

ADVANCED COUNTING AND BIJECTIONS FOR AMC 12: SOLUTIONS

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1. SOLUTIONS

Beginner.

1. (2018 AMC 12B, Problem #5) How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

(A) 128 (B) 192 (C) 224 (D) 240 (E) 256

Strategy. Choose a nonempty subset of the primes, then any subset of the remaining numbers.

Solution. The primes in the set are 2, 3, 5, 7; the other elements are 4, 6, 8, 9. The four primes have $2^4 - 1 = 15$ nonempty subsets. The four other elements have $2^4 = 16$ subsets, including the empty set. Taking the union of one choice of each type gives every requested subset exactly once. Thus there are $15 \cdot 16 = 240$ subsets.

Answer: (D) 240. □

2. (2006 AMC 12B, Problem #9) How many even three-digit integers have the property that their digits, read left to right, are in strictly increasing order?

(A) 21 (B) 34 (C) 51 (D) 72 (E) 150

Strategy. Fix the final even digit and choose the two smaller digits.

Solution. The last digit can be 2, 4, 6, or 8; it cannot be 0, since it is larger than the other two digits. The earlier two digits must be positive and smaller than the last digit, since a leading zero is not allowed. For last digit d , choose two digits from $1, \dots, d-1$; their increasing order is then forced. The count is

$$\binom{1}{2} + \binom{3}{2} + \binom{5}{2} + \binom{7}{2} = 0 + 3 + 10 + 21 = 34.$$

Answer: (B) 34. □

3. (1986 AHSME, Problem #10) The 120 permutations of AHSME are arranged in dictionary order, as if each were an ordinary five-letter word. The last letter of the 86th word in this list is

(A) A (B) H (C) S (D) M (E) E

Strategy. Locate the word by counting blocks with common initial letters.

Solution. The alphabetical order is A, E, H, M, S. There are $4! = 24$ words beginning with each letter. Positions 73 through 96 begin with M, so word 86 is the 14th in this block. The second-letter blocks, each of size $3! = 6$, are A, E, H, S. The 14th word has second letter H and is the 2nd word beginning MH. The remaining letters are A, E, S; their first two orders are AES and ASE. Hence the word is MHASE, whose last letter is E.

Answer: (E) E. □

4. (1990 AHSME, Problem #17) How many of the numbers 100, 101, $\dots$, 999 have three different digits in increasing order or in decreasing order?

(A) 120 (B) 168 (C) 204 (D) 216 (E) 240

Strategy. Encode each number by its set of digits and its direction of order.

Solution. An increasing three-digit number cannot contain 0, since that digit would have to come first. Choosing three digits from 1 through 9 gives exactly one increasing number, so there are $\binom{9}{3} = 84$.

A decreasing number can contain 0. Each three-element subset of $\{0, 1, \dots, 9\}$ gives exactly one decreasing three-digit number, because its largest digit is positive. There are $\binom{10}{3} = 120$ such numbers. No number with three different digits is both increasing and decreasing. The total is $84 + 120 = 204$.

Answer: (C) 204. □

5. (2005 AMC 12A, Problem #11) How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

(A) 41 (B) 42 (C) 43 (D) 44 (E) 45

Strategy. Choose the first and last digits; the middle digit is then determined.

Solution. Write the number as $100a + 10b + c$, where $1 \leq a \leq 9$ and $0 \leq c \leq 9$. The condition is $b = (a + c)/2$. This is an integer exactly when a and c have the same parity, and then it is a digit, since it lies between 0 and 9. For each of the 9 choices of a , exactly 5 choices of c have the required parity. Each pair gives one number, so the count is $9 \cdot 5 = 45$.

Answer: (E) 45. □

6. (1998 AIME, Problem #7) Let n be the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $x_1 + x_2 + x_3 + x_4 = 98$. Find $\frac{n}{100}$.

Strategy. Write each odd number as $2y + 1$ and count the resulting nonnegative solutions by stars and bars.

Solution. Write $x_i = 2y_i + 1$. The map $y \mapsto 2y + 1$ is a bijection from the nonnegative integers to the positive odd integers, so the quadruples correspond to nonnegative solutions of

$$2(y_1 + y_2 + y_3 + y_4) + 4 = 98, \quad \text{that is,} \quad y_1 + y_2 + y_3 + y_4 = 47.$$

By stars and bars this equation has $\binom{50}{3} = 19600$ solutions. Hence $n = 19600$ and $\frac{n}{100} = 196$.

Answer: 196. □

Intermediate.

