POLYNOMIALS: FACTOR THEOREM, RATIONAL ROOTS, AND VIETA FOR AMC 12

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1. Polynomial Division

This section divides one polynomial by another and relates the remainder upon division by $x-a$ to the value of the polynomial at a.

1.1. Long Division.

Definition 1.1.1. A **polynomial** in x is an expression

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and the **coefficients** $a_0, a_1, \ldots, a_n$ are real numbers. If $a_n \neq 0$, the **degree** of P is n, written $\deg P$, and a_n is the **leading coefficient**. The polynomial P is **monic** if its leading coefficient is 1. The number a_0 is the **constant term**. The polynomial whose coefficients are all 0 is the **zero polynomial**; it has no degree. A **root** (or **zero**) of P is a number r with $P(r) = 0$.

Substituting $x = 0$ and $x = 1$ gives $P(0) = a_0$, the constant term, and $P(1) = a_n + a_{n-1} + \cdots + a_0$, the sum of the coefficients. The degree of a product of two polynomials is the sum of their degrees, because the product of the leading terms is the leading term of the product. The degree of a sum or a difference is at most the larger of the two degrees, and it is smaller when the leading terms cancel. For example, $(x^3 + 2x^2 - 1) - (x^3 - 5x + 4) = 2x^2 + 5x - 5$ has degree 2.

Theorem 1.1.2 (Division Algorithm)**.** *Let P and D be polynomials, where D is not the zero polynomial. There are unique polynomials Q and R such that*

$$P(x) = D(x)Q(x) + R(x),$$

*where R is the zero polynomial or $\deg R < \deg D$. The polynomial Q is the **quotient** and R is the **remainder** upon division of P by D. If R is the zero polynomial, then D is a **factor** of P, and P is **divisible** by D.*

Proof. Long division gives Q and R. Each step subtracts from what is left of P a multiple $cx^k D(x)$ that cancels the leading term of what is left, so the degree of what is left drops. The process stops when what is left is 0 or has degree less than $\deg D$. What is left is then R, and the multiples cx^k add up to Q. For uniqueness, suppose that $DQ_1 + R_1 = DQ_2 + R_2$. Then $D(Q_1 - Q_2) = R_2 - R_1$. The right side is 0 or has degree less than $\deg D$, while the left side, if it is not 0, has degree at least $\deg D$. So both sides are 0, which gives $Q_1 = Q_2$ and $R_1 = R_2$. □

Remark 1.1.3 (Integer and Rational Coefficients)**.** Each step of long division divides the leading coefficient of what is left by the leading coefficient of D. So if P and D have integer coefficients and D is monic, then Q and R have integer coefficients. If P and D have rational coefficients, then so do Q and R.

Example.

Problem 1.1.4. Find the quotient and the remainder when $2x^4 + x^3 - 7x^2 + 8x - 3$ is divided by $x^2 + 2x - 1$.

Strategy. At each step, divide the leading term of what is left by x^2, the leading term of the divisor, and subtract that multiple of the divisor. Stop when what is left has degree less than 2.

Solution. The three steps are

$$(2x^4 + x^3 - 7x^2 + 8x - 3) - 2x^2(x^2 + 2x - 1) = -3x^3 - 5x^2 + 8x - 3,$$
$$(-3x^3 - 5x^2 + 8x - 3) - (-3x)(x^2 + 2x - 1) = x^2 + 5x - 3,$$
$$(x^2 + 5x - 3) - 1 \cdot (x^2 + 2x - 1) = 3x - 2.$$

The last expression has degree $1 < 2$, so it is the remainder, and the multiples used add up to the quotient $2x^2 - 3x + 1$. As a check,

$$(x^2 + 2x - 1)(2x^2 - 3x + 1) + (3x - 2) = (2x^4 + x^3 - 7x^2 + 5x - 1) + (3x - 2),$$

which is the original polynomial.

Answer: quotient $2x^2 - 3x + 1$, remainder $3x - 2$. □

1.2. The Remainder Theorem.

Theorem 1.2.1 (Remainder Theorem)**.** *The remainder when a polynomial $P(x)$ is divided by $x - a$ is $P(a)$.*

Proof. By the Division Algorithm, $P(x) = (x - a)Q(x) + r$, where the remainder r is a constant because it is 0 or has degree less than 1. Substituting $x = a$ gives $P(a) = r$. □

For example, the remainder when $x^3 - 2x + 5$ is divided by $x - 2$ is $2^3 - 2 \cdot 2 + 5 = 9$, and the remainder when it is divided by $x + 1 = x - (-1)$ is $(-1)^3 - 2 \cdot (-1) + 5 = 6$.

