SYMMETRIC SUMS, NEWTON'S SUMS, AND ROOT TRANSFORMATIONS FOR AMC 12: PRACTICE SET SOLUTIONS

ANDY JO

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1. Solutions

Beginner.

1. (1996 AHSME, Problem #23) The sum of the lengths of the twelve edges of a rectangular box is 140, and the distance from one corner of the box to the farthest corner is 21. The total surface area of the box is

(A) 776 **(B)** 784 **(C)** 798 **(D)** 800 **(E)** 812

Strategy. Let the edges have lengths a, b, c. The edges give $e_1 = a + b + c$, the diagonal gives $a^2 + b^2 + c^2$, and the surface area is $2e_2$, so part (a) of the Three Numbers proposition gives the area without the individual edges.

Solution. Let the edge lengths be a, b and c, and let $e_1 = a + b + c$ and $e_2 = ab + bc + ca$. The twelve edges consist of four edges of each length, so $4e_1 = 140$ and $e_1 = 35$. The corner farthest from a given corner is the opposite one. By the Pythagorean theorem, applied once on a face and once more with the third edge, the distance between them is $\sqrt{a^2 + b^2 + c^2}$, so $a^2 + b^2 + c^2 = 21^2 = 441$. The total surface area is $2(ab + bc + ca) = 2e_2$, and part (a) of the Three Numbers proposition says that $a^2 + b^2 + c^2 = e_1^2 - 2e_2$. Therefore

$$2e_2 = e_1^2 - (a^2 + b^2 + c^2) = 1225 - 441 = 784.$$

Such a box exists: the edges 14, 14 and 7 have sum 35 and sum of squares $196 + 196 + 49 = 441$.

Answer: (B) 784. □

2. (2007 AMC 10A, Problem #20) Suppose that the number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$?

(A) 164 **(B)** 172 **(C)** 192 **(D)** 194 **(E)** 212

Strategy. The numbers a and a^{-1} have sum 4 and product 1. Newton's sums for two numbers give their power sums $a^k + a^{-k}$ one at a time.

Solution. The equation contains a^{-1}, so $a \neq 0$. The two numbers a and a^{-1} have $e_1 = a + a^{-1} = 4$ and $e_2 = a \cdot a^{-1} = 1$. Let $p_k = a^k + a^{-k}$. By Newton's sums for two numbers, $p_k = e_1 p_{k-1} - e_2 p_{k-2} = 4p_{k-1} - p_{k-2}$ for $k \geq 2$, with $p_0 = 2$ and $p_1 = 4$. Hence

$$p_2 = 16 - 2 = 14, \qquad p_3 = 56 - 4 = 52, \qquad p_4 = 208 - 14 = 194.$$

As a check, the numbers a^2 and a^{-2} have sum 14 and product 1, so the sum of their squares is $14^2 - 2 \cdot 1 = 194$.

Answer: (D) 194. □

3. (2005 AMC 12B, Problem #12) The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of m, n, and p is zero. What is the value of $\dfrac{n}{p}$?

(A) 1 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

Strategy. Doubling the roots of $x^2 + px + m$ is the case $k = 2$ of the Shifted and Scaled Roots theorem, so the monic quadratic with the doubled roots is $x^2 + 2px + 4m$. Compare its coefficients with those of $x^2 + mx + n$.

Solution. Let $P(x) = x^2 + px + m$ have roots r and s. By part (b) of the Shifted and Scaled Roots theorem with $k = 2$, the polynomial

$$2^2 P\Big(\frac{x}{2}\Big) = x^2 + 2px + 4m$$

is monic with roots $2r$ and $2s$. The polynomial $x^2 + mx + n$ is also monic with roots $2r$ and $2s$. Both are equal to $(x-2r)(x-2s)$, so their coefficients agree: $m = 2p$ and $n = 4m$. Hence $n = 8p$, and since $p \neq 0$, $\dfrac{n}{p} = 8$. For example, $x^2 - x - 2$ has the roots 2 and -1, and $x^2 - 2x - 8$ has the roots 4 and -2; here $p = -1$, $m = -2$ and $n = -8$.