7. (2023 AMC 12B, Problem #8) How many nonempty subsets B of $\{0, 1, 2, 3, \dots, 12\}$ have the property that the number of elements in B is equal to the least element of B ? For example, $B = \{4, 6, 8, 11\}$ satisfies the condition.

(A) 256 (B) 136 (C) 108 (D) 144 (E) 156

Strategy. Fix the least element, which also fixes the size of the subset.

Solution. Let the least element be k . The subset is nonempty and has size k , so $k \geq 1$. After including k , choose $k - 1$ further elements from $k + 1, \dots, 12$, a set of size $12 - k$. This gives $\binom{12-k}{k-1}$ subsets. This is nonzero only when $k - 1 \leq 12 - k$, so $1 \leq k \leq 6$. The total is

$$\binom{11}{0} + \binom{10}{1} + \binom{9}{2} + \binom{8}{3} + \binom{7}{4} + \binom{6}{5} = 1 + 10 + 36 + 56 + 35 + 6 = 144.$$

Answer: (D) 144. □

8. (2018 AMC 12A, Problem #13) How many nonnegative integers can be written in the form

$$\begin{aligned} & a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 \\ & + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 \\ & + a_1 \cdot 3^1 + a_0 \cdot 3^0, \end{aligned}$$

where $a_i \in \{-1, 0, 1\}$ for $0 \leq i \leq 7$?

- (A) 512 (B) 729 (C) 1094 (D) 3281 (E) 59,048

Strategy. Add 1 to every coefficient to obtain ordinary base-3 digits.

Solution. Put $b_i = a_i + 1$. Each b_i is 0, 1, or 2, so the eight coefficients $b_7, \dots, b_0$ are the base-3 representation, allowing leading zeros, of an integer M from 0 to $3^8 - 1 = 6560$. Every such M arises, and from exactly one choice of coefficients, because base-3 representations exist and are unique.

The original integer is

$$M - (1 + 3 + \dots + 3^7) = M - \frac{3^8 - 1}{2} = M - 3280.$$

So the integers that can be written in the given form are exactly -3280 through 3280 . Of these, 3281 are nonnegative, including zero.

Answer: (D) 3281. □

9. (2011 AMC 12A, Problem #16) Each vertex of convex pentagon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?

- (A) 2520 (B) 2880 (C) 3120 (D) 3250 (E) 3750

Strategy. Read the pentagon diagonals as a five-vertex cycle and split according to repeated colors.

Solution. The diagonals, in order, are AC, CE, EB, BD, DA , so the vertices A, C, E, B, D form a cycle in which neighbors must have different colors. Choose distinct colors for A and C in $6 \cdot 5 = 30$ ways. Fix these colors and count the remaining choices.

If E has the same color as A , then B has 5 choices and D has 4, since D must differ from both B and A . This gives 20.

Otherwise, E has 4 choices, distinct from both A and C . If B has the color of A , then D has 5 choices. If B differs from both E and A , it has 4 choices, and D has 4. This gives $4(5 + 4 \cdot 4) = 84$.

The cases are disjoint and include all colorings. The total is $30(20 + 84) = 3120$.

Answer: (C) 3120. □

10. (2010 AMC 12A, Problem #18) A 16-step path is to go from $(-4, -4)$ to $(4, 4)$ with each step increasing either the x -coordinate or the y -coordinate by 1. How many such paths stay outside or on the boundary of the square $-2 \leq x \leq 2, -2 \leq y \leq 2$ at each step?

- (A) 92 (B) 144 (C) 1568 (D) 1698 (E) 12,800

Strategy. Split the paths according to where they first step from the line $y = -2$ to the line $y = -1$.

Solution. The forbidden points are the lattice points strictly inside the square, those with $-2 < x < 2$ and $-2 < y < 2$. Every path rises from $y = -2$ to $y = -1$ exactly once, at some value of x . The point $(x, -1)$ it reaches is outside the square only if $x \geq 2$ or $x \leq -2$.

Suppose $x \geq 2$. Before this step every point has $y \leq -2$, and after it every point has $x \geq 2$, so the whole path avoids the forbidden points. Thus every path in this case is allowed.

Suppose instead $x \leq -2$. The path then steps from $x = -2$ to $x = -1$ later, at a height $y \geq -1$, and the point it reaches is outside the square only if $y \geq 2$. Reflection across the line $y = x$ exchanges these two cases, so it is a bijection between them. Count the first case and double.

