SYMMETRIC SUMS, NEWTON'S SUMS, AND ROOT TRANSFORMATIONS FOR AMC 12: SOLUTIONS

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1. Solutions

Beginner.

1. (2002 AMC 12B, Problem #19) If a, b, and c are positive real numbers such that $a(b+c) = 152$, $b(c+a) = 162$, and $c(a+b) = 170$, then abc is

(A) 672 **(B)** 688 **(C)** 704 **(D)** 720 **(E)** 750

Strategy. The three left sides add up to twice the elementary symmetric sum $ab+bc+ca$. Subtracting each equation from this sum gives one product of two of the numbers.

Solution. Adding the three equations gives $2(ab+bc+ca) = 152+162+170 = 484$, so $ab+bc+ca = 242$. Since $a(b+c) = ab+ca$, subtracting gives $bc = 242-152 = 90$. In the same way $ca = 242-162 = 80$ and $ab = 242-170 = 72$. Multiplying,

$$(abc)^2 = 72 \cdot 80 \cdot 90 = 518400 = 720^2,$$

and since a, b, c are positive, $abc = 720$. As a check, $a = 8$, $b = 9$ and $c = 10$ satisfy the three equations.

Answer: (D) 720. □

2. (1981 AHSME, Problem #8) For all positive numbers x, y, z, the product

$$(x+y+z)^{-1}(x^{-1}+y^{-1}+z^{-1})(xy+yz+zx)^{-1}\big((xy)^{-1}+(yz)^{-1}+(zx)^{-1}\big)$$

equals

(A) $x^{-2}y^{-2}z^{-2}$ **(B)** $x^{-2}+y^{-2}+z^{-2}$ **(C)** $(x+y+z)^{-1}$ **(D)** $\dfrac{1}{xyz}$

(E) $\dfrac{1}{xy+yz+zx}$

Strategy. Each of the four factors is a quotient of elementary symmetric sums of x, y, z.

Solution. Let $e_1 = x+y+z$, $e_2 = xy+yz+zx$ and $e_3 = xyz$. By part (e) of the Three Numbers proposition, $x^{-1}+y^{-1}+z^{-1} = e_2/e_3$ and $(xy)^{-1}+(yz)^{-1}+(zx)^{-1} = e_1/e_3$. The product is therefore

$$\frac{1}{e_1}\cdot\frac{e_2}{e_3}\cdot\frac{1}{e_2}\cdot\frac{e_1}{e_3} = \frac{1}{e_3^2} = x^{-2}y^{-2}z^{-2}.$$

Answer: (A) $x^{-2}y^{-2}z^{-2}$. □

3. (2021 AMC 10B, Problem #15) The real number x satisfies the equation

$$x+\frac{1}{x} = \sqrt{5}.$$

What is the value of $x^{11}-7x^7+x^3$?

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** $\sqrt{5}$

Strategy. The numbers x and $1/x$ have sum $\sqrt{5}$ and product 1. Factoring x^7 out of the expression leaves $x^4+x^{-4}-7$, and x^4+x^{-4} is a power sum of x and $1/x$.

Solution. Write $x^{11}-7x^7+x^3 = x^7\big(x^4+x^{-4}-7\big)$. The numbers x and $1/x$ have $e_1 = \sqrt{5}$ and $e_2 = 1$, so their power sums satisfy $p_k = \sqrt{5}\,p_{k-1}-p_{k-2}$, with $p_0 = 2$ and $p_1 = \sqrt{5}$. Then

$$p_2 = 5-2 = 3, \qquad p_3 = 3\sqrt{5}-\sqrt{5} = 2\sqrt{5}, \qquad p_4 = 2\sqrt{5}\cdot\sqrt{5}-3 = 7.$$

So $x^4 + x^{-4} = 7$, and the expression equals $x^7 \cdot 0 = 0$.