Answer: (D) 8. □

4. (1960 AHSME, Problem #31) For $x^2 + 2x + 5$ to be a factor of $x^4 + px^2 + q$, the values of p and q must be, respectively:

(A) -2, 5 **(B)** 5, 25 **(C)** 10, 20 **(D)** 6, 25 **(E)** 14, 25

Strategy. The polynomial $x^4 + px^2 + q$ is $Q(x^2)$ with $Q(y) = y^2 + py + q$. If r and s are the roots of $x^2 + 2x + 5$, then r^2 and s^2 must be the roots of Q, and the Squared Roots proposition gives the monic quadratic with these roots.

Solution. Let r and s be the roots of $x^2 + 2x + 5$. By Vieta's formulas $r + s = -2$ and $rs = 5$, and $r \neq s$ because the discriminant $4 - 20$ is not 0. Let $Q(y) = y^2 + py + q$, so that $x^4 + px^2 + q = Q(x^2)$.

If $x^2 + 2x + 5$ is a factor of $Q(x^2)$, then r and s are roots of $Q(x^2)$, so $Q(r^2) = Q(s^2) = 0$. The numbers r^2 and s^2 are different, because $r^2 - s^2 = (r - s)(r + s)$ and neither factor is 0. By the Roots and Degree theorem, the monic quadratic Q is then $(y - r^2)(y - s^2)$, the monic quadratic with roots r^2 and s^2. Since $x^2 + 2x + 5 = (x^2 + 5) + x \cdot 2$, the Squared Roots proposition with $E(y) = y + 5$ and $O(y) = 2$ gives

$$Q(y) = (-1)^2\big(E(y)^2 - yO(y)^2\big) = (y + 5)^2 - 4y = y^2 + 6y + 25.$$

So $p = 6$ and $q = 25$. As a check, $r^2 + s^2 = (r + s)^2 - 2rs = -6$ and $r^2s^2 = 25$, and $x^4 + 6x^2 + 25 = (x^2 + 5)^2 - (2x)^2 = (x^2 + 2x + 5)(x^2 - 2x + 5)$, so these values work.

Answer: (D) 6, 25. □

5. (1983 AHSME, Problem #20) If $\tan\alpha$ and $\tan\beta$ are the roots of $x^2 - px + q = 0$, and $\cot\alpha$ and $\cot\beta$ are the roots of $x^2 - rx + s = 0$, then rs is necessarily

(A) pq **(B)** $\dfrac{1}{pq}$ **(C)** $\dfrac{p}{q^2}$ **(D)** $\dfrac{q}{p^2}$ **(E)** $\dfrac{p}{q}$

Strategy. Since $\cot\alpha = 1/\tan\alpha$ and $\cot\beta = 1/\tan\beta$, each root of the second equation is the reciprocal of a root of the first. Reverse the coefficients of $x^2 - px + q$ and divide by the new leading coefficient.

Solution. Both $\tan\alpha$ and $\cot\alpha$ are defined, so $\tan\alpha \neq 0$ and $\cot\alpha = 1/\tan\alpha$; the same holds for β. By Vieta's formulas $q = \tan\alpha\tan\beta \neq 0$. Let $P(x) = x^2 - px + q$. Its constant term is not 0, so by the Reciprocal Roots theorem the polynomial with the coefficients in reverse order,

$$x^2P\Big(\frac{1}{x}\Big) = qx^2 - px + 1,$$

has the roots $\cot\alpha$ and $\cot\beta$. Dividing by q, the monic quadratic with these roots is $x^2 - \frac{p}{q}x + \frac{1}{q}$. It is also $x^2 - rx + s$, so $r = \frac{p}{q}$ and $s = \frac{1}{q}$. Therefore $rs = \dfrac{p}{q^2}$.

Answer: (C) $\dfrac{p}{q^2}$. □

Intermediate.