For a fixed x in the first case, the initial path from $(-4, -4)$ to $(x, -2)$ has $x + 4$ right steps and two up steps, giving $\binom{x+6}{2}$ choices. After the up step to $(x, -1)$, the path to $(4, 4)$ has $4 - x$ right steps and five up steps, giving $\binom{9-x}{5}$ choices. With $x = 2, 3, 4$, the count is

$$2 \left(\binom{8}{2} \binom{7}{5} + \binom{9}{2} \binom{6}{5} + \binom{10}{2} \binom{5}{5} \right) = 2(588 + 216 + 45) = 1698.$$

Answer: (D) 1698. □

11. (2018 AMC 12A, Problem #15) A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?

(A) 510 (B) 1022 (C) 8190 (D) 8192 (E) 65,534

Strategy. Group the squares that rotations and reflections force to have the same color.

Solution. Give the square centers coordinates (x, y) , where each coordinate is an integer from -3 through 3 . The required rotations and reflections change signs and interchange coordinates. Hence two squares are forced to have the same color exactly when the unordered pairs $\{|x|, |y|\}$ agree.

A group is specified uniquely by $0 \leq a \leq b \leq 3$, where a and b are the smaller and larger absolute coordinates. There are $1 + 2 + 3 + 4 = 10$ groups. Any assignment of black or white to these groups gives one symmetric code, and reading their colors recovers that assignment. There are $2^{10} = 1024$ assignments. Exactly two use only one color, so the required number is $1024 - 2 = 1022$.

Answer: (B) 1022. □

12. (1970 AHSME, Problem #31) If a number is selected at random from the set of all five-digit numbers in which the sum of the digits is equal to 43, what is the probability that this number will be divisible by 11?

(A) $\frac{2}{5}$ (B) $\frac{1}{5}$ (C) $\frac{1}{6}$ (D) $\frac{1}{11}$ (E) $\frac{1}{15}$

Strategy. Replace each digit d by $9 - d$, so that the digit sum 43 becomes a small sum.

Solution. Write the number with digits $d_1 d_2 d_3 d_4 d_5$ and put $e_i = 9 - d_i$. The e_i are non-negative integers with

$$e_1 + e_2 + e_3 + e_4 + e_5 = 45 - 43 = 2.$$

Conversely, any nonnegative solution of this equation has every $e_i \leq 2$, so every $d_i = 9 - e_i$ is a digit and $d_1 \geq 7$ is nonzero. By stars and bars there are $\binom{6}{4} = 15$ such numbers, and each is equally likely.

A number is divisible by 11 exactly when its alternating digit sum $d_1 - d_2 + d_3 - d_4 + d_5$ is divisible by 11. This sum equals $9 - (e_1 - e_2 + e_3 - e_4 + e_5)$. Since the e_i are nonnegative with sum 2, the expression in parentheses lies between -2 and 2 , so the alternating digit sum lies between 7 and 11 . It is divisible by 11 only when it equals 11, that is, when $e_2 + e_4 = 2$ and $e_1 = e_3 = e_5 = 0$. The pairs $(e_2, e_4) = (2, 0), (1, 1), (0, 2)$ give the 3 numbers 97999, 98989, 99979. The probability is $\frac{3}{15} = \frac{1}{5}$.

Answer: (B) $\frac{1}{5}$. □

Advanced.

13. (1997 AHSME, Problem #30) For positive integers n , denote by $D(n)$ the number of pairs of different adjacent digits in the binary (base two) representation of n . For example, $D(3) = D(11_2) = 0$, $D(21) = D(10101_2) = 4$, and $D(97) = D(1100001_2) = 2$. For how many positive integers n less than or equal to 97 does $D(n) = 2$?

(A) 16 (B) 20 (C) 26 (D) 30 (E) 35

Strategy. Encode a binary number by its first digit and the two positions where the digit changes.

Solution. A positive integer's binary representation starts with 1. A representation of length ℓ with exactly two changes is determined by choosing two of its $\ell - 1$ gaps. For lengths 1 through 6, the count is

$$\binom{0}{2} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = 0 + 0 + 1 + 3 + 6 + 10 = 20.$$

The remaining numbers have seven digits and are at most $97 = 1100001_2$. With exactly two changes, they have the form $1^a 0^b 1^c$, meaning a ones, then b zeros, then c ones, where a, b, c are positive and sum to 7.

If $a = 1$, the first two digits are 10, so the number is smaller than 1100001_2 . There are 5 choices for (b, c) , namely $(1, 5)$ through $(5, 1)$. If $a = 2$, then $b + c = 5$. The choice $b = 4, c = 1$ gives $1100001_2 = 97$ itself. If $b \leq 3$, the number agrees with 1100001_2 in its first $b + 2$ digits and has a 1 in the next place, where 1100001_2 has a 0, so it exceeds 97. If $a \geq 3$, the number starts 111 and exceeds 97. Thus there are $20 + 5 + 1 = 26$ numbers.