When $a \neq b$, the remainder upon division by $(x - a)(x - b)$ is 0 or has degree at most 1, so it can be written as $cx + d$. Substituting $x = a$ and $x = b$ into

$$P(x) = (x - a)(x - b)Q(x) + cx + d$$

gives $ca + d = P(a)$ and $cb + d = P(b)$, two linear equations for c and d. So the remainder is the polynomial whose graph is the line through $(a, P(a))$ and $(b, P(b))$.

Example.

Problem 1.2.2 (1976 AHSME, Problem #19)**.** A polynomial $p(x)$ has remainder three when divided by $x - 1$ and remainder five when divided by $x - 3$. The remainder when $p(x)$ is divided by $(x - 1)(x - 3)$ is

(A) $x - 2$ **(B)** $x + 2$ **(C)** 2 **(D)** 8 **(E)** 15

Strategy. The divisor has degree 2, so the remainder has the form $cx + d$. The Remainder Theorem gives $p(1)$ and $p(3)$.

Solution. By the Division Algorithm, $p(x) = (x - 1)(x - 3)Q(x) + cx + d$ for some polynomial Q and numbers c and d. By the Remainder Theorem, $p(1) = 3$ and $p(3) = 5$. Substituting $x = 1$ and $x = 3$ gives

$$c + d = 3, \qquad 3c + d = 5,$$

so $c = 1$ and $d = 2$. The remainder is $x + 2$.

Answer: (B) $x + 2$. □

1.3. The Factor Theorem.

Theorem 1.3.1 (Factor Theorem)**.** *The polynomial $x - a$ is a factor of $P(x)$ if and only if $P(a) = 0$.*

Proof. By the Remainder Theorem, the remainder upon division by $x - a$ is $P(a)$, and it is 0 exactly when $P(a) = 0$. □

Theorem 1.3.2 (Roots and Degree)**.** *If $r_1, r_2, \ldots, r_k$ are distinct roots of a polynomial P, then*

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_k)Q(x)$$

for some polynomial Q. Consequently, a polynomial of degree n has at most n distinct roots.

Proof. Since $P(r_1) = 0$, the Factor Theorem gives $P(x) = (x - r_1)P_1(x)$. For $i \geq 2$, substituting $x = r_i$ gives $0 = (r_i - r_1)P_1(r_i)$, and $r_i \neq r_1$, so $P_1(r_i) = 0$. Repeating the argument with P_1 and the roots $r_2, \ldots, r_k$ gives the factorization. If P has degree n and k distinct roots, then Q is not the zero polynomial, and comparing degrees gives $n = k + \deg Q \geq k$. □

Corollary 1.3.3 (Identity)**.** *If two polynomials of degree at most n take equal values at $n + 1$ different numbers, then they are equal: their coefficients agree.*

Proof. Their difference is the zero polynomial or has degree at most n, and it has $n + 1$ distinct roots. By the Roots and Degree theorem it cannot have degree at most n, so it is the zero polynomial. □

Corollary 1.3.4 (Integer Values)**.** *If P has integer coefficients and $a \neq b$ are integers, then $a - b$ divides $P(a) - P(b)$.*

Proof. The polynomial $P(x) - P(b)$ has integer coefficients and the root b, so $P(x) - P(b) = (x - b)Q(x)$ by the Factor Theorem. The divisor $x - b$ is monic with integer coefficients, so Q has integer coefficients by Remark 1.1.3. Hence $P(a) - P(b) = (a - b)Q(a)$, where $Q(a)$ is an integer. □

Example.

Problem 1.3.5 (1961 AHSME, Problem #22)**.** If $3x^3 - 9x^2 + kx - 12$ is divisible by $x - 3$, then it is also divisible by:

(A) $3x^2 - x + 4$ **(B)** $3x^2 - 4$ **(C)** $3x^2 + 4$ **(D)** $3x - 4$ **(E)** $3x + 4$

Strategy. The value at $x = 3$ must be 0, which gives k. Grouping the terms in pairs then shows the other factor.

Solution. By the Factor Theorem, the polynomial is 0 at $x = 3$:

$$3 \cdot 27 - 9 \cdot 9 + 3k - 12 = 3k - 12 = 0,$$

so $k = 4$. Grouping terms,

$$3x^3 - 9x^2 + 4x - 12 = 3x^2(x - 3) + 4(x - 3) = (x - 3)(3x^2 + 4).$$

So the polynomial is also divisible by $3x^2 + 4$.