Answer: (B) 0. □

4. (2019 AMC 12A, Problem #17) Let s_k denote the sum of the kth powers of the roots of the polynomial $x^3 - 5x^2 + 8x - 13$. In particular, $s_0 = 3$, $s_1 = 5$, and $s_2 = 9$. Let a, b, and c be real numbers such that $s_{k+1} = a\,s_k + b\,s_{k-1} + c\,s_{k-2}$ for $k = 2, 3, \ldots$. What is $a + b + c$?

(A) -6 **(B)** 0 **(C)** 6 **(D)** 10 **(E)** 26

Strategy. Newton's sums for $x^3 - 5x^2 + 8x - 13$ give a relation among four consecutive power sums.

Solution. For this cubic the coefficients in Newton's sums are 1, -5, 8, -13. Part (b) gives, for every $k \geq 4$,

$$s_k - 5s_{k-1} + 8s_{k-2} - 13s_{k-3} = 0.$$

For $k = 3$, part (a) gives $s_3 - 5s_2 + 8s_1 + 3 \cdot (-13) = 0$, and since $s_0 = 3$ this is again $s_3 - 5s_2 + 8s_1 - 13s_0 = 0$. So for every $k \geq 2$,

$$s_{k+1} = 5s_k - 8s_{k-1} + 13s_{k-2},$$

which gives $a = 5$, $b = -8$, $c = 13$ and $a + b + c = 10$.

Answer: (D) 10. □

5. (1958 AHSME, Problem #41) The roots of $Ax^2 + Bx + C = 0$ are r and s. For the roots of

$$x^2 + px + q = 0$$

to be r^2 and s^2, p must equal:

(A) $\dfrac{B^2 - 4AC}{A^2}$ **(B)** $\dfrac{B^2 - 2AC}{A^2}$ **(C)** $\dfrac{2AC - B^2}{A^2}$ **(D)** $B^2 - 2C$ **(E)** $2C - B^2$

Strategy. By Vieta's formulas, p is minus the sum of the new roots, $r^2 + s^2$, which is a symmetric expression in r and s.

Solution. By Vieta's formulas for $Ax^2 + Bx + C$, $r + s = -B/A$ and $rs = C/A$. The roots of $x^2 + px + q$ are r^2 and s^2, so

$$p = -(r^2 + s^2) = -\big((r+s)^2 - 2rs\big) = -\Big(\frac{B^2}{A^2} - \frac{2C}{A}\Big) = \frac{2AC - B^2}{A^2}.$$

Answer: (C) $\dfrac{2AC - B^2}{A^2}$. □

Intermediate.

6. (1980 AHSME, Problem #27) The sum

$$\sqrt[3]{5 + 2\sqrt{13}} + \sqrt[3]{5 - 2\sqrt{13}}$$

equals

(A) $\dfrac{3}{2}$ **(B)** $\dfrac{\sqrt[3]{65}}{4}$ **(C)** $\dfrac{1 + \sqrt[6]{13}}{2}$ **(D)** $\sqrt[3]{2}$ **(E)** none of these

Strategy. Let u and v be the two cube roots and $x = u + v$. The sum $u^3 + v^3$ and the product uv are known, so the identity for the sum of two cubes gives a cubic equation for x.

Solution. Let $u = \sqrt[3]{5+2\sqrt{13}}$ and $v = \sqrt[3]{5-2\sqrt{13}}$ (real cube roots), and let $x = u + v$. Then $u^3 + v^3 = 10$, and

$$uv = \sqrt[3]{(5+2\sqrt{13})(5-2\sqrt{13})} = \sqrt[3]{25-52} = \sqrt[3]{-27} = -3.$$

The identity $u^3 + v^3 = (u+v)^3 - 3uv(u+v)$ gives $10 = x^3 + 9x$, so

$$x^3 + 9x - 10 = (x-1)(x^2 + x + 10) = 0.$$

The quadratic factor has discriminant $1 - 40 < 0$, so it has no real root, and $x = 1$. None of the choices (A) to (D) equals 1: $\frac{3}{2} \neq 1$; $\frac{\sqrt[3]{65}}{4} = 1$ would need $65 = 64$; $\frac{1+\sqrt[6]{13}}{2} = 1$ would need $\sqrt[6]{13} = 1$; and $\sqrt[3]{2} \neq 1$.

