so that the two coordinates can be chosen independently.

Solution. Each step changes $x+y$ by 1, so after ten steps $x+y$ is even. Each step also changes $|x|+|y|$ by at most 1, so an endpoint satisfies $|x|+|y| \leq 10$. Conversely, every integer point with $x+y$ even and $|x|+|y| \leq 10$ is reachable: take $|x|+|y|$ steps directly to it, and use the remaining even number of steps in back-and-forth pairs.

Set $u = x+y$ and $v = x-y$. Checking the cases where x and y have the same sign or opposite signs gives

$$\max(|u|, |v|) = |x| + |y|.$$

Also $v = u - 2y$ has the same parity as u . Hence an endpoint gives a pair (u, v) of even integers from -10 to 10 . Conversely, for such u and v the inverse map

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}$$

gives integers with $x+y = u$ even and $|x|+|y| = \max(|u|, |v|) \leq 10$, which is a reachable endpoint. There are 11 even integers from -10 to 10 , so the number of endpoints is $11^2 = 121$.

Answer: (B) 121. □

2.2. Stars and Bars.

Theorem 2.2.1. Let k be a positive integer and n a nonnegative integer. The equation $x_1 + x_2 + \cdots + x_k = n$ has $\binom{n+k-1}{k-1}$ solutions in nonnegative integers.

Proof. Write a solution as a row of n stars and $k-1$ bars: x_1 stars, a bar, x_2 stars, a bar, and so on, ending with x_k stars. For example, with $k=3$ and $n=5$, the solution $(2, 0, 3)$ becomes $\star\star || \star\star\star$. Conversely, a row of n stars and $k-1$ bars gives a solution: x_1 is the number of stars before the first bar, x_i is the number between the $(i-1)$ st and i th bars, and x_k is the number after the last bar. A row is determined by choosing which $k-1$ of its $n+k-1$ places hold bars. □

For solutions in positive integers, put $y_i = x_i - 1$. Then $y_1 + \cdots + y_k = n - k$ with every $y_i \geq 0$, so there are $\binom{n-1}{k-1}$ positive solutions when $n \geq k$. Any lower bound $x_i \geq c_i$ is removed the same way.

A nondecreasing sequence $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n$ is determined by how many of its terms equal each of $1, 2, \dots, n$. These n numbers are nonnegative integers with sum k , so there are $\binom{n+k-1}{k}$ such sequences.

Suppose also that every x_i is at most m . Two methods handle this upper bound.

- Replace each x_i by $m - x_i$. This is a bijection from the solutions with sum n to the solutions with sum $km - n$, still with $0 \leq x_i \leq m$. If $km - n \leq m$, the upper bounds hold automatically for the new equation.
- Subtract the solutions in which some variable exceeds m . If $x_1 \geq m + 1$, put $x'_1 = x_1 - (m + 1)$; the new equation has sum $n - m - 1$. When $n \leq 2m + 1$, no two variables can both exceed m , so these cases do not overlap.

For example, $x_1 + x_2 + x_3 = 10$ has $\binom{12}{2} = 66$ solutions in nonnegative integers. For each i , the solutions with $x_i \geq 7$ correspond to nonnegative solutions with sum 3, of which there are $\binom{5}{2} = 10$. No two variables can both be at least 7. Hence $66 - 3 \cdot 10 = 36$ solutions have every $x_i \leq 6$.

Example.

Problem 2.2.2 (2018 AMC 12B, Problem #22). Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many such polynomials satisfy $P(-1) = -9$?

- (A) 110 (B) 143 (C) 165 (D) 220 (E) 286

Strategy. Write the condition as an equation in the four coefficients, then replace the coefficients with a minus sign by 9 minus themselves.

Solution. Write $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are digits. Different choices of (a, b, c, d) give different polynomials, and $P(-1) = -a + b - c + d$. Put $a' = 9 - a$ and $c' = 9 - c$. These are again digits, and the replacement is reversible. Then

$$P(-1) = -(9 - a') + b - (9 - c') + d = a' + b + c' + d - 18,$$

so the condition $P(-1) = -9$ becomes

$$a' + b + c' + d = 9.$$

Four nonnegative integers with sum 9 are automatically at most 9, so the upper bounds hold. By stars and bars the number of solutions is $\binom{12}{3} = 220$.

