SYMMETRIC SUMS, NEWTON'S SUMS, AND ROOT TRANSFORMATIONS FOR AMC 12

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Contents

1. Symmetric Sums 2
1.1. Elementary Symmetric Sums 2
1.2. Symmetric Expressions in Three Roots 4
2. Newton's Sums 5
2.1. Power Sums 5
2.2. Sums of Values of a Polynomial 7
3. Root Transformations 7
3.1. Shifted and Scaled Roots 8
3.2. Reciprocal Roots 9
3.3. Squared Roots 10
3.4. Removing the Second-Highest Term 11
4. Problems 12

1. Symmetric Sums

This section writes symmetric expressions in the roots of a polynomial through its coefficients, without finding the roots.

1.1. Elementary Symmetric Sums.

Definition 1.1.1. A polynomial expression in the numbers $r_1, r_2, \ldots, r_n$ is **symmetric** if it does not change when the numbers are rearranged in any order. The **elementary symmetric sums** of $r_1, \ldots, r_n$ are

$$e_1 = r_1 + r_2 + \cdots + r_n, \qquad e_2 = \sum_{i<j} r_i r_j, \qquad \ldots, \qquad e_n = r_1 r_2 \cdots r_n;$$

in general, e_k is the sum of the $\binom{n}{k}$ products of the numbers taken k at a time. The **power sums** are $p_k = r_1^k + r_2^k + \cdots + r_n^k$ for $k \geq 1$.

For two numbers x and y, $e_1 = x + y$ and $e_2 = xy$. For three numbers r, s, t,

$$e_1 = r + s + t, \qquad e_2 = rs + st + tr, \qquad e_3 = rst.$$

The expressions $r^2 + s^2 + t^2$ and $(r + s)(s + t)(t + r)$ are symmetric in r, s, t. The expression $r^2 s + s^2 t + t^2 r$ is not: exchanging r and s turns it into $s^2 r + r^2 t + t^2 s$.

The elementary symmetric sums are the coefficients of a product of linear factors. Each term in the expansion of $(1 + r_1 x)(1 + r_2 x) \cdots (1 + r_n x)$ takes 1 or $r_i x$ from each factor, and taking $r_i x$ from exactly k factors gives a product of k of the numbers times x^k. Each such product arises once, so

$$(1 + r_1 x)(1 + r_2 x) \cdots (1 + r_n x) = 1 + e_1 x + e_2 x^2 + \cdots + e_n x^n.$$

In this notation, Vieta's formulas say that if $P(x) = a_n x^n + \cdots + a_1 x + a_0 = a_n (x - r_1) \cdots (x - r_n)$, then $e_k = (-1)^k a_{n-k}/a_n$.

Remark 1.1.2 (Non-real Roots)**.** A polynomial of degree n need not have n real roots; $x^2 + 1$ has none. When non-real (complex) numbers are allowed, every polynomial of degree $n \geq 1$ factors as $a_n (x - r_1)(x - r_2) \cdots (x - r_n)$. This is the Fundamental Theorem of Algebra, used here without proof. In this handout the roots of a polynomial of degree n are always this full list of n numbers, real or not, each listed as many times as its multiplicity. Vieta's formulas hold for this list, and every computation below uses only the coefficients.

Theorem 1.1.3 (Symmetric Polynomials)**.** *Every symmetric polynomial in $r_1, \ldots, r_n$ can be written as a polynomial in $e_1, \ldots, e_n$.*

The proof is omitted. Together with Vieta's formulas, the theorem says that every symmetric polynomial expression in the roots of a polynomial can be computed from the coefficients alone. Each formula used in this handout is proved directly, without this theorem.

For any n numbers, squaring e_1 gives each square r_i^2 once and each product $r_i r_j$ with $i < j$ twice, so

$$p_2 = e_1^2 - 2e_2, \qquad \text{or} \qquad e_2 = \frac{e_1^2 - p_2}{2}.$$

For two numbers x and y, this and the expansion $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$ give

$$x^2 + y^2 = e_1^2 - 2e_2, \qquad x^3 + y^3 = e_1^3 - 3e_1 e_2, \qquad (x - y)^2 = e_1^2 - 4e_2.$$

Conversely, two numbers with sum e_1 and product e_2 are the roots of $t^2 - e_1 t + e_2$, since $(t - x)(t - y) = t^2 - (x + y)t + xy$.

Examples.

Problem 1.1.4 (2004 AIME I, Problem #7)**.** Let C be the coefficient of x^2 in the expansion of the product

$$(1-x)(1+2x)(1-3x)\cdots(1+14x)(1-15x).$$

Find $|C|$.

Strategy. The product has the form $(1+a_1x)(1+a_2x)\cdots(1+a_{15}x)$ with $a_k=(-1)^k k$, so C is the elementary symmetric sum e_2 of $a_1,\ldots,a_{15}$. Find e_1 and the sum of the squares first.