Answer: (E) none of these. □

7. (2005 AMC 12B, Problem #23) Let S be the set of ordered triples (x, y, z) of real numbers for which

$$\log_{10}(x+y) = z$$

and

$$\log_{10}(x^2 + y^2) = z + 1.$$

There are real numbers a and b such that for all ordered triples (x, y, z) in S we have $x^3 + y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$. What is the value of $a + b$?

(A) $\frac{15}{2}$ **(B)** $\frac{29}{2}$ **(C)** 15 **(D)** $\frac{39}{2}$ **(E)** 24

Strategy. Let $T = 10^z$. The two equations give $x + y = T$ and $x^2 + y^2 = 10T$, hence xy, and then $x^3 + y^3$ by the identities for two numbers.

Solution. By the definition of the logarithm, $x + y = 10^z$ and $x^2 + y^2 = 10^{z+1}$. Let $T = 10^z$, so that $e_1 = x + y = T$ and $x^2 + y^2 = 10T$. Then

$$e_2 = xy = \frac{e_1^2 - (x^2 + y^2)}{2} = \frac{T^2 - 10T}{2},$$

and

$$x^3 + y^3 = e_1^3 - 3e_1e_2 = T^3 - \frac{3T(T^2 - 10T)}{2} = -\frac{1}{2}T^3 + 15T^2.$$

Since $T^3 = 10^{3z}$ and $T^2 = 10^{2z}$, this is $a \cdot 10^{3z} + b \cdot 10^{2z}$ with $a = -\frac{1}{2}$ and $b = 15$, so $a + b = \frac{29}{2}$.

Answer: (B) $\frac{29}{2}$. □

8. (2008 AIME II, Problem #7) Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

Strategy. The cubic has no x^2 term, so $r + s + t = 0$, and each sum of two roots is minus the third root.

Solution. By Vieta's formulas for $8x^3 + 0x^2 + 1001x + 2008$, $r + s + t = 0$ and $rst = -\frac{2008}{8} = -251$. Hence $r + s = -t$, $s + t = -r$ and $t + r = -s$, so

$$(r+s)^3 + (s+t)^3 + (t+r)^3 = -(r^3 + s^3 + t^3).$$

Since $r + s + t = 0$, part (c) of the Three Numbers proposition gives $r^3 + s^3 + t^3 = 3rst = -753$. The required sum is 753.

Answer: 753. □

9. (1981 AHSME, Problem #30) If a, b, c, d are the solutions of the equation $x^4 - bx - 3 = 0$, then an equation whose solutions are

$$\frac{a+b+c}{d^2}, \quad \frac{a+b+d}{c^2}, \quad \frac{a+c+d}{b^2}, \quad \frac{b+c+d}{a^2}$$

is

(A) $3x^4 + bx + 1 = 0$ **(B)** $3x^4 - bx + 1 = 0$ **(C)** $3x^4 + bx^3 - 1 = 0$
(D) $3x^4 - bx^3 - 1 = 0$ **(E)** none of these

Strategy. The equation has no x^3 term, so the four solutions add up to 0, and each numerator is minus the remaining solution. The new solutions are then $-1/a$, $-1/b$, $-1/c$, $-1/d$.