Answer: (D) 220. □

2.3. Pairing Outcomes. A bijection from a set to itself can show that two parts of the set have the same size. Suppose a rule, applied twice, returns the original object. Then each object is either unchanged by the rule or paired with a different object. If the rule sends every object with property A to an object with property B , and every object with property B to an object with property A , then the two properties hold for equally many objects. Objects with neither property are counted separately.

Example.

Problem 2.3.1 (2008 AMC 12A, Problem #21). A permutation $(a_1, a_2, a_3, a_4, a_5)$ of $(1, 2, 3, 4, 5)$ is *heavy-tailed* if $a_1 + a_2 < a_4 + a_5$. What is the number of heavy-tailed permutations?

- (A) 36 (B) 40 (C) 44 (D) 48 (E) 52

Strategy. Reverse each permutation to pair the smaller-tail and larger-tail cases. Count equal sums separately.

Solution. Reversing the five entries interchanges the sums $a_1 + a_2$ and $a_4 + a_5$. Reversal is its own inverse, so the number of permutations with $a_1 + a_2 < a_4 + a_5$ equals the number with $a_1 + a_2 > a_4 + a_5$.

The two sums can be equal only when $15 - a_3$ is even, so a_3 is 1, 3, or 5. For $a_3 = 1$, the remaining numbers split as $\{2, 5\}$ and $\{3, 4\}$. For $a_3 = 3$, they split as $\{1, 5\}$ and $\{2, 4\}$. For

$a_3 = 5$, they split as $\{1, 4\}$ and $\{2, 3\}$. In each case this is the only equal-sum split, because each pair has sum $(15 - a_3)/2$ and the smallest remaining number determines its partner.

Each split gives 2 choices for which pair comes first and 2 orders within each pair, or 8 permutations. There are $3 \cdot 8 = 24$ equal-sum permutations. The other $5! - 24 = 96$ divide into two classes of equal size, so $96/2 = 48$ are heavy-tailed.

Answer: (D) 48. □

3. GAPS AND RUNS

3.1. Separated Selections. Suppose $1 \leq a_1 < a_2 < \cdots < a_k \leq n$, and successive selected numbers must differ by at least d , where $d \geq 1$. Define

$$b_i = a_i - (d - 1)(i - 1), \quad 1 \leq i \leq k.$$

Then $1 \leq b_1 < b_2 < \cdots < b_k \leq n - (d - 1)(k - 1)$. Conversely, any increasing selection in this shorter range gives a permitted selection by

$$a_i = b_i + (d - 1)(i - 1).$$

These operations undo each other. Thus the number of k -element selections is

$$\binom{n - (d - 1)(k - 1)}{k}.$$

If $n - (d - 1)(k - 1) < k$, there are no such selections. For $k = 0$ the formula gives 1, the empty selection. For selections with no two adjacent numbers $d = 2$; for selections with at least two unused numbers between any two selected ones $d = 3$.

Example.

Problem 3.1.1. How many 4-element subsets of $\{1, 2, \dots, 12\}$ contain no consecutive integers?

Strategy. Subtract 0, 1, 2, 3 from the selected numbers in increasing order.

Solution. Write a valid subset as $a_1 < a_2 < a_3 < a_4$ and put $b_i = a_i - (i - 1)$. The b_i form an arbitrary 4-element subset of $\{1, \dots, 9\}$. Conversely, adding 0, 1, 2, 3 to its increasing entries gives a valid original subset. For example, $\{1, 4, 5, 9\}$ maps back to $\{1, 5, 7, 12\}$. The bijection gives $\binom{9}{4} = 126$ subsets.

Answer: 126. □

3.2. Runs in a String. A **run** is a maximal consecutive block of equal symbols. For example, 1110011 has three runs, of lengths 3, 2, 2. A binary string is determined by its first symbol and its run lengths, because successive runs alternate symbols.

Alternatively, choose the positions where the symbol changes. A binary string of length n has $n - 1$ gaps between neighboring symbols. Given its first symbol and a choice of r gaps, the string that changes symbol at exactly those gaps is determined. Conversely, a string determines its first symbol and the gaps where adjacent symbols differ. Therefore there are $\binom{n-1}{r}$ binary strings of length n with a given first symbol and exactly r changes.