6. (2014 AMC 12B, Problem #16) Let P be a cubic polynomial with $P(0) = k$, $P(1) = 2k$, and $P(-1) = 3k$. What is $P(2) + P(-2)$?

(A) 0 **(B)** k **(C)** $6k$ **(D)** $7k$ **(E)** $14k$

Strategy. The sums $P(1)+P(-1)$ and $P(2)+P(-2)$ are sums of values of P over the roots of $x^2 - 1$ and of $x^2 - 4$. For two roots t and $-t$ the odd power sums are 0, so only the coefficient of x^2 and the constant term matter.

Solution. Write $P(x) = ax^3 + bx^2 + cx + d$. For a number $t \neq 0$, the roots of $x^2 - t^2$ are t and $-t$, and their power sums are $p_1 = 0$, $p_2 = 2t^2$ and $p_3 = 0$. By the formula for sums of values of a polynomial,

$$P(t) + P(-t) = ap_3 + bp_2 + cp_1 + 2d = 2bt^2 + 2d.$$

Since $P(0) = d$, $d = k$. With $t = 1$, $P(1)+P(-1) = 2b+2k = 5k$, so $2b = 3k$. With $t = 2$,

$$P(2) + P(-2) = 8b + 2d = 4 \cdot 3k + 2k = 14k.$$

Answer: (E) $14k$. □

7. (2003 AIME I, Problem #4) Given that $\log_{10} \sin x + \log_{10} \cos x = -1$ and that $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, find n.

Strategy. For the two numbers $\sin x$ and $\cos x$, the first equation gives the product e_2, and $\sin^2 x + \cos^2 x = 1$ gives the sum of the squares. The identity $p_2 = e_1^2 - 2e_2$ then gives $(\sin x + \cos x)^2$.

Solution. The logarithms in the first equation are defined, so $\sin x > 0$ and $\cos x > 0$. By the product rule for logarithms, the first equation says that $\log_{10}(\sin x \cos x) = -1$. So the two numbers $\sin x$ and $\cos x$ have $e_2 = \sin x \cos x = \frac{1}{10}$, and the sum of their squares is $p_2 = \sin^2 x + \cos^2 x = 1$. Since $p_2 = e_1^2 - 2e_2$,

$$(\sin x + \cos x)^2 = e_1^2 = p_2 + 2e_2 = 1 + \frac{1}{5} = \frac{6}{5}.$$

Since $\sin x + \cos x > 0$, the power and quotient rules for logarithms give

$$\log_{10}(\sin x + \cos x) = \frac{1}{2} \log_{10} \frac{12}{10} = \frac{1}{2}(\log_{10} 12 - 1).$$

Comparing with the second equation, $\log_{10} n = \log_{10} 12$, so $n = 12$.

Answer: 12. □

8. (1977 AHSME, Problem #23) If the solutions of the equation $x^2 + px + q = 0$ are the cubes of the solutions of the equation $x^2 + mx + n = 0$, then

(A) $p = m^3 + 3mn$ **(B)** $p = m^3 - 3mn$ **(C)** $p + q = m^3$ **(D)** $\left(\frac{m}{n}\right)^3 = \frac{p}{q}$

(E) none of these

Strategy. By Vieta's formulas, p is minus the sum of the cubes of the solutions of $x^2 + mx + n = 0$. Newton's sums for two numbers give that sum from $e_1 = -m$ and $e_2 = n$.