Solution. The constant term of $x^4 - bx - 3$ is -3, so no solution is 0. The coefficient of x^3 is 0, so by Vieta's formulas $a + b + c + d = 0$. Hence $a + b + c = -d$, and

$$\frac{a+b+c}{d^2} = \frac{-d}{d^2} = -\frac{1}{d};$$

in the same way the other three numbers are $-1/c$, $-1/b$ and $-1/a$. Let $P(x) = x^4 - bx - 3$. By the Reciprocal Roots theorem combined with the Shifted and Scaled Roots theorem, the roots of $x^4 P(-1/x)$ are $-1/a$, $-1/b$, $-1/c$, $-1/d$, and

$$x^4 P\Big(-\frac{1}{x}\Big) = x^4\Big(\frac{1}{x^4} + \frac{b}{x} - 3\Big) = 1 + bx^3 - 3x^4.$$

Multiplying by -1, an equation with the required solutions is $3x^4 - bx^3 - 1 = 0$.
Answer: (D) $3x^4 - bx^3 - 1 = 0$. □

10. (2005 AIME I, Problem #6) Let P be the product of the nonreal roots of $x^4 - 4x^3 + 6x^2 - 4x = 2005$. Find $\lfloor P \rfloor$. (The notation $\lfloor P \rfloor$ denotes the greatest integer that is less than or equal to P.)

Strategy. The coefficient of x^3 is -4, so the substitution $x = y + 1$ removes the cubic term. The shifted equation has only two terms.

Solution. The roots of the equation are the roots of $f(x) = x^4 - 4x^3 + 6x^2 - 4x - 2005$. By the Removing the Second-Highest Term theorem with $h = \frac{-4}{4 \cdot 1} = -1$, substitute $x = y + 1$:

$$\begin{aligned} f(y+1) = (y^4 + 4y^3 + 6y^2 + 4y + 1) - (4y^3 + 12y^2 + 12y + 4) \\ + (6y^2 + 12y + 6) - (4y + 4) - 2005 = y^4 - 2006. \end{aligned}$$

Now $y^4 - 2006 = \big(y^2 - \sqrt{2006}\big)\big(y^2 + \sqrt{2006}\big)$. The first factor gives the two real roots $y = \pm\sqrt[4]{2006}$. The second factor is positive for every real y, so its two roots are the non-real roots. In terms of x, they are the roots of

$$(x-1)^2 + \sqrt{2006} = x^2 - 2x + 1 + \sqrt{2006},$$

whose product is $1 + \sqrt{2006}$ by Vieta's formulas. So $P = 1 + \sqrt{2006}$. Since $44^2 = 1936 < 2006 < 2025 = 45^2$, the number P lies between 45 and 46, and $\lfloor P \rfloor = 45$.
Answer: 45. □

Advanced.

11. (2001 AIME I, Problem #3) Find the sum of the roots, real and non-real, of the equation

$$x^{2001} + \big(\tfrac{1}{2} - x\big)^{2001} = 0,$$

given that there are no multiple roots.

Strategy. Replacing x by $\frac{1}{2}-x$ does not change the equation, so the roots come in pairs r and $\frac{1}{2}-r$. The degree gives the number of pairs.

Solution. Let $f(x)=x^{2001}+\left(\frac{1}{2}-x\right)^{2001}$. By the binomial theorem,

$$\left(\frac{1}{2}-x\right)^{2001}=-x^{2001}+2001\cdot\frac{1}{2}\,x^{2000}-\cdots,$$

so the x^{2001} terms cancel and f has degree 2000 and leading coefficient $\frac{2001}{2}$. So f has 2000 roots counted with multiplicity, and by hypothesis they are distinct.

Since $f\left(\frac{1}{2}-x\right)=\left(\frac{1}{2}-x\right)^{2001}+x^{2001}=f(x)$, and the roots of $f\left(\frac{1}{2}-x\right)$ are the numbers $\frac{1}{2}-r$ for the roots r of f (Shifted and Scaled Roots theorem), the number $\frac{1}{2}-r$ is a root whenever r is. The two numbers r and $\frac{1}{2}-r$ are different unless $r=\frac{1}{4}$, and $\frac{1}{4}$ is not a root, because $f\left(\frac{1}{4}\right)=2\left(\frac{1}{4}\right)^{2001}\neq 0$. So the 2000 roots split into 1000 pairs $\left\{r,\frac{1}{2}-r\right\}$, each with sum $\frac{1}{2}$, and the sum of all the roots is $1000\cdot\frac{1}{2}=500$.