Restrictions on run lengths can also be counted directly. Consider strings that start and end with 0, have no two adjacent zeros, and have no three adjacent ones. Each run of ones has length 1 or 2, and consecutive runs of ones are separated by a single zero. If there are m runs of ones, there are $m + 1$ zeros. If t of the runs have length 2, the string has length $(m + 1) + m + t = 2m + t + 1$, and there are $\binom{m}{t}$ ways to choose which runs have length 2. Summing $\binom{m}{t}$ over all m and t with $2m + t + 1$ equal to the required length gives the count.

Example.

Problem 3.2.1. How many strings of length 10 formed from the digits 0 and 1 have exactly 4 runs?

Strategy. Choose the first digit and the three gaps where the digit changes.

Solution. A string with 4 runs changes digit at exactly 3 of its 9 gaps. Choose the first digit in 2 ways and the 3 gaps in $\binom{9}{3} = 84$ ways. These choices determine the string, and every string with 4 runs arises from exactly one choice. Hence there are $2 \cdot 84 = 168$ strings.

Answer: 168. □

4. LATTICE PATHS

4.1. Encoding Steps. A **lattice point** has integer coordinates. A path from $(0, 0)$ to (a, b) taking only unit steps right or up has a right steps and b up steps. Encode these by a word with a letters R and b letters U. Reading the word traces the path, and reading a path records its word, so the correspondence is reversible. Hence the number of paths is

$$\frac{(a+b)!}{a!b!} = \binom{a+b}{a}.$$

A path through a specified point splits uniquely into a path to the point and a path from the point. Count each part and multiply. To count paths that avoid a region, split them according to where they first cross a chosen line, so that every path belongs to exactly one case.

Example.

Problem 4.1.1. How many paths from $(0, 0)$ to $(5, 4)$ take only unit steps right or up and pass through $(2, 1)$?

Strategy. Encode the two parts of the path separately.

Solution. The first part has two right steps and one up step, giving $\binom{3}{2} = 3$ words. The second part has three right steps and three up steps, giving $\binom{6}{3} = 20$ words. Concatenating the two words gives exactly one required path. Splitting a path when it reaches $(2, 1)$ recovers those words, so there are $3 \cdot 20 = 60$ paths.

Answer: 60. □

5. PARKING FUNCTIONS

This section counts parking functions by placing the spaces on a circle and dividing by the number of rotations.

5.1. Counting by Rotation. There are n cars, arriving in a fixed order, and n spaces in a row labeled $0, 1, \dots, n-1$. Each car has a preferred space. It parks in the first empty space at or after its preference; if there is none, it fails to park. A list of the n preferences is a **parking function** when all cars park.

Lemma 5.1.1. Sort the preferences as $b_1 \leq b_2 \leq \dots \leq b_n$. The list is a parking function exactly when $b_i \leq i-1$ for every i .

Proof. First, whether all cars park does not depend on the order of arrival. To see this, interchange two neighboring cars in the arrival order. Look at their first available spaces before either parks. Filling an empty space other than a car's first available space does not change that car's first available space. So if the two first available spaces differ, the same two spaces are filled in either

order. If they coincide, the two cars fill that space and the next available space, in either order; if there is no next space, one fails in either order. If a car has no available space even before the other parks, it fails in either order. In every case the later cars find the same empty spaces. Successive interchanges allow any ordering of the cars.

Now let the cars arrive in sorted order. If $b_i \leq i - 1$ for all i , induction shows that the i th car parks in space $i - 1$, because earlier cars have filled exactly spaces 0 through $i - 2$. Conversely, if $b_i \geq i$ for some i , at least $n - i + 1$ cars have preferences at least i , but only $n - i$ spaces have labels at least i . These cars cannot all park. $\square$

Theorem 5.1.2. *The number of parking functions for n cars is $(n + 1)^{n-1}$.*

Proof. Place $n + 1$ labeled spaces $0, 1, \dots, n$ on a circle, and let each car choose any one of them as its preference. Cars move forward around the circle until an empty space is found. All n cars park, leaving exactly one space empty. There are $(n + 1)^n$ preference lists.

Add the same number modulo $n + 1$ to every preference. This rotates the whole parking process by that number and therefore moves the empty space by the same number. The $n + 1$ rotations of a list are distinct, since even its first preference changes under a nonzero rotation. Their empty spaces are all different, so exactly one of the $n + 1$ rotations leaves space n empty.