Solution. Write the product as $(1+a_1x)(1+a_2x)\cdots(1+a_{15}x)$, where $a_k=(-1)^k k$; that is, $a_1=-1$, $a_2=2$, $a_3=-3$, $\ldots$, $a_{15}=-15$. The coefficient of x^2 is the second elementary symmetric sum of $a_1,\ldots,a_{15}$, so $C=e_2$. Pairing consecutive terms,

$$e_1=(-1+2)+(-3+4)+\cdots+(-13+14)-15=7-15=-8,$$

and since $a_k^2=k^2$,

$$p_2=1^2+2^2+\cdots+15^2=\frac{15\cdot16\cdot31}{6}=1240.$$

Therefore

$$C=e_2=\frac{e_1^2-p_2}{2}=\frac{64-1240}{2}=-588,$$

and $|C|=588$.

Answer: 588. □

Problem 1.1.5 (1983 AIME, Problem #5)**.** Suppose that the sum of the squares of two complex numbers x and y is 7 and the sum of the cubes is 10. What is the largest real value that $x+y$ can have?

Strategy. Write both conditions with $e_1=x+y$ and $e_2=xy$, and eliminate e_2.

Solution. Let $e_1=x+y$ and $e_2=xy$. The two conditions are

$$e_1^2-2e_2=7 \qquad \text{and} \qquad e_1^3-3e_1e_2=10.$$

The first gives $e_2=(e_1^2-7)/2$. Substituting into the second,

$$e_1^3-\tfrac{3}{2}e_1(e_1^2-7)=10, \qquad \text{that is,} \qquad e_1^3-21e_1+20=0.$$

Since $e_1=1$ is a root, $e_1^3-21e_1+20=(e_1-1)(e_1^2+e_1-20)=(e_1-1)(e_1-4)(e_1+5)$, so $x+y$ is 1, 4 or -5. The value 4 occurs: let x and y be the two roots of $t^2-4t+\frac{9}{2}$, which are non-real (the discriminant is $16-18<0$), as the problem allows. Then $e_1=4$ and $e_2=\frac{9}{2}$, so $x^2+y^2=16-9=7$ and $x^3+y^3=64-54=10$. The largest real value of $x+y$ is 4.

Answer: 4. □

1.2. **Symmetric Expressions in Three Roots.** The identities below write the most common symmetric expressions in three numbers r, s, t through $e_1 = r+s+t$, $e_2 = rs+st+tr$ and $e_3 = rst$. When r, s, t are the roots of a cubic, Vieta's formulas then give their values.

Proposition 1.2.1 (Three Numbers)**.** Let r, s, t be numbers, and let $e_1 = r+s+t$, $e_2 = rs+st+tr$ and $e_3 = rst$. Then:

(a) $r^2+s^2+t^2 = e_1^2 - 2e_2$.
(b) $r^2s + r^2t + s^2r + s^2t + t^2r + t^2s = e_1e_2 - 3e_3$.
(c) $r^3+s^3+t^3 = e_1^3 - 3e_1e_2 + 3e_3$. Equivalently,
$$r^3+s^3+t^3-3rst = (r+s+t)(r^2+s^2+t^2-rs-st-tr),$$
so $r^3+s^3+t^3 = 3rst$ whenever $r+s+t=0$.
(d) $r^2s^2+s^2t^2+t^2r^2 = e_2^2 - 2e_1e_3$.
(e) If $rst \neq 0$, then $\dfrac{1}{r}+\dfrac{1}{s}+\dfrac{1}{t} = \dfrac{e_2}{e_3}$ and $\dfrac{1}{rs}+\dfrac{1}{st}+\dfrac{1}{tr} = \dfrac{e_1}{e_3}$.
(f) $(r+s)(s+t)(t+r) = e_1e_2 - e_3$.

Proof. Part (a) is the case $n=3$ of $p_2 = e_1^2 - 2e_2$. For (b), the product $e_1e_2 = (r+s+t)(rs+st+tr)$ has nine terms: the six terms of the sum in (b), each once, and rst three times. For (c), the product $e_1(r^2+s^2+t^2)$ is $r^3+s^3+t^3$ plus the sum in (b), so by (a) and (b)
$$r^3+s^3+t^3 = e_1(e_1^2-2e_2) - (e_1e_2-3e_3) = e_1^3 - 3e_1e_2 + 3e_3.$$
Then $r^3+s^3+t^3-3rst = e_1(e_1^2-3e_2) = e_1\big((r^2+s^2+t^2)-e_2\big)$, which is the factorization. For (d), squaring e_2 gives r^2s^2, s^2t^2, t^2r^2 once each and twice each of $rs\cdot st$, $st\cdot tr$, $tr\cdot rs$, whose sum is $rst(r+s+t) = e_1e_3$. Part (e) follows by writing each sum over the common denominator rst. For (f), $r+s = e_1-t$, $s+t = e_1-r$ and $t+r = e_1-s$, so the product is the value at $x=e_1$ of $(x-r)(x-s)(x-t) = x^3 - e_1x^2 + e_2x - e_3$, which is $e_1^3 - e_1^3 + e_1e_2 - e_3$. □

Products of the form $(c-r)(c-s)(c-t)$ are values of the polynomial: if $P(x) = a(x-r)(x-s)(x-t)$, then
$$(c-r)(c-s)(c-t) = \frac{P(c)}{a}$$
for every number c. For example, $(r+1)(s+1)(t+1) = -P(-1)/a$. The proof of (f) is the case $c = e_1$: the product $(r+s)(s+t)(t+r)$ equals $P(e_1)/a$.

Examples.