Answer: (C) $3x^2 + 4$. □

2. Roots

This section finds rational roots, treats repeated roots, and shows that, for a polynomial with rational coefficients, irrational roots of the form $a + b\sqrt{c}$ come in pairs.

2.1. **Rational Roots.**

Theorem 2.1.1 (Rational Root Theorem)**.** *Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ have integer coefficients, with $a_n \neq 0$ and $a_0 \neq 0$. If p/q is a root of P, where p and q are integers with no common factor greater than 1 and $q > 0$, then p divides a_0 and q divides a_n.*

Proof. Multiplying $P(p/q) = 0$ by q^n gives

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0.$$

Every term except $a_0 q^n$ is a multiple of p, so p divides $a_0 q^n$. Since p and q have no common prime factor, p divides a_0. In the same way, every term except $a_n p^n$ is a multiple of q, so q divides $a_n p^n$, and therefore q divides a_n. □

Corollary 2.1.2 (Integer Roots)**.** *An integer root of a polynomial with integer coefficients divides the constant term. If the polynomial is monic, every rational root is an integer.*

To find the rational roots of a polynomial with integer coefficients, list the candidates $\pm p/q$ allowed by the theorem and test them. When a root r is found, divide by $x - r$: the quotient has lower degree, and its roots are the remaining roots. For a root p/q, dividing by $qx - p$, a constant multiple of $x - p/q$, gives a quotient with the same roots. If $a_0 = 0$, first factor out the largest power of x.

Examples.

Problem 2.1.3. Find all roots of the polynomial $2x^3 - 3x^2 - 11x + 6$.

Strategy. A rational root p/q in lowest terms has p dividing 6 and q dividing 2. Test the candidates, and divide out a root once one is found.

Solution. By the Rational Root Theorem, the candidates are ± 1, ± 2, ± 3, ± 6, $\pm\frac{1}{2}$ and $\pm\frac{3}{2}$. Let $P(x) = 2x^3 - 3x^2 - 11x + 6$. Then $P(1) = -6$, $P(-1) = 12$, $P(2) = -12$, and

$$P(-2) = -16 - 12 + 22 + 6 = 0.$$

Dividing by $x + 2$,

$$2x^3 - 3x^2 - 11x + 6 = (x + 2)(2x^2 - 7x + 3) = (x + 2)(2x - 1)(x - 3).$$

The roots are -2, $\frac{1}{2}$ and 3.

Answer: -2, $\frac{1}{2}$, and 3. □

Problem 2.1.4 (2017 AMC 12A, Problem #21)**.** A set S is constructed as follows. To begin, $S = \{0, 10\}$. Repeatedly, as long as possible, if x is an integer root of some polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for some $n \geq 1$, all of whose coefficients a_i are elements of S, then x is put into S. When no more elements can be added to S, how many elements does S have?

(A) 4 **(B)** 5 **(C)** 7 **(D)** 9 **(E)** 11

Strategy. An integer root divides the constant term. Show that every element of S is 0 or a divisor of 10, and then show that each of these numbers is put into S.

Solution. Let $T = \{0, \pm 1, \pm 2, \pm 5, \pm 10\}$. Suppose that every element of S so far lies in T, and let $m \neq 0$ be an integer root of a polynomial with coefficients in S, not the zero polynomial. Dividing the polynomial by the largest power of x that divides it gives a polynomial that still has the root m, has integer coefficients, and has a nonzero constant term a in S. By the Integer Roots corollary, m divides a, and a is one of ± 1, ± 2, ± 5, ± 10, so m is in T. Hence S never leaves T, and S has at most 9 elements.

All nine elements of T are put into S, in this order, each by a polynomial whose coefficients are already in S:

- $10x + 10$ has the root -1;
- $-x^3 - x + 10$ has the root 2, since $-8 - 2 + 10 = 0$;
- $2x + 10$ has the root -5;
- $-x^2 - x + 2 = -(x + 2)(x - 1)$ has the roots -2 and 1;
- $x - 5$ has the root 5, and $x + 10$ has the root -10.

So S ends with 9 elements.

Answer: (D) 9. □

2.2. **Multiple Roots.**

Definition 2.2.1. A root r of a polynomial P, not the zero polynomial, has **multiplicity** m if $(x-r)^m$ is a factor of P but $(x-r)^{m+1}$ is not. A root of multiplicity 1, 2 or 3 is a **simple**, **double** or **triple** root.