Solution. Let u and v be the solutions of $x^2 + mx + n = 0$. By Vieta's formulas, $e_1 = u + v = -m$ and $e_2 = uv = n$. The solutions of $x^2 + px + q = 0$ are u^3 and v^3, so by Vieta's formulas again $p = -(u^3 + v^3)$ and $q = u^3v^3 = n^3$. Newton's sums for two numbers give $u^2 + v^2 = e_1(u + v) - 2e_2 = e_1^2 - 2e_2$ and

$$u^3 + v^3 = e_1(u^2 + v^2) - e_2(u + v) = e_1^3 - 3e_1e_2 = -m^3 + 3mn.$$

Hence $p = m^3 - 3mn$ for every such pair of equations, which is choice (B). The other choices do not always hold. For example, the solutions of $x^2 - 3x + 2 = 0$ are 1 and 2, and the equation with the solutions 1 and 8 is $x^2 - 9x + 8 = 0$; so $m = -3$, $n = 2$, $p = -9$ and $q = 8$. Then $m^3 + 3mn = -45 \neq p$, $p + q = -1 \neq m^3$, and $\left(\frac{m}{n}\right)^3 = -\frac{27}{8} \neq \frac{p}{q}$.

Answer: (B) $p = m^3 - 3mn$. □

9. (1990 AHSME, Problem #30) If $R_n = \frac{1}{2}(a^n + b^n)$, where $a = 3 + 2\sqrt{2}$, $b = 3 - 2\sqrt{2}$, and $n = 0, 1, 2, \ldots$, then R_{12345} is an integer. Its units digit is

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

Strategy. The numbers a and b have sum 6 and product 1, so Newton's sums for two numbers give $R_n = 6R_{n-1} - R_{n-2}$. Each units digit then depends only on the two before it, so the units digits repeat.

Solution. Here $a + b = 6$ and $ab = 9 - 8 = 1$, so a and b are the roots of $t^2 - 6t + 1$, with $e_1 = 6$ and $e_2 = 1$. Newton's sums for two numbers give

$$a^n + b^n = 6\big(a^{n-1} + b^{n-1}\big) - \big(a^{n-2} + b^{n-2}\big) \qquad (n \geq 2),$$

and dividing by 2, $R_n = 6R_{n-1} - R_{n-2}$, with $R_0 = 1$ and $R_1 = 3$. So every R_n is an integer, and the units digit of R_n is determined by the units digits of R_{n-1} and R_{n-2}. The next values are $R_2 = 17$, $R_3 = 99$, $R_4 = 577$, $R_5 = 3363$, $R_6 = 19601$ and $R_7 = 114243$, so the units digits of $R_0, R_1, \ldots, R_7$ are

$$1,\ 3,\ 7,\ 9,\ 7,\ 3,\ 1,\ 3.$$

The digits $1, 3$ of R_6, R_7 are those of R_0, R_1, so the units digits repeat with period 6. Since $12345 = 6 \cdot 2057 + 3$, the units digit of R_{12345} is that of $R_3 = 99$, namely 9.

Answer: (E) 9. □

10. (2013 AIME I, Problem #5) The real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$ can be written in the form $\dfrac{\sqrt[3]{a} + \sqrt[3]{b} + 1}{c}$, where a, b, and c are positive integers. Find $a + b + c$.

Strategy. Reverse the coefficients and change every sign to get a monic cubic whose roots are the reciprocals of the roots. Removing the second-highest term of that cubic leaves $z^3 - 9$.

Solution. Let $P(x) = 8x^3 - 3x^2 - 3x - 1$. Its constant term is not 0, so by the Reciprocal Roots theorem the roots of the reversed polynomial $x^3P(1/x) = -x^3 - 3x^2 - 3x + 8$ are the reciprocals of the roots of P. Multiplying by -1 and writing y for the variable, they are the roots of

$$Q(y) = y^3 + 3y^2 + 3y - 8.$$

By the Removing the Second-Highest Term theorem with $h = \frac{3}{3 \cdot 1} = 1$, substitute $y = z - 1$:

$$Q(z - 1) = (z^3 - 3z^2 + 3z - 1) + (3z^2 - 6z + 3) + (3z - 3) - 8 = z^3 - 9.$$

Let $u = \sqrt[3]{9}$. Then $z^3 - 9 = (z - u)(z^2 + uz + u^2)$, and the quadratic factor has the negative discriminant $u^2 - 4u^2$, so u is the only real root of $z^3 - 9$. Hence $u - 1$ is the only real root of Q, and the real root of P is its reciprocal. Since $u^3 - 1 = 8$ and $u^3 - 1 = (u - 1)(u^2 + u + 1)$,

$$\frac{1}{u - 1} = \frac{u^2 + u + 1}{8} = \frac{\sqrt[3]{81} + \sqrt[3]{9} + 1}{8}.$$

So the form holds with $a = 81$, $b = 9$ and $c = 8$, and $a + b + c = 98$.