Problem 1.2.2. Let r, s, and t be the roots of $x^3 - 4x^2 + x + 5$. Find $\dfrac{1}{r}+\dfrac{1}{s}+\dfrac{1}{t}$ and $r^2s+r^2t+s^2r+s^2t+t^2r+t^2s$.

Strategy. The roots are not rational, so write both expressions with e_1, e_2, e_3 and use Vieta's formulas.

Solution. Let $P(x) = x^3-4x^2+x+5$. By the Rational Root Theorem the only rational candidates are ±1 and ±5, and $P(1)=3$, $P(-1)=-1$, $P(5)=35$, $P(-5)=-225$, so no root is rational. By Vieta's formulas, $e_1 = r+s+t = 4$, $e_2 = rs+st+tr = 1$ and $e_3 = rst = -5$; in particular no root is 0. By part (e) of the Three Numbers proposition,
$$\frac{1}{r}+\frac{1}{s}+\frac{1}{t} = \frac{e_2}{e_3} = -\frac{1}{5},$$

and by part (b),

$$r^2s + r^2t + s^2r + s^2t + t^2r + t^2s = e_1e_2 - 3e_3 = 4 + 15 = 19.$$

Answer: $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = -\frac{1}{5}$ and $r^2s + r^2t + s^2r + s^2t + t^2r + t^2s = 19$. □

Problem 1.2.3 (1996 AIME, Problem #5)**.** Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b$, $b + c$, and $c + a$. Find t.

Strategy. The constant term t is minus the product of the roots of the second cubic. Since $a+b+c$ is known, each of $a + b$, $b + c$, $c + a$ is a known number minus a root of the first cubic, so the product is a value of the first polynomial.

Solution. Let $P(x) = x^3+3x^2+4x-11 = (x-a)(x-b)(x-c)$. By Vieta's formulas $a+b+c = -3$, so

$$a + b = -3 - c, \qquad b + c = -3 - a, \qquad c + a = -3 - b.$$

Hence

$$(a + b)(b + c)(c + a) = (-3 - a)(-3 - b)(-3 - c) = P(-3) = -27 + 27 - 12 - 11 = -23.$$

By Vieta's formulas for the second cubic, the product of its roots is $-t$, so $t = -(a+b)(b+c)(c+a) = 23$.

Answer: 23. □

2. Newton's Sums

Newton's sums find the power sums $p_1, p_2, p_3, \ldots$ of the roots of a polynomial one at a time from its coefficients.

2.1. **Power Sums.** As in Section 1.1, p_k is the sum of the kth powers of the roots. Let $p_0 = n$, since $r^0 = 1$ for each of the n roots.

Theorem 2.1.1 (Newton's Sums)**.** *Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $a_n \neq 0$, have roots $r_1, \ldots, r_n$, and let $p_k = r_1^k + \cdots + r_n^k$, with $p_0 = n$.*

(a) For $1 \leq k \leq n$,

$$a_np_k + a_{n-1}p_{k-1} + \cdots + a_{n-k+1}p_1 + k\,a_{n-k} = 0.$$

(b) For $k > n$,

$$a_np_k + a_{n-1}p_{k-1} + \cdots + a_1p_{k-n+1} + a_0p_{k-n} = 0.$$

For a cubic $ax^3 + bx^2 + cx + d$ the formulas read

$$\begin{aligned} ap_1 + b &= 0,\\ ap_2 + bp_1 + 2c &= 0,\\ ap_3 + bp_2 + cp_1 + 3d &= 0,\\ ap_k + bp_{k-1} + cp_{k-2} + dp_{k-3} &= 0 \qquad (k \geq 4). \end{aligned}$$

For two numbers x and y, the roots of $t^2 - e_1t + e_2$, they read $p_1 = e_1$ and $p_k = e_1p_{k-1} - e_2p_{k-2}$ for $k \geq 2$, with $p_0 = 2$.

Proof. By Vieta's formulas, $a_{n-k} = (-1)^k a_n e_k$. First let $k \geq n$. Each root r satisfies $a_nr^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0$. Multiplying by r^{k-n} gives

$$a_nr^k + a_{n-1}r^{k-1} + \cdots + a_0r^{k-n} = 0,$$

and adding these n equations, one for each root, gives (b) when $k > n$. When $k = n$, the last terms add up to $a_0p_0 = n\,a_0$, which gives (a) for $k = n$.

Now let $k \leq 3$. After division by a_n, the formulas of (a) for $k = 1, 2, 3$ read

$$p_1 - e_1 = 0, \qquad p_2 - e_1 p_1 + 2e_2 = 0, \qquad p_3 - e_1 p_2 + e_2 p_1 - 3e_3 = 0.$$

The first is Vieta's formula for the sum of the roots, and the second is $p_2 = e_1^2 - 2e_2$. For the third, expanding the products gives

$$e_1 p_2 = p_3 + \sum_{i \neq j} r_i^2 r_j, \qquad e_2 p_1 = \sum_{i \neq j} r_i^2 r_j + 3e_3,$$

because in $e_2 p_1$ each product of three different roots arises three times, once for each choice of the root taken from p_1. Subtracting gives $e_1 p_2 - e_2 p_1 = p_3 - 3e_3$. The cases $4 \leq k < n$, which occur only for polynomials of degree at least 5, follow from a longer expansion of the same kind. □

The formulas are used in order: each one gives the next power sum from the ones before it. Part (b) says that the power sums satisfy the same relation as the powers of any single root.