If r has multiplicity m, then $P(x) = (x-r)^m Q(x)$ with $Q(r) \neq 0$, since otherwise the Factor Theorem would give one more factor $x - r$.

Theorem 2.2.2 (Factorization with Multiplicities)**.** *If the distinct roots $r_1, \ldots, r_k$ of a polynomial P have multiplicities $m_1, \ldots, m_k$, then*

$$P(x) = (x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_k)^{m_k}Q(x)$$

for some polynomial Q, so $m_1 + m_2 + \cdots + m_k \leq \deg P$.

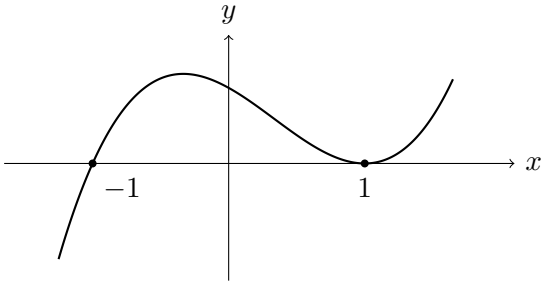
Proof. Write $P(x) = (x-r_1)^{m_1}P_1(x)$, and for $i \geq 2$ write $P(x) = (x-r_i)^{m_i}S(x)$. Setting $x = r_i$ in both gives $(r_i - r_1)^{m_1}P_1(r_i) = 0$, and $r_i \neq r_1$, so $P_1(r_i) = 0$ and $P_1(x) = (x-r_i)P_2(x)$ by the Factor Theorem. Now P equals $x - r_i$ times $(x-r_1)^{m_1}P_2(x)$ and also $x - r_i$ times $(x-r_i)^{m_i-1}S(x)$, so these two quotients are equal by the uniqueness of the quotient in the Division Algorithm. Repeating this step m_i times shows that $(x-r_i)^{m_i}$ is a factor of P_1. The same argument applied to P_1 and the roots $r_2, \ldots, r_k$, and so on, gives the factorization. The polynomial Q is not the zero polynomial, because P is not, so comparing degrees gives the inequality. □

For example, $x^2(x-3)^3(x+1)$ has the double root 0, the triple root 3 and the simple root -1, and $2+3+1=6$ is its degree.

Theorem 2.2.3 (Sign near a Root)**.** *Let r be a root of P of multiplicity m. If m is odd, then $P(x)$ changes sign as x increases through r. If m is even, then $P(x)$ has the same sign just to the left and just to the right of r.*

Proof. Write $P(x) = (x-r)^m Q(x)$ with $Q(r) \neq 0$. The graph of a polynomial is unbroken, so a polynomial changes sign only at its roots. Since Q has finitely many roots and r is not one of them, $Q(x)$ keeps the sign of $Q(r)$ for all x close enough to r. The factor $(x-r)^m$ changes sign at r exactly when m is odd. □

In a graph, the curve crosses the x-axis at a root of odd multiplicity and touches it without crossing at a root of even multiplicity. In particular, a polynomial that is never negative has only roots of even multiplicity. The graph of $y = (x+1)(x-1)^2$ below crosses the axis at the simple root -1 and touches it at the double root 1.



Example.

Problem 2.2.4 (1980 AHSME, Problem #24)**.** For some real number r, the polynomial

$$8x^3 - 4x^2 - 42x + 45$$

is divisible by $(x-r)^2$. Which of the following numbers is closest to r?

(A) 1.22 **(B)** 1.32 **(C)** 1.42 **(D)** 1.52 **(E)** 1.62

Strategy. Find the roots with the Rational Root Theorem. The complete factorization shows which root is repeated.

Solution. Let $P(x) = 8x^3 - 4x^2 - 42x + 45$. The candidates for rational roots are $\pm p/q$ with p dividing 45 and q dividing 8. The candidates are tested in order of absolute value. None of the eighteen candidates smaller than $\frac{3}{2}$ in absolute value is a root (for example, $P(1) = 7$, $P(-1) = 75$, $P(\frac{1}{2}) = 24$ and $P(-\frac{1}{2}) = 64$), but

$$P\Big(\frac{3}{2}\Big) = 27 - 9 - 63 + 45 = 0.$$

Dividing by $2x - 3$,

$$8x^3 - 4x^2 - 42x + 45 = (2x - 3)(4x^2 + 4x - 15) = (2x - 3)^2(2x + 5).$$

If $(x - r)^2$ divides P, then r is a root of multiplicity at least 2. The roots are $\frac{3}{2}$, of multiplicity 2, and $-\frac{5}{2}$, which is simple because $(2x-3)^2$ is not 0 at $x = -\frac{5}{2}$. Hence $r = \frac{3}{2} = 1.5$, and the closest of the given numbers is 1.52.