In such a list no car prefers or passes space n : a car reaching that empty space would park there. Removing space n therefore gives a parking function for the row $0, 1, \dots, n - 1$. Conversely, a parking function leaves space n empty when the extra space is restored. These two operations are inverse, so the number of parking functions is $(n + 1)^n / (n + 1) = (n + 1)^{n-1}$. $\square$

For example, with 3 cars there are $4^2 = 16$ parking functions. Their sorted preferences can only be 000, 001, 002, 011, or 012, which have respectively 1, 3, 3, 3, and 6 orders. This gives the same total.

6. PROBLEMS

Beginner.

- (2018 AMC 12B, Problem #5) How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?
(A) 128 (B) 192 (C) 224 (D) 240 (E) 256
- (2006 AMC 12B, Problem #9) How many even three-digit integers have the property that their digits, read left to right, are in strictly increasing order?
(A) 21 (B) 34 (C) 51 (D) 72 (E) 150
- (1986 AHSME, Problem #10) The 120 permutations of AHSME are arranged in dictionary order, as if each were an ordinary five-letter word. The last letter of the 86th word in this list is
(A) A (B) H (C) S (D) M (E) E
- (1990 AHSME, Problem #17) How many of the numbers 100, 101, $\dots$, 999 have three different digits in increasing order or in decreasing order?
(A) 120 (B) 168 (C) 204 (D) 216 (E) 240
- (2005 AMC 12A, Problem #11) How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?
(A) 41 (B) 42 (C) 43 (D) 44 (E) 45
- (1998 AIME, Problem #7) Let n be the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $x_1 + x_2 + x_3 + x_4 = 98$. Find $\frac{n}{100}$.

Intermediate.

- (2023 AMC 12B, Problem #8) How many nonempty subsets B of $\{0, 1, 2, 3, \dots, 12\}$ have the property that the number of elements in B is equal to the least element of B ? For example, $B = \{4, 6, 8, 11\}$ satisfies the condition.
(A) 256 (B) 136 (C) 108 (D) 144 (E) 156
- (2018 AMC 12A, Problem #13) How many nonnegative integers can be written in the form

$$\begin{aligned} & a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 \\ & + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 \\ & + a_1 \cdot 3^1 + a_0 \cdot 3^0, \end{aligned}$$

where $a_i \in \{-1, 0, 1\}$ for $0 \leq i \leq 7$?

- (A) 512 (B) 729 (C) 1094 (D) 3281 (E) 59,048
- (2011 AMC 12A, Problem #16) Each vertex of convex pentagon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?
(A) 2520 (B) 2880 (C) 3120 (D) 3250 (E) 3750

10. (2010 AMC 12A, Problem #18) A 16-step path is to go from $(-4, -4)$ to $(4, 4)$ with each step increasing either the x -coordinate or the y -coordinate by 1. How many such paths stay outside or on the boundary of the square $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ at each step?
 (A) 92 (B) 144 (C) 1568 (D) 1698 (E) 12,800
11. (2018 AMC 12A, Problem #15) A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?
 (A) 510 (B) 1022 (C) 8190 (D) 8192 (E) 65,534
12. (1970 AHSME, Problem #31) If a number is selected at random from the set of all five-digit numbers in which the sum of the digits is equal to 43, what is the probability that this number will be divisible by 11?
 (A) $\frac{2}{5}$ (B) $\frac{1}{5}$ (C) $\frac{1}{6}$ (D) $\frac{1}{11}$ (E) $\frac{1}{15}$

Advanced.

13. (1997 AHSME, Problem #30) For positive integers n , denote by $D(n)$ the number of pairs of different adjacent digits in the binary (base two) representation of n . For example, $D(3) = D(11_2) = 0$, $D(21) = D(10101_2) = 4$, and $D(97) = D(1100001_2) = 2$. For how many positive integers n less than or equal to 97 does $D(n) = 2$?
 (A) 16 (B) 20 (C) 26 (D) 30 (E) 35
14. (2010 AMC 12B, Problem #17) The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?
 (A) 18 (B) 24 (C) 36 (D) 42 (E) 60
15. (2007 AMC 12A, Problem #25) Call a set of integers *spacy* if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \dots, 12\}$, including the empty set, are spacy?
 (A) 121 (B) 123 (C) 125 (D) 127 (E) 129
16. (2022 AMC 12A, Problem #24) How many strings of length 5 formed from the digits 0, 1, 2, 3, 4 are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j ? (For example, 02214 satisfies the condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)
 (A) 500 (B) 625 (C) 1089 (D) 1199 (E) 1296