Examples.

Problem 2.1.2 (1975 AHSME, Problem #27)**.** If p, q, and r are distinct roots of $x^3 - x^2 + x - 2 = 0$, then $p^3 + q^3 + r^3$ equals

(A) -1 **(B)** 1 **(C)** 3 **(D)** 5 **(E)** none of these

Strategy. Apply Newton's sums to the cubic for $k = 1, 2, 3$.

Solution. The three distinct roots p, q, r are all the roots of $x^3 - x^2 + x - 2$. In Newton's sums for a cubic, $a = 1$, $b = -1$, $c = 1$ and $d = -2$, so

$$\begin{aligned} (p + q + r) - 1 &= 0, \\ (p^2 + q^2 + r^2) - (p + q + r) + 2 &= 0, \\ (p^3 + q^3 + r^3) - (p^2 + q^2 + r^2) + (p + q + r) - 6 &= 0. \end{aligned}$$

Hence $p + q + r = 1$, $p^2 + q^2 + r^2 = 1 - 2 = -1$, and $p^3 + q^3 + r^3 = -1 - 1 + 6 = 4$. None of the choices (A) to (D) equals 4. The negative sum of squares shows that not all three roots are real; the formulas hold all the same.

Answer: (E) none of these. □

Problem 2.1.3. Find the sum of the fifth powers of the four roots of $x^4 - x^3 - 2x + 1$.

Strategy. Compute the power sums one at a time: part (a) of Newton's sums for $k = 1, 2, 3, 4$, then part (b) for $k = 5$.

Solution. The only rational candidates are ± 1, and the polynomial takes the values -1 and 5 there, so its roots are not rational. Here $a_4 = 1$, $a_3 = -1$, $a_2 = 0$, $a_1 = -2$ and $a_0 = 1$. Part (a) of Newton's sums gives

$$\begin{aligned} p_1 - 1 = 0, &\quad \text{so } p_1 = 1; \\ p_2 - p_1 + 2 \cdot 0 = 0, &\quad \text{so } p_2 = 1; \\ p_3 - p_2 + 0 \cdot p_1 + 3 \cdot (-2) = 0, &\quad \text{so } p_3 = 7; \\ p_4 - p_3 + 0 \cdot p_2 - 2p_1 + 4 \cdot 1 = 0, &\quad \text{so } p_4 = 5. \end{aligned}$$

Part (b) with $k = 5$ gives

$$p_5 - p_4 + 0 \cdot p_3 - 2p_2 + 1 \cdot p_1 = 0, \qquad \text{so} \qquad p_5 = 5 + 2 - 1 = 6.$$

Answer: 6. □

Problem 2.1.4 (2020 AMC 10A, Problem #14)**.** Real numbers x and y satisfy $x + y = 4$ and $x \cdot y = -2$. What is the value of

$$x + \frac{x^3}{y^2} + \frac{y^3}{x^2} + y?$$

(A) 360 **(B)** 400 **(C)** 420 **(D)** 440 **(E)** 480

Strategy. Over a common denominator, the two fractions give $(x^5 + y^5)/(xy)^2$. Newton's sums for two numbers give $x^5 + y^5$.

Solution. Over the common denominator x^2y^2,

$$\frac{x^3}{y^2} + \frac{y^3}{x^2} = \frac{x^5 + y^5}{x^2y^2}.$$

For x and y, $e_1 = 4$ and $e_2 = -2$, so the power sums satisfy $p_k = 4p_{k-1} + 2p_{k-2}$ for $k \geq 2$, with $p_0 = 2$ and $p_1 = 4$. Then

$$p_2 = 16 + 4 = 20, \quad p_3 = 80 + 8 = 88, \quad p_4 = 352 + 40 = 392, \quad p_5 = 1568 + 176 = 1744.$$

Since $x^2y^2 = 4$, the value is $4 + \frac{1744}{4} = 4 + 436 = 440$.

Answer: (D) 440. □

2.2. **Sums of Values of a Polynomial.** If $P(x) = c_m x^m + \cdots + c_1 x + c_0$ and $z_1, \ldots, z_n$ are the roots of Q, then

$$P(z_1) + P(z_2) + \cdots + P(z_n) = c_m p_m + \cdots + c_1 p_1 + n\, c_0,$$

where p_k is the sum of the kth powers of $z_1, \ldots, z_n$. When P has degree n or more, divide first. By the Division Algorithm, $P = QS + R$, where R is 0 or has degree less than n. Since $Q(z_i) = 0$, the values $P(z_i)$ and $R(z_i)$ are equal for every root, so the sum of the values of P equals the sum of the values of R, which needs only $p_1, \ldots, p_{n-1}$.

Example.

Problem 2.2.1 (2003 AIME II, Problem #9)**.** Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$.

Strategy. Divide P by Q. At each root of Q, the value of P equals the value of the remainder, and the sum of those values needs only p_1 and p_2.