Answer: (D) 1.52. □

2.3. **Irrational Conjugate Roots.**

Theorem 2.3.1 (Conjugate Roots)**.** *Let P have rational coefficients, and let a, b, c be rational numbers with $b \neq 0$ and $\sqrt{c}$ irrational. If $a + b\sqrt{c}$ is a root of P, then its* ***conjugate*** *$a - b\sqrt{c}$ is also a root, and P is divisible by*

$$(x - a)^2 - b^2c = x^2 - 2ax + a^2 - b^2c.$$

Proof. Let

$$M(x) = x^2 - 2ax + a^2 - b^2c = (x - a - b\sqrt{c})(x - a + b\sqrt{c}).$$

It has rational coefficients, and its roots are $a \pm b\sqrt{c}$. By the Division Algorithm, $P = MQ + R$ with $R(x) = ux + v$, where u and v are rational by Remark 1.1.3. Substituting $x = a + b\sqrt{c}$ gives $0 = u(a + b\sqrt{c}) + v$, so $ub\sqrt{c} = -(ua + v)$. If $u \neq 0$, then $\sqrt{c} = -(ua + v)/(ub)$ would be rational, which is false. So $u = 0$, and then $v = 0$. Hence R is the zero polynomial, M divides P, and the root $a - b\sqrt{c}$ of M is also a root of P. □

Integer coefficients are rational, so the theorem applies to polynomials with integer coefficients. The hypothesis on the coefficients is needed: $x - \sqrt{2}$ has the root $\sqrt{2}$ but not $-\sqrt{2}$.

Example.

Problem 2.3.2. Find the monic polynomial of least degree with rational coefficients that has $\sqrt{2}$ and $1 - \sqrt{3}$ as roots.

Strategy. Rational coefficients force the conjugates $-\sqrt{2}$ and $1 + \sqrt{3}$ to be roots as well.

Solution. Let P be such a polynomial. By the Conjugate Roots theorem, $-\sqrt{2}$ and $1 + \sqrt{3}$ are also roots of P, so P has the four distinct roots $\pm\sqrt{2}$ and $1 \pm \sqrt{3}$, and $\deg P \geq 4$ by the Roots and Degree theorem. The monic quartic

$$(x^2 - 2)\big((x - 1)^2 - 3\big) = (x^2 - 2)(x^2 - 2x - 2) = x^4 - 2x^3 - 4x^2 + 4x + 4$$

has rational coefficients and these four roots. It is the only monic quartic with these roots: by the Roots and Degree theorem, such a quartic is $(x - \sqrt{2})(x + \sqrt{2})(x - 1 - \sqrt{3})(x - 1 + \sqrt{3})$ times a monic polynomial of degree 0, which is 1.

Answer: $x^4 - 2x^3 - 4x^2 + 4x + 4$. □

3. Vieta's Formulas

Vieta's formulas express the coefficients of a polynomial through its roots, so sums and products of roots can be found without solving the equation.

3.1. **Sums and Products of Roots.** Suppose that a polynomial of degree n factors as $P(x) = a_n(x-r_1)(x-r_2)\cdots(x-r_n)$. Then $r_1, \ldots, r_n$ are the roots of P, each listed as many times as its multiplicity. Such a factorization exists when P has n distinct roots, by the Roots and Degree theorem, and more generally when the multiplicities of its roots add up to n, by the Factorization with Multiplicities theorem (Theorem 2.2.2).

Theorem 3.1.1 (Vieta's Formulas)**.** *Let*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x-r_1)(x-r_2)\cdots(x-r_n),$$

where $a_n \neq 0$. For $k = 1, \ldots, n$, let e_k be the sum of the products of the roots taken k at a time:

$$e_1 = r_1 + r_2 + \cdots + r_n, \qquad e_2 = \sum_{i<j} r_i r_j, \qquad \ldots, \qquad e_n = r_1 r_2 \cdots r_n.$$