Answer: 98. □

Advanced.

11. (1999 AHSME, Problem #30) The number of ordered pairs of integers (m, n) for which $mn \geq 0$ and

$$m^3 + n^3 + 99mn = 33^3$$

is equal to

(A) 2 **(B)** 3 **(C)** 33 **(D)** 35 **(E)** 99

Strategy. With $t = -33$, the equation reads $m^3+n^3+t^3-3mnt = 0$. Part (c) of the Three Numbers proposition factors the left side, and each factor gives its own pairs.

Solution. Let $t = -33$. Then $t^3 = -33^3$ and $-3mnt = 99mn$, so the equation is

$$m^3 + n^3 + t^3 - 3mnt = 0.$$

By part (c) of the Three Numbers proposition, the left side equals

$$(m + n + t)(m^2 + n^2 + t^2 - mn - nt - tm).$$

The second factor equals $\frac{1}{2}\big((m-n)^2+(n-t)^2+(t-m)^2\big)$, so it is 0 only when $m = n = t$, which gives the pair $(-33, -33)$. This pair has $mn = 1089 \geq 0$, so it counts.

The first factor is 0 when $m+n = 33$. If one of m and n were negative, the other would be greater than 33, and mn would be negative. So $0 \leq m \leq 33$ and $n = 33 - m$, which gives 34 pairs, none of them equal to $(-33, -33)$. The total is $34 + 1 = 35$.

Answer: (D) 35. □

12. (2014 AIME I, Problem #9) Let $x_1 < x_2 < x_3$ be the three real roots of the equation $\sqrt{2014}\,x^3 - 4029x^2 + 2 = 0$. Find $x_2(x_1 + x_3)$.

Strategy. Scale the roots by $\sqrt{2014}$. Since $4029 = 2 \cdot 2014 + 1$, the scaled cubic has integer coefficients and the root 1, and the sum of the other two roots follows from Vieta's formulas.

Solution. Let $s = \sqrt{2014}$ and $P(x) = sx^3 - 4029x^2 + 2$. By part (b) of the Shifted and Scaled Roots theorem with $k = s$, the polynomial

$$s^3 P\Big(\frac{x}{s}\Big) = sx^3 - 4029s\,x^2 + 2s^3$$

has the roots sx_1, sx_2, sx_3. Dividing by s, using $s^2 = 2014$ and writing y for the variable, these are the roots of $y^3 - 4029y^2 + 4028$. Its coefficients add up to 0, so $y = 1$ is a root, and the Factor Theorem gives

$$y^3 - 4029y^2 + 4028 = (y - 1)(y^2 - 4028y - 4028).$$

At $y = 1$ the quadratic factor equals $1 - 8056 = -8055 < 0$, so it has two real roots and 1 lies strictly between them. Scaling by $s > 0$ keeps the order $x_1 < x_2 < x_3$, so $sx_2 = 1$ is the middle root, and sx_1 and sx_3 are the roots of the quadratic factor, whose sum is 4028 by Vieta's formulas. Therefore

$$x_2(x_1 + x_3) = \frac{sx_2\,(sx_1 + sx_3)}{s^2} = \frac{1 \cdot 4028}{2014} = 2.$$

Answer: 2. □

13. (2019 AIME I, Problem #10) For distinct complex numbers z_1, z_2, …, z_{673}, the polynomial

$$(x-z_1)^3(x-z_2)^3\cdots(x-z_{673})^3$$

can be expressed as $x^{2019}+20x^{2018}+19x^{2017}+g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left|\sum_{1\le j<k\le 673} z_jz_k\right|$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Strategy. The polynomial has 2019 roots, each z_j listed three times. Read the first two elementary symmetric sums of this list from the coefficients, and write them in terms of $S_1=\sum z_j$ and $S_2=\sum_{j<k} z_jz_k$ with the identity $p_2=e_1^2-2e_2$.