Solution. Long division gives

$$x^6 - x^5 - x^3 - x^2 - x = (x^4 - x^3 - x^2 - 1)(x^2 + 1) + (x^2 - x + 1),$$

since $(x^4 - x^3 - x^2 - 1)(x^2 + 1) = x^6 - x^5 - x^3 - 2x^2 - 1$. Because $Q(z_i) = 0$, the value $P(z_i)$ equals $z_i^2 - z_i + 1$, so

$$P(z_1) + P(z_2) + P(z_3) + P(z_4) = p_2 - p_1 + 4,$$

where p_k is the sum of the kth powers of $z_1, \ldots, z_4$. Newton's sums for Q, with $a_4 = 1$, $a_3 = -1$ and $a_2 = -1$, give $p_1 - 1 = 0$ and $p_2 - p_1 - 2 = 0$, so $p_1 = 1$ and $p_2 = 3$. The sum is $3 - 1 + 4 = 6$.

Answer: 6. □

3. Root Transformations

Given the roots $r_1, \ldots, r_n$ of a polynomial P, this section builds a polynomial whose roots are $f(r_1), \ldots, f(r_n)$, where $f(r)$ is $r + h$, kr, $1/r$ or r^2. The new polynomial can be found from the elementary symmetric sums of the new roots, which Section 1 computes from the coefficients of P, or, when r can be recovered from $y = f(r)$, by substituting into P.

3.1. **Shifted and Scaled Roots.**

Theorem 3.1.1 (Shifted and Scaled Roots)**.** *Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ have degree n and roots $r_1, \ldots, r_n$, and let h and k be numbers with $k \neq 0$.*

(a) The polynomial $P(x-h)$ has degree n, leading coefficient a_n, and roots $r_1 + h, \ldots, r_n + h$.

(b) The polynomial

$$k^n P(x/k) = a_n x^n + k\, a_{n-1}x^{n-1} + k^2 a_{n-2}x^{n-2} + \cdots + k^n a_0$$

has degree n, leading coefficient a_n, and roots $kr_1, \ldots, kr_n$.

Proof. $P(x-h) = a_n(x-h-r_1)\cdots(x-h-r_n) = a_n\big(x-(r_1+h)\big)\cdots\big(x-(r_n+h)\big)$. Similarly, $k^n P(x/k) = a_n k^n (x/k - r_1)\cdots(x/k - r_n) = a_n(x-kr_1)\cdots(x-kr_n)$, and the term $a_j(x/k)^j$ of $P(x/k)$ becomes $k^{n-j}a_j x^j$ after multiplication by k^n. □

Combining the two parts, the roots of $P\big((x-h)/k\big)$ are $kr_1 + h, \ldots, kr_n + h$. In particular, the roots of $P(-x)$ are $-r_1, \ldots, -r_n$, and the roots of $P(h-x)$ are $h - r_1, \ldots, h - r_n$. A shift by h adds nh to the sum of the roots.

Examples.

Problem 3.1.2 (1963 AHSME, Problem #14)**.** Given the equations $x^2 + kx + 6 = 0$ and $x^2 - kx + 6 = 0$. If, when the roots of the equations are suitably listed, each root of the second equation is 5 more than the corresponding root of the first equation, then k equals:

(A) 5 **(B)** -5 **(C)** 7 **(D)** -7 **(E)** none of these

Strategy. If $P(x) = x^2 + kx + 6$ has roots r and s, then the monic quadratic with roots $r+5$ and $s+5$ is $P(x-5)$. Compare its coefficients with those of $x^2 - kx + 6$.

Solution. Let $P(x) = x^2 + kx + 6$ have roots r and s. The second polynomial is monic with roots $r+5$ and $s+5$, and so is $P(x-5)$, by the Shifted and Scaled Roots theorem. A monic quadratic is determined by its two roots, so

$$x^2 - kx + 6 = (x-5)^2 + k(x-5) + 6 = x^2 + (k-10)x + 31 - 5k.$$

Comparing coefficients, $k - 10 = -k$ and $31 - 5k = 6$, and both give $k = 5$. As a check, $x^2 + 5x + 6$ has the roots -2 and -3, and $x^2 - 5x + 6$ has the roots 3 and 2, which are 5 more.

Answer: (A) 5. □

Problem 3.1.3 (1961 AHSME, Problem #29)**.** Let the roots of $ax^2 + bx + c = 0$ be r and s. The equation with roots $ar + b$ and $as + b$ is:

(A) $x^2 - bx - ac = 0$ **(B)** $x^2 - bx + ac = 0$ **(C)** $x^2 + 3bx + ca + 2b^2 = 0$
(D) $x^2 + 3bx - ca + 2b^2 = 0$ **(E)** $x^2 + bx(2-a) + a^2c + b^2(a+1) = 0$

Strategy. The new roots are $ar + b$ and $as + b$: each root of the given equation is scaled by a and shifted by b. So substitute $(x-b)/a$ for x, which is allowed because $a \neq 0$.

Solution. Let $P(x) = ax^2 + bx + c$; since the equation is quadratic, $a \neq 0$. By the Shifted and Scaled Roots theorem, with scale factor a and shift b, the roots of $P\big((x-b)/a\big)$ are $ar + b$ and $as + b$. Multiplying by a to make it monic,

$$a\,P\Big(\frac{x-b}{a}\Big) = (x-b)^2 + b(x-b) + ac = x^2 - bx + ac.$$

So the required equation is $x^2 - bx + ac = 0$. As a check by Vieta's formulas, $r + s = -b/a$ and $rs = c/a$, so $(ar+b) + (as+b) = a(r+s) + 2b = b$ and $(ar+b)(as+b) = a^2 rs + ab(r+s) + b^2 = ac - b^2 + b^2 = ac$.