Then

$$e_k = (-1)^k \frac{a_{n-k}}{a_n} \qquad (k = 1, 2, \ldots, n).$$

Proof. Expand $a_n(x-r_1)(x-r_2)\cdots(x-r_n)$. Each term of the expansion takes either x or $-r_i$ from each factor. Taking $-r_i$ from exactly k factors and x from the other $n-k$ gives $(-1)^k$ times a product of k of the roots times x^{n-k}, and each product of k roots arises once. So the coefficient of x^{n-k} is $a_n(-1)^k e_k$. It equals a_{n-k} by the Identity corollary, since the two forms of P are equal for every x. □

For a cubic $ax^3 + bx^2 + cx + d$ with roots r, s, t,

$$r + s + t = -\frac{b}{a}, \qquad rs + st + tr = \frac{c}{a}, \qquad rst = -\frac{d}{a}.$$

For a quartic $ax^4 + bx^3 + cx^2 + dx + e$ with roots r, s, t, u, the sum of the roots is $-b/a$, the sum of the six products of two roots is c/a, the sum of the four products of three roots is $-d/a$, and $rstu = e/a$.

Remark 3.1.2 (Vieta in Reverse)**.** Three numbers u, v, w with sum s_1, sum of products in pairs s_2 and product s_3 are the three roots of

$$t^3 - s_1 t^2 + s_2 t - s_3 = 0,$$

because expanding $(t-u)(t-v)(t-w)$ gives $t^3 - (u+v+w)t^2 + (uv+vw+wu)t - uvw$.

Examples.

Problem 3.1.3 (1966 AHSME, Problem #30)**.** If three of the roots of $x^4 + ax^2 + bx + c = 0$ are 1, 2, and 3, then the value of $a + c$ is:

(A) 35 **(B)** 24 **(C)** -12 **(D)** -61 **(E)** -63

Strategy. The coefficient of x^3 is 0, so the four roots add to 0. Find the fourth root first.

Solution. By the Roots and Degree theorem, $x^4 + ax^2 + bx + c = (x-1)(x-2)(x-3)(x-s)$ for some number s, since the remaining factor is monic of degree 1. So the roots are 1, 2, 3 and s. The coefficient of x^3 is 0, so by Vieta's formulas $1+2+3+s = 0$, and $s = -6$. Then a is the sum of the products of pairs of 1, 2, 3, -6:

$$a = 2 + 3 - 6 + 6 - 12 - 18 = -25,$$

and c is the product of the four roots, $c = 1 \cdot 2 \cdot 3 \cdot (-6) = -36$. So $a + c = -61$. As a check, $(x^2 - 3x + 2)(x^2 + 3x - 18) = x^4 - 25x^2 + 60x - 36$.

Answer: (D) -61. □

Problem 3.1.4 (1976 AHSME, Problem #30)**.** How many distinct ordered triples (x, y, z) satisfy the equations

$$x + 2y + 4z = 12,$$
$$xy + 4yz + 2xz = 22,$$
$$xyz = 6?$$

(A) none **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 6

Strategy. Set $u = x$, $v = 2y$ and $w = 4z$. The three equations then give the sum of u, v, w, the sum of their products in pairs, and their product.

Solution. With $u = x$, $v = 2y$ and $w = 4z$,

$$u + v + w = 12, \qquad uv + vw + wu = 2xy + 8yz + 4xz = 2 \cdot 22 = 44, \qquad uvw = 8xyz = 48.$$

By Vieta in reverse, u, v, w are the roots of $t^3 - 12t^2 + 44t - 48 = 0$. Since $2^3 - 12 \cdot 2^2 + 44 \cdot 2 - 48 = 0$, the number 2 is a root, and

$$t^3 - 12t^2 + 44t - 48 = (t - 2)(t^2 - 10t + 24) = (t - 2)(t - 4)(t - 6).$$

So u, v, w are 2, 4, 6 in some order. Conversely, every order satisfies the three equations, because the sum, the sum of the products in pairs and the product do not depend on the order. The $3! = 6$ orders give 6 different triples $(x, y, z) = (u, v/2, w/4)$.

Answer: (E) 6. □

3.2. **Expressions in the Roots.** The coefficients of a polynomial also determine the following two expressions in its roots.

The sum of the squares of the roots is

$$r_1^2 + r_2^2 + \cdots + r_n^2 = e_1^2 - 2e_2,$$

because squaring e_1 gives each square once and each product $r_i r_j$ with $i < j$ twice. For three roots, $r^2 + s^2 + t^2 = (r + s + t)^2 - 2(rs + st + tr)$.