As a check with Vieta's formulas, the coefficient of x^{1999} in f is $\binom{2001}{2}\left(\frac{1}{2}\right)^2(-1)^{1999}=-500250$, so the sum of the roots is $500250\big/\frac{2001}{2}=500$.

Answer: 500. □

12. (2015 AIME II, Problem #14) Let x and y be real numbers satisfying $x^4y^5+y^4x^5=810$ and $x^3y^6+y^3x^6=945$. Evaluate $2x^3+(xy)^3+2y^3$.

Strategy. Both given expressions and the required one are symmetric in x and y, so by the Symmetric Polynomials theorem each can be written with $e_1=x+y$ and $e_2=xy$.

Solution. Let $e_1=x+y$ and $e_2=xy$. The two equations are

$$x^4y^4(x+y)=e_2^4e_1=810 \qquad \text{and} \qquad x^3y^3(x^3+y^3)=e_2^3\left(e_1^3-3e_1e_2\right)=945.$$

The second is $(e_1e_2)^3-3e_2^4e_1=945$, so $(e_1e_2)^3=945+3\cdot 810=3375$ and $e_1e_2=15$. Then

$$e_2^3=\frac{e_2^4e_1}{e_1e_2}=\frac{810}{15}=54, \qquad e_1^3=\frac{(e_1e_2)^3}{e_2^3}=\frac{3375}{54}=\frac{125}{2}.$$

The required value is

$$2(x^3+y^3)+(xy)^3=2\left(e_1^3-3e_1e_2\right)+e_2^3=2\left(\frac{125}{2}-45\right)+54=35+54=89.$$

Answer: 89. □

13. (2019 AIME I, Problem #8) Let x be a real number such that $\sin^{10}x+\cos^{10}x=\frac{11}{36}$. Then $\sin^{12}x+\cos^{12}x=\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Strategy. Let $u=\sin^2x$ and $v=\cos^2x$. Then $u+v=1$, and the given and required sums are the power sums p_5 and p_6 of u and v. Newton's sums express them through $t=uv$.

Solution. Let $u=\sin^2x$, $v=\cos^2x$ and $t=uv$. Then $e_1=u+v=1$ and $e_2=t$, so the power sums $p_k=u^k+v^k$ satisfy $p_k=p_{k-1}-tp_{k-2}$, with $p_0=2$ and $p_1=1$:

$$p_2=1-2t, \qquad p_3=1-3t,$$
$$p_4=1-4t+2t^2, \qquad p_5=1-5t+5t^2,$$
$$p_6=1-6t+9t^2-2t^3.$$

The condition $p_5 = \frac{11}{36}$ gives $5t^2 - 5t + \frac{25}{36} = 0$, that is, $36t^2 - 36t + 5 = (6t-1)(6t-5) = 0$. Since $(u-v)^2 = e_1^2 - 4e_2 = 1 - 4t \geq 0$, $t \leq \frac{1}{4}$, so $t = \frac{1}{6}$. Then

$$p_6 = p_5 - tp_4 = \frac{11}{36} - \frac{1}{6}\Big(1 - \frac{4}{6} + \frac{2}{36}\Big) = \frac{11}{36} - \frac{1}{6}\cdot\frac{7}{18} = \frac{33}{108} - \frac{7}{108} = \frac{13}{54}.$$

Since 13 and 54 are relatively prime, $m + n = 67$.

Answer: 67. □

14. (1990 AIME, Problem #15) Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

Strategy. Let $s_n = ax^n + by^n$. The numbers x and y are the roots of $t^2 - (x+y)t + xy$. As in the proof of Newton's sums, multiply $x^2 = (x+y)x - xy$ by ax^n and $y^2 = (x+y)y - xy$ by by^n, and add. This gives a relation among three consecutive s_n whose coefficients $x+y$ and xy are unknown.