Answer: (B) $x^2 - bx + ac = 0$. □

Problem 3.1.4 (1993 AIME, Problem #5)**.** Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x - n)$. What is the coefficient of x in $P_{20}(x)$?

Strategy. Each step is a shift: $P_n(x) = P_{n-1}(x - n)$. After twenty steps the total shift is $1 + 2 + \cdots + 20$, so $P_{20}(x) = P_0(x - 210)$. Only three terms of $P_0(x - 210)$ contribute to the coefficient of x.

Solution. By induction, $P_n(x) = P_0\big(x - (1 + 2 + \cdots + n)\big)$: this holds for $n = 1$, and if it holds for $n - 1$, then

$$P_n(x) = P_{n-1}(x - n) = P_0\big(x - n - (1 + 2 + \cdots + (n - 1))\big).$$

Since $1 + 2 + \cdots + 20 = 210$, $P_{20}(x) = P_0(x - 210)$; its roots are those of P_0 increased by 210. In

$$P_0(x - 210) = (x - 210)^3 + 313(x - 210)^2 - 77(x - 210) - 8,$$

the terms in x are $3 \cdot 210^2 x$ from the cube, $313 \cdot (-2 \cdot 210)x$ from the square, and $-77x$. The coefficient of x is

$$3 \cdot 44100 - 626 \cdot 210 - 77 = 132300 - 131460 - 77 = 763.$$

Answer: 763. □

3.2. **Reciprocal Roots.**

Theorem 3.2.1 (Reciprocal Roots)**.** *Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ have degree n and roots $r_1, \ldots, r_n$, and suppose that $a_0 \neq 0$. Then no root is 0, and the polynomial with the coefficients in reverse order,*

$$x^n P(1/x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n,$$

has degree n and roots $1/r_1, \ldots, 1/r_n$.

Proof. Since $P(0) = a_0 \neq 0$, the number 0 is not a root. Then

$$x^n P(1/x) = a_n x^n \Big(\frac{1}{x} - r_1\Big) \cdots \Big(\frac{1}{x} - r_n\Big) = a_n (1 - r_1 x) \cdots (1 - r_n x).$$

Each factor $1 - r_i x$ equals $-r_i(x - 1/r_i)$, so $x^n P(1/x) = a_n(-r_1) \cdots (-r_n)\,(x - 1/r_1) \cdots (x - 1/r_n)$, and the constant $a_n(-r_1) \cdots (-r_n) = (-1)^n a_n e_n$ equals a_0 by Vieta's formulas. □

Combined with the Shifted and Scaled Roots theorem with $k = -1$, the roots of $x^n P(-1/x)$ are $-1/r_1, \ldots, -1/r_n$.

Example.

Problem 3.2.2 (2021 AMC 12B, Problem #16)**.** Let $g(x)$ be a polynomial with leading coefficient 1, whose three roots are the reciprocals of the three roots of $f(x) = x^3 + ax^2 + bx + c$, where $1 < a < b < c$. What is $g(1)$ in terms of a, b, and c?

(A) $\dfrac{1 + a + b + c}{c}$ **(B)** $1 + a + b + c$ **(C)** $\dfrac{1 + a + b + c}{c^2}$ **(D)** $\dfrac{a + b + c}{c^2}$

(E) $\dfrac{1 + a + b + c}{a + b + c}$

Strategy. Reverse the coefficients of f to get a cubic with the reciprocal roots, then divide by its leading coefficient c to make it monic.

Solution. Let r_1, r_2, r_3 be the roots of f. Since $c > 1$, the constant term of f is not 0, so by the Reciprocal Roots theorem the roots of $cx^3 + bx^2 + ax + 1$ are $1/r_1$, $1/r_2$, $1/r_3$. The monic cubic with these roots is therefore

$$g(x) = \Big(x - \frac{1}{r_1}\Big)\Big(x - \frac{1}{r_2}\Big)\Big(x - \frac{1}{r_3}\Big) = \frac{cx^3 + bx^2 + ax + 1}{c},$$

and $g(1) = \dfrac{c + b + a + 1}{c} = \dfrac{1 + a + b + c}{c}$. As a check through values of f,

$$g(1) = \frac{(r_1 - 1)(r_2 - 1)(r_3 - 1)}{r_1 r_2 r_3} = \frac{-f(1)}{-c} = \frac{1 + a + b + c}{c},$$

since $f(1) = (1 - r_1)(1 - r_2)(1 - r_3)$ and $r_1 r_2 r_3 = -c$.