If $P(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$, then substituting a number c gives

$$P(c) = a(c - r_1)(c - r_2) \cdots (c - r_n).$$

Changing the sign of each of the n factors gives

$$(r_1 - c)(r_2 - c) \cdots (r_n - c) = (-1)^n \frac{P(c)}{a}.$$

For a cubic, $(r - c)(s - c)(t - c) = -P(c)/a$.

Example.

Problem 3.2.1. The polynomial $x^3 - 2x^2 - 5x + 1$ has three real roots r, s, and t. Find $r^2 + s^2 + t^2$ and $(r + 1)(s + 1)(t + 1)$.

Strategy. The roots are not rational: the only candidates are ± 1, and neither is a root. Write both expressions in terms of $r + s + t$, $rs + st + tr$ and rst, or through a value of the polynomial.

Solution. Let $P(x) = x^3 - 2x^2 - 5x + 1$, so $P(x) = (x-r)(x-s)(x-t)$ by the Roots and Degree theorem. By Vieta's formulas, $r+s+t = 2$, $rs+st+tr = -5$ and $rst = -1$. Hence

$$r^2 + s^2 + t^2 = (r+s+t)^2 - 2(rs+st+tr) = 4 + 10 = 14.$$

For the product, substitute $x = -1$:

$$P(-1) = (-1-r)(-1-s)(-1-t) = -(r+1)(s+1)(t+1).$$

Since $P(-1) = -1-2+5+1 = 3$, it follows that $(r+1)(s+1)(t+1) = -3$. As a check, expanding gives $(r+1)(s+1)(t+1) = rst + (rs+st+tr) + (r+s+t) + 1 = -1 - 5 + 2 + 1 = -3$.

Answer: $r^2 + s^2 + t^2 = 14$ and $(r+1)(s+1)(t+1) = -3$. □

4. Building Polynomials

A polynomial is determined by its degree, its leading coefficient, and enough of its roots or values. This section uses this fact to find unknown coefficients and unknown polynomials.

4.1. **Comparing Coefficients.** By the Identity corollary (Corollary 1.3.3), two polynomials that are equal for every x, or for more values of x than the larger of their degrees, have the same coefficient for each power of x. This gives a method for finding unknown coefficients: write the unknown polynomial in a form that shows the given information (for example, a known factor times an unknown factor of known degree), expand, and set the coefficients of each power of x equal, starting with the highest power.

Examples.

Problem 4.1.1 (1965 AHSME, Problem #19)**.** If $x^4 + 4x^3 + 6px^2 + 4qx + r$ is exactly divisible by $x^3 + 3x^2 + 9x + 3$, the value of $(p+q)r$ is:

(A) -18 **(B)** 12 **(C)** 15 **(D)** 27 **(E)** 45

Strategy. The quotient of a monic quartic by a monic cubic is a monic linear polynomial $x + m$. Expand and compare coefficients.

Solution. Exact divisibility means that $x^4 + 4x^3 + 6px^2 + 4qx + r = (x^3 + 3x^2 + 9x + 3)(x + m)$ for some number m. Expanding the right side,

$$(x^3 + 3x^2 + 9x + 3)(x + m) = x^4 + (m+3)x^3 + (3m+9)x^2 + (9m+3)x + 3m.$$

Comparing the coefficients of x^3 gives $m + 3 = 4$, so $m = 1$. Then $6p = 3m + 9 = 12$, $4q = 9m + 3 = 12$ and $r = 3m = 3$, so $p = 2$, $q = 3$, $r = 3$, and $(p+q)r = 15$.

Answer: (C) 15. □

Problem 4.1.2 (2017 AMC 12A, Problem #23)**.** For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

(A) -9009 **(B)** -8008 **(C)** -7007 **(D)** -6006 **(E)** -5005

Strategy. Every root of g is a root of f, so by the Roots and Degree theorem f is g times a monic linear polynomial. Expand the product and set its coefficients equal to those of f.

Solution. Let r_1, r_2, r_3 be the three distinct roots of g, so that $g(x) = (x-r_1)(x-r_2)(x-r_3)$ by the Roots and Degree theorem. They are also roots of f, so by the same theorem $f(x) = (x-r_1)(x-r_2)(x-r_3)Q(x) = g(x)Q(x)$, where Q is monic of degree 1, say $Q(x) = x - r$. Expanding,

$$f(x) = (x^3 + ax^2 + x + 10)(x - r) = x^4 + (a-r)x^3 + (1-ar)x^2 + (10-r)x - 10r.$$

Setting these coefficients equal to those of $f(x) = x^4 + x^3 + bx^2 + 100x + c$ gives

$$a - r = 1, \qquad 1 - ar = b, \qquad 10 - r = 100, \qquad -10r = c.$$

The third equation gives $r = -90$, so $a = -89$, $b = 1 - ar = -8009$ and $c = 900$. Since $f(1)$ is the sum of the coefficients of f,

$$f(1) = 1 + 1 + b + 100 + c = 102 - 8009 + 900 = -7007.$$

As a check, $g(x) = x^3 - 89x^2 + x + 10$ takes the values -81, 10, -77, 99 at $x = -1$, 0, 1, 89, so it does have three distinct real roots.