Answer: (C) 26. □

14. (2010 AMC 12B, Problem #17) The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

(A) 18 (B) 24 (C) 36 (D) 42 (E) 60

Strategy. Use transposition to fix the position of 2, then split according to the position of 3.

Solution. The top-left entry must be 1, and the bottom-right entry must be 9. The entry 2 is smaller than every other entry except 1, so every entry above it or to its left is 1. Hence 2 is immediately right of or immediately below 1. Transposing an array interchanges these two possibilities, so assume that 2 is at the top-middle position and double the count. The same reasoning shows that 3 is either top-right or middle-left.

If 3 is top-right, the first row is 1, 2, 3. The entry 4 is the smallest one left, so it is the middle-left entry, since every other open position has an open position above it or to its left. Write the array as

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & x & y \\ z & w & 9 \end{pmatrix}.$$

The entries x, y, z, w are 5, 6, 7, 8 with $x < y$, $z < w$, and $x < w$. Choose $\{x, y\}$ in $\binom{4}{2} = 6$ ways; the orders in both rows are then forced. The condition $x < w$ fails only for $\{x, y\} = \{7, 8\}$, since then $w \leq 6 < 7 = x$. This case gives 5 arrays.

If 3 is middle-left, write the array as

$$\begin{pmatrix} 1 & 2 & a \\ 3 & b & c \\ d & e & 9 \end{pmatrix}.$$

The entries a, b, c, d, e are 4, 5, 6, 7, 8 in some order, with $a < c$, $d < e$, $b < c$, and $b < e$. If $b = 4$, choose the two entries for a, c in $\binom{4}{2} = 6$ ways; their order and the order of d, e are forced. If $b = 5$, the entry 4 is either a or d , since c and e exceed b . If $a = 4$, then c is any of 6, 7, 8 and the order of d, e is forced, giving 3; likewise $d = 4$ gives 3, for 6 total. If $b = 6$, then c, e are 7, 8 in 2 orders and a, d are 4, 5 in 2 orders, giving 4. Larger b would leave fewer than two larger entries for c, e . This case gives $6 + 6 + 4 = 16$.

Restoring the two possible positions of 2, the total is $2(5 + 16) = 42$.

Answer: (D) 42. □

15. (2007 AMC 12A, Problem #25) Call a set of integers *spacy* if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \dots, 12\}$, including the empty set, are spacy?

(A) 121 (B) 123 (C) 125 (D) 127 (E) 129

Strategy. Fix the number of selected integers and use the separated-selection bijection with $d = 3$.

Solution. A set is spacy exactly when any two of its elements differ by at least 3. For a nonempty spacy set, list its k elements as $a_1 < \dots < a_k$; the condition is $a_{i+1} - a_i \geq 3$. Subtracting $2(i-1)$ from a_i gives an arbitrary k -element subset of $\{1, \dots, 14 - 2k\}$, and adding these amounts back gives the inverse.

Such subsets exist only when $k \leq 14 - 2k$, that is, $k \leq 4$. Including the empty set, the total is

$$1 + \binom{12}{1} + \binom{10}{2} + \binom{8}{3} + \binom{6}{4} = 1 + 12 + 45 + 56 + 15 = 129.$$

Answer: (E) 129. □

16. (2022 AMC 12A, Problem #24) How many strings of length 5 formed from the digits 0, 1, 2, 3, 4 are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j ? (For example, 02214 satisfies the condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

(A) 500 (B) 625 (C) 1089 (D) 1199 (E) 1296

Strategy. Read the five digits as parking preferences and use the count of parking functions.

Solution. Sort the five digits as $b_1 \leq b_2 \leq \dots \leq b_5$. For each $j = 1, 2, 3, 4$, at least j digits are less than j exactly when $b_j \leq j - 1$. The remaining condition $b_5 \leq 4$ holds because every digit belongs to $\{0, 1, 2, 3, 4\}$.

Read the string, in its original order, as the preferences of 5 cars for spaces 0, 1, 2, 3, 4. By the lemma on sorted preferences, the strings in the problem are exactly the parking functions for 5 cars. By the theorem on parking functions, their number is

$$(5 + 1)^{5-1} = 6^4 = 1296.$$

Answer: (E) 1296.

□

2. ANSWER KEY

Problem	Letter	Value
1	D	240
2	B	34
3	E	E
4	C	204
5	E	45
6		196
7	D	144
8	D	3281
9	C	3120
10	D	1698
11	B	1022
12	B	$\frac{1}{5}$
13	C	26
14	D	42
15	E	129
16	E	1296