Answer: (A) $\dfrac{1 + a + b + c}{c}$. □

3.3. **Squared Roots.** By Vieta's formulas, the monic polynomial with roots $r_1^2, \ldots, r_n^2$ has as its coefficients, with alternating signs, the elementary symmetric sums of $r_1^2, \ldots, r_n^2$. These are symmetric in $r_1, \ldots, r_n$, so Section 1 computes them. For three roots r, s, t, parts (a) and (d) of the Three Numbers proposition give

$$r^2 + s^2 + t^2 = e_1^2 - 2e_2, \qquad r^2 s^2 + s^2 t^2 + t^2 r^2 = e_2^2 - 2e_1 e_3, \qquad r^2 s^2 t^2 = e_3^2,$$

so the monic cubic with roots r^2, s^2, t^2 is

$$y^3 - (e_1^2 - 2e_2)y^2 + (e_2^2 - 2e_1 e_3)y - e_3^2.$$

The same polynomial can also be built directly from P, without computing symmetric sums.

Proposition 3.3.1 (Squared Roots)**.** Let P be a monic polynomial of degree n with roots $r_1, \ldots, r_n$, and write $P(x) = E(x^2) + xO(x^2)$ by separating the even and the odd powers of x. Then $(-1)^n\big(E(y)^2 - yO(y)^2\big)$ is the monic polynomial with roots $r_1^2, \ldots, r_n^2$.

Proof. Since $P(-x) = E(x^2) - xO(x^2)$,

$$P(x)P(-x) = E(x^2)^2 - x^2 O(x^2)^2.$$

On the other hand, $(x - r_i)(-x - r_i) = -(x^2 - r_i^2)$, so $P(x)P(-x) = (-1)^n(x^2 - r_1^2)\cdots(x^2 - r_n^2)$. Hence $Q(y) = (-1)^n\big(E(y)^2 - yO(y)^2\big)$ equals $(y - r_1^2)\cdots(y - r_n^2)$ for every $y \geq 0$, by taking $x = \sqrt{y}$. Two polynomials that agree at infinitely many numbers are equal, so $Q(y) = (y - r_1^2)\cdots(y - r_n^2)$. □

Example.

Problem 3.3.2. Find the monic cubic polynomial whose roots are the squares of the roots of $x^3 - 3x^2 - x + 2$.

Strategy. Compute the elementary symmetric sums of the squares from e_1, e_2, e_3. Check the result with the Squared Roots proposition.

Solution. Let r, s, t be the roots. By Vieta's formulas $e_1 = 3$, $e_2 = -1$ and $e_3 = -2$. By parts (a) and (d) of the Three Numbers proposition,

$$r^2 + s^2 + t^2 = 9 + 2 = 11, \qquad r^2 s^2 + s^2 t^2 + t^2 r^2 = 1 - 2 \cdot 3 \cdot (-2) = 13, \qquad r^2 s^2 t^2 = 4,$$

so the monic cubic with roots r^2, s^2, t^2 is $x^3 - 11x^2 + 13x - 4$. As a check, $x^3 - 3x^2 - x + 2 = (-3x^2 + 2) + x(x^2 - 1)$, so $E(y) = -3y + 2$ and $O(y) = y - 1$, and the Squared Roots proposition gives

$$\begin{aligned} -\big(E(y)^2 - yO(y)^2\big) &= y(y-1)^2 - (3y-2)^2 \\ &= y^3 - 2y^2 + y - 9y^2 + 12y - 4 = y^3 - 11y^2 + 13y - 4. \end{aligned}$$

Answer: $x^3 - 11x^2 + 13x - 4$. □

3.4. **Removing the Second-Highest Term.**

Theorem 3.4.1 (Removing the Second-Highest Term)**.** *Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ have degree n and roots $r_1, \ldots, r_n$, and let $h = \dfrac{a_{n-1}}{na_n}$. Then the polynomial $P(y-h)$ has no y^{n-1} term. Its roots are $r_1 + h, \ldots, r_n + h$, and their sum is 0.*

Proof. By the Shifted and Scaled Roots theorem, $P(y-h)$ has leading coefficient a_n and roots $r_i + h$. By Vieta's formulas their sum is $(r_1 + \cdots + r_n) + nh = -\dfrac{a_{n-1}}{a_n} + \dfrac{a_{n-1}}{a_n} = 0$, and the coefficient of y^{n-1} is $-a_n$ times this sum. □

For a monic cubic $x^3 + ax^2 + bx + c$, the substitution $x = y - a/3$ gives a cubic of the form $y^3 + uy + v$. Its roots y_1, y_2, y_3 add up to 0, so by part (c) of the Three Numbers proposition, $y_1^3 + y_2^3 + y_3^3 = 3y_1y_2y_3 = -3v$.

Example.

Problem 3.4.2. Let r, s, and t be the roots of $x^3 + 6x^2 + 5x - 9$. Find $(r+2)^3 + (s+2)^3 + (t+2)^3$.

Strategy. Here $h = 6/3 = 2$, so substituting $x = y - 2$ gives a cubic without a y^2 term whose roots are $r + 2$, $s + 2$, $t + 2$. Their sum is 0, so the sum of their cubes is three times their product.