Solution. Let $s_n = ax^n + by^n$, $e_1 = x + y$ and $e_2 = xy$. The numbers x and y are the roots of $t^2 - e_1 t + e_2$, so $x^2 = e_1 x - e_2$ and $y^2 = e_1 y - e_2$. Multiplying the first by ax^n, the second by by^n, and adding,

$$s_{n+2} = e_1 s_{n+1} - e_2 s_n \qquad (n \geq 1).$$

With $n = 1$ and $n = 2$ this gives

$$16 = 7e_1 - 3e_2, \qquad 42 = 16e_1 - 7e_2.$$

Multiplying the first equation by 7 and the second by 3 and subtracting gives $112 - 126 = 49e_1 - 48e_1$, so $e_1 = -14$, and then $3e_2 = 7(-14) - 16 = -114$, so $e_2 = -38$. Therefore

$$ax^5 + by^5 = s_5 = e_1 s_4 - e_2 s_3 = -14 \cdot 42 + 38 \cdot 16 = -588 + 608 = 20.$$

Answer: 20. □

15. (2014 AIME II, Problem #5) Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r + 4$ and $s - 3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

Strategy. Let u be the third root of p and w the third root of q. Neither cubic has an x^2 term, so each set of roots adds up to 0. The two cubics have the same coefficient a of x, and their constant terms differ by 240.

Solution. Since r and s are two of the three roots of p, the roots of p are r, s and a third number u, and likewise the roots of q are $r+4$, $s-3$ and a third number w. Neither cubic has an x^2 term, so by Vieta's formulas

$$u = -(r+s), \qquad w = -(r+4) - (s-3) = -(r+s+1).$$

The coefficient of x is a in both cubics. For p it is $rs + u(r+s) = rs - (r+s)^2$, and for q it is $(r+4)(s-3) + w(r+s+1) = (r+4)(s-3) - (r+s+1)^2$. Setting these equal and expanding,

$$rs - (r+s)^2 = rs - 3r + 4s - 12 - (r+s)^2 - 2(r+s) - 1,$$

so $5r - 2s + 13 = 0$ and $s = \frac{5r+13}{2}$.

The constant terms give $b = -rsu = rs(r+s)$ and $b + 240 = -(r+4)(s-3)w = (r+4)(s-3)(r+s+1)$. Subtracting,

$$(r+4)(s-3)(r+s+1) - rs(r+s) = 240.$$

Substituting $s = \frac{5r+13}{2}$, so that $s - 3 = \frac{5r+7}{2}$, $r + s = \frac{7r+13}{2}$ and $r + s + 1 = \frac{7r+15}{2}$, and multiplying by 4,

$$(r+4)(5r+7)(7r+15) - r(5r+13)(7r+13) = 960.$$

The left side is $(35r^3 + 264r^2 + 601r + 420) - (35r^3 + 156r^2 + 169r) = 108r^2 + 432r + 420$, so $108r^2 + 432r - 540 = 0$, that is, $r^2 + 4r - 5 = (r-1)(r+5) = 0$.

If $r = 1$, then $s = 9$, $u = -10$ and $b = rs(r+s) = 90$. If $r = -5$, then $s = -6$, $u = 11$ and $b = (-5)(-6)(-11) = -330$. Both cases occur: $x^3 - 91x + 90$ has the roots 1, 9, -10 and $x^3 - 91x + 330$ has the roots 5, 6, -11; $x^3 - 91x - 330$ has the roots -5, -6, 11 and $x^3 - 91x - 90$ has the roots -1, -9, 10. The possible values of $|b|$ are 90 and 330, and their sum is 420.

Answer: 420. □

2. Answer Key

Problem	Letter	Value
1	D	720
2	A	$x^{-2}y^{-2}z^{-2}$
3	B	0
4	D	10
5	C	$\frac{2AC-B^2}{A^2}$
6	E	none of these
7	B	$\frac{29}{2}$
8		753
9	D	$3x^4 - bx^3 - 1 = 0$
10		45
11		500
12		89
13		67
14		20
15		420