Solution. The polynomial is monic of degree 2019, and its roots are the 2019 numbers $z_1, z_1, z_1, \ldots, z_{673}, z_{673}, z_{673}$, each z_j three times. They may be non-real; Vieta's formulas hold for this list all the same. Let E_1 and E_2 be the first two elementary symmetric sums of the list and P_2 the sum of the squares of its numbers. By Vieta's formulas, $E_1=-20$ and $E_2=19$.

Let $S_1=z_1+\cdots+z_{673}$ and $S_2=\sum_{1\le j<k\le 673} z_jz_k$. Since each z_j appears three times, $E_1=3S_1$, and by the identity $p_2=e_1^2-2e_2$ for $z_1,\ldots,z_{673}$,

$$P_2=3\big(z_1^2+\cdots+z_{673}^2\big)=3\big(S_1^2-2S_2\big).$$

The same identity for the list of 2019 roots gives

$$E_2=\frac{E_1^2-P_2}{2}=\frac{9S_1^2-3S_1^2+6S_2}{2}=3S_1^2+3S_2.$$

Now $S_1=-\frac{20}{3}$, so $19=3\cdot\frac{400}{9}+3S_2=\frac{400}{3}+3S_2$. Hence $3S_2=\frac{57-400}{3}=-\frac{343}{3}$ and $S_2=-\frac{343}{9}$. The required value is $|S_2|=\frac{343}{9}$, and since $343=7^3$ and $9=3^2$ are relatively prime, $m+n=343+9=352$.

Answer: 352. □

14. (2024 AIME II, Problem #11) Find the number of triples of nonnegative integers (a,b,c) satisfying $a+b+c=300$ and

$$a^2b+a^2c+b^2a+b^2c+c^2a+c^2b=6{,}000{,}000.$$

Strategy. By part (b) of the Three Numbers proposition, the six-term sum is $e_1e_2-3e_3$. With $e_1=300$ the condition becomes a linear relation between e_2 and e_3, and that relation says that the product $(100-a)(100-b)(100-c)$ is 0.

Solution. Let $e_1=a+b+c=300$, $e_2=ab+bc+ca$ and $e_3=abc$. By part (b) of the Three Numbers proposition, the left side of the second equation is $e_1e_2-3e_3=300e_2-3e_3$, so the condition is $300e_2-3e_3=6{,}000{,}000$, that is,

$$100e_2-e_3=2{,}000{,}000.$$

The product $(100-a)(100-b)(100-c)$ is the value at $x=100$ of $(x-a)(x-b)(x-c)=x^3-e_1x^2+e_2x-e_3$, so

$$(100-a)(100-b)(100-c)=10^6-3\cdot10^6+100e_2-e_3=100e_2-e_3-2{,}000{,}000.$$

Hence a triple with $a+b+c=300$ satisfies the condition exactly when one of the factors $100-a$, $100-b$, $100-c$ is 0, that is, when 100 is among the numbers a, b, c.

There are 201 triples with $a = 100$, since then $b+c = 200$ and b can be any of $0, 1, \ldots, 200$. Likewise there are 201 triples with $b = 100$ and 201 with $c = 100$. If two of the numbers equal 100, then so does the third, so any two of these three sets have only the triple $(100, 100, 100)$ in common. By inclusion-exclusion, the number of triples is

$$3 \cdot 201 - 3 \cdot 1 + 1 = 601.$$

Answer: 601. □

15. (2026 AIME II, Problem #11) Find the greatest integer n such that the cubic polynomial

$$x^3 - \frac{n}{6}x^2 + (n-11)x - 400$$

has roots α^2, β^2, and γ^2, where α, β, and γ are complex numbers, and there are exactly seven different possible values for $\alpha + \beta + \gamma$.