Solution. By the Removing the Second-Highest Term theorem with $h = \frac{6}{3 \cdot 1} = 2$, the numbers $r + 2$, $s + 2$, $t + 2$ are the roots of

$$\begin{aligned} (y-2)^3 + 6(y-2)^2 + 5(y-2) - 9 &= (y^3 - 6y^2 + 12y - 8) + (6y^2 - 24y + 24) + (5y - 10) - 9 \\ &= y^3 - 7y - 3, \end{aligned}$$

and their sum is 0. By part (c) of the Three Numbers proposition, the sum of their cubes is three times their product, and by Vieta's formulas their product is 3. So

$$(r+2)^3 + (s+2)^3 + (t+2)^3 = 3 \cdot 3 = 9.$$

As a check, expanding gives $(r+2)^3 + (s+2)^3 + (t+2)^3 = p_3 + 6p_2 + 12p_1 + 24$, and Newton's sums give $p_1 = -6$, $p_2 = 36 - 10 = 26$ and $p_3 = -6 \cdot 26 + 5 \cdot 6 + 27 = -99$, so the sum is $-99 + 156 - 72 + 24 = 9$.

Answer: 9. □

4. Problems

Beginner.

1. (2002 AMC 12B, Problem #19) If a, b, and c are positive real numbers such that $a(b+c) = 152$, $b(c+a) = 162$, and $c(a+b) = 170$, then abc is

(A) 672 **(B)** 688 **(C)** 704 **(D)** 720 **(E)** 750

2. (1981 AHSME, Problem #8) For all positive numbers x, y, z, the product

$$(x+y+z)^{-1}(x^{-1}+y^{-1}+z^{-1})(xy+yz+zx)^{-1}\big((xy)^{-1}+(yz)^{-1}+(zx)^{-1}\big)$$

equals

(A) $x^{-2}y^{-2}z^{-2}$ **(B)** $x^{-2}+y^{-2}+z^{-2}$ **(C)** $(x+y+z)^{-1}$ **(D)** $\dfrac{1}{xyz}$

(E) $\dfrac{1}{xy+yz+zx}$

3. (2021 AMC 10B, Problem #15) The real number x satisfies the equation

$$x + \frac{1}{x} = \sqrt{5}.$$

What is the value of $x^{11} - 7x^7 + x^3$?

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** $\sqrt{5}$

4. (2019 AMC 12A, Problem #17) Let s_k denote the sum of the kth powers of the roots of the polynomial $x^3 - 5x^2 + 8x - 13$. In particular, $s_0 = 3$, $s_1 = 5$, and $s_2 = 9$. Let a, b, and c be real numbers such that $s_{k+1} = a\,s_k + b\,s_{k-1} + c\,s_{k-2}$ for $k = 2, 3, \ldots$. What is $a+b+c$?

(A) -6 **(B)** 0 **(C)** 6 **(D)** 10 **(E)** 26

5. (1958 AHSME, Problem #41) The roots of $Ax^2 + Bx + C = 0$ are r and s. For the roots of

$$x^2 + px + q = 0$$

to be r^2 and s^2, p must equal:

(A) $\dfrac{B^2 - 4AC}{A^2}$ **(B)** $\dfrac{B^2 - 2AC}{A^2}$ **(C)** $\dfrac{2AC - B^2}{A^2}$ **(D)** $B^2 - 2C$

(E) $2C - B^2$

Intermediate.

6. (1980 AHSME, Problem #27) The sum

$$\sqrt[3]{5+2\sqrt{13}} + \sqrt[3]{5-2\sqrt{13}}$$

equals

(A) $\dfrac{3}{2}$ **(B)** $\dfrac{\sqrt[3]{65}}{4}$ **(C)** $\dfrac{1+\sqrt[6]{13}}{2}$ **(D)** $\sqrt[3]{2}$ **(E)** none of these

7. (2005 AMC 12B, Problem #23) Let S be the set of ordered triples (x, y, z) of real numbers for which

$$\log_{10}(x+y) = z$$

and

$$\log_{10}(x^2+y^2) = z+1.$$

There are real numbers a and b such that for all ordered triples (x, y, z) in S we have $x^3+y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$. What is the value of $a+b$?

(A) $\frac{15}{2}$ **(B)** $\frac{29}{2}$ **(C)** 15 **(D)** $\frac{39}{2}$ **(E)** 24

8. (2008 AIME II, Problem #7) Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

9. (1981 AHSME, Problem #30) If a, b, c, d are the solutions of the equation $x^4 - bx - 3 = 0$, then an equation whose solutions are

$$\frac{a+b+c}{d^2}, \quad \frac{a+b+d}{c^2}, \quad \frac{a+c+d}{b^2}, \quad \frac{b+c+d}{a^2}$$

is

(A) $3x^4+bx+1=0$ **(B)** $3x^4-bx+1=0$ **(C)** $3x^4+bx^3-1=0$
(D) $3x^4-bx^3-1=0$ **(E)** none of these

10. (2005 AIME I, Problem #6) Let P be the product of the nonreal roots of $x^4-4x^3+6x^2-4x=2005$. Find $\lfloor P \rfloor$. (The notation $\lfloor P \rfloor$ denotes the greatest integer that is less than or equal to P.)

Advanced.

11. (2001 AIME I, Problem #3) Find the sum of the roots, real and non-real, of the equation

$$x^{2001} + \left(\tfrac{1}{2} - x\right)^{2001} = 0,$$

given that there are no multiple roots.

12. (2015 AIME II, Problem #14) Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

13. (2019 AIME I, Problem #8) Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

14. (1990 AIME, Problem #15) Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

15. (2014 AIME II, Problem #5) Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r+4$ and $s-3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.