Strategy. Changing the signs of all three of α, β, γ turns each possible value of $\alpha+\beta+\gamma$ into its negative, so an odd number of values, such as seven, must include 0. For a choice with $\alpha + \beta + \gamma = 0$, the cubic with roots α^2, β^2, γ^2 depends only on e_2 and e_3, and comparing it with the given cubic ties n to e_2 in two ways.

Solution. Let r_1, r_2, r_3 be the roots of the cubic. Their product is 400, so none of them is 0. For each i let u_i be a root of $x^2 - r_i$; the roots of $x^2 - r_i$ are then u_i and $-u_i$, which are different. In some order, α, β, γ are $\pm u_1$, $\pm u_2$, $\pm u_3$, so the possible values of $\alpha + \beta + \gamma$ are the eight numbers $\pm u_1 \pm u_2 \pm u_3$, and changing all three signs turns a value v into $-v$.

Two of these sums are equal exactly when the terms in the places where their signs differ, taken with the signs of the first sum, add up to 0. For one place this would make some u_i equal to 0, which is impossible. For two places it says that $u_i = \pm u_j$ for some $i \neq j$, so $r_i = r_j$. For three places the sums are v and $-v$, equal exactly when $v = 0$.

If two roots are equal, say $r_1 = r_2$ with $u_2 = u_1$, then the four sums with opposite signs on u_1 and u_2 take only the two values $\pm u_3$, so there are at most six values. If the roots are distinct, the only possible coincidence is $v = -v$ with $v = 0$, and it happens for at most one pair, since two sums equal to 0 that are not negatives of each other would differ in one or two places. So there are exactly seven values precisely when the roots are distinct and some choice gives $\alpha + \beta + \gamma = 0$.

For such a choice, let $e_1 = \alpha + \beta + \gamma = 0$, $e_2 = \alpha\beta + \beta\gamma + \gamma\alpha$ and $e_3 = \alpha\beta\gamma$. By parts (a) and (d) of the Three Numbers proposition, the monic cubic with roots α^2, β^2, γ^2 is

$$x^3 - (e_1^2 - 2e_2)x^2 + (e_2^2 - 2e_1e_3)x - e_3^2 = x^3 + 2e_2x^2 + e_2^2x - e_3^2.$$

Comparing coefficients with the given cubic, $\frac{n}{6} = -2e_2$ and $n - 11 = e_2^2$. So $e_2 = -\frac{n}{12}$ and $n - 11 = \frac{n^2}{144}$, that is,

$$n^2 - 144n + 1584 = (n-12)(n-132) = 0, \qquad \text{so} \qquad n = 12 \text{ or } n = 132.$$

For $n = 132$, let α, β, γ be the roots of $t^3 - 11t - 20$. This cubic has no t^2 term, so $\alpha + \beta + \gamma = 0$. By the Squared Roots proposition with $E(y) = -20$ and $O(y) = y - 11$, the monic cubic with roots α^2, β^2, γ^2 is

$$-\big(E(y)^2 - yO(y)^2\big) = y(y-11)^2 - 400 = y^3 - 22y^2 + 121y - 400,$$

which is the given cubic for $n = 132$, since $\frac{132}{6} = 22$ and $132 - 11 = 121$. Its roots are distinct, since it equals $(y-16)(y^2 - 6y + 25)$ and the quadratic factor, with the negative discriminant $36 - 100$, has two different non-real roots. So for $n = 132$ there are exactly seven values, and the greatest n is 132.

Answer: 132. □

2. Answer Key

Problem	Letter	Value
1	B	784
2	D	194
3	D	8
4	D	6, 25
5	C	$\frac{p}{q^2}$
6	E	$14k$
7		12
8	B	$p = m^3 - 3mn$
9	E	9
10		98
11	D	35
12		2
13		352
14		601
15		132