Answer: (C) -7007. □

4.2. **Polynomials with Given Values.** If $P(a_1) = P(a_2) = \cdots = P(a_k) = c$ for distinct numbers $a_1, \ldots, a_k$, then $a_1, \ldots, a_k$ are roots of $P(x) - c$, so by the Roots and Degree theorem

$$P(x) - c = (x - a_1)(x - a_2)\cdots(x - a_k)Q(x)$$

for some polynomial Q. More generally, if P agrees with a simpler polynomial g at $a_1, \ldots, a_k$ (for example, $g(x) = x$), the same holds for $P(x) - g(x)$. When $\deg g < \deg P$, subtracting g does not change the degree or the leading coefficient, so the degree and leading coefficient of P also determine those of Q. If P has integer coefficients and c and the a_i are integers, then Q has integer coefficients by Remark 1.1.3, because $(x - a_1)\cdots(x - a_k)$ is monic with integer coefficients.

Examples.

Problem 4.2.1. A monic cubic polynomial P satisfies $P(1) = 1$, $P(2) = 2$, and $P(3) = 3$. Find $P(4)$.

Strategy. The polynomial $P(x) - x$ has the roots 1, 2, 3.

Solution. Subtracting x does not change the leading term, so $P(x) - x$ is monic of degree 3. It has the roots 1, 2, 3, so by the Roots and Degree theorem $P(x) - x = (x-1)(x-2)(x-3)$. Hence

$$P(4) = 4 + (4-1)(4-2)(4-3) = 4 + 6 = 10.$$

Answer: 10. □

Problem 4.2.2 (2010 AMC 12B, Problem #21)**.** Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that

$$P(1) = P(3) = P(5) = P(7) = a,$$

and

$$P(2) = P(4) = P(6) = P(8) = -a.$$

What is the smallest possible value of a?

(A) 105 **(B)** 315 **(C)** 945 **(D)** 7! **(E)** 8!

Strategy. Subtracting a from P turns 1, 3, 5, 7 into roots. Once these roots are factored out, the remaining factor has integer coefficients, so its values at 2, 4, 6, 8 are integers, and each of them is a times a known fraction.

Solution. Since P has integer coefficients, $a = P(1)$ is an integer, and so the polynomial $P(x) - a$ also has integer coefficients. It is 0 at 1, 3, 5, 7, so by the Roots and Degree theorem

$$P(x) - a = (x-1)(x-3)(x-5)(x-7)Q(x).$$

The product $(x-1)(x-3)(x-5)(x-7)$ is monic with integer coefficients, so Q has integer coefficients by Remark 1.1.3, and $Q(2)$, $Q(4)$, $Q(6)$, $Q(8)$ are integers. At $x = 2$, 4, 6, 8 the left side is $-2a$, and the product of the four factors is -15, 9, -15, 105. So

$$Q(2) = \frac{2a}{15}, \qquad Q(4) = -\frac{2a}{9}, \qquad Q(6) = \frac{2a}{15}, \qquad Q(8) = -\frac{2a}{105}.$$

Hence 9, 15 and 105 divide $2a$. They are odd, so they divide a, and so does $\operatorname{lcm}(9, 15, 105) = 315$. Thus $a \geq 315$.

For $a = 315$ the required values are $Q(2) = Q(6) = 42$, $Q(4) = -70$ and $Q(8) = -6$. Try $Q(x) = 42 + (x-2)(x-6)L(x)$ with L linear. Then $Q(4) = 42 - 4L(4)$ and $Q(8) = 42 + 12L(8)$, so $L(4) = 28$ and $L(8) = -4$, which gives $L(x) = 60 - 8x$. This Q has integer coefficients, so

$$P(x) = 315 + (x-1)(x-3)(x-5)(x-7)\big(42 + (x-2)(x-6)(60-8x)\big)$$

has integer coefficients, and it takes the value 315 at 1, 3, 5, 7 and the value -315 at 2, 4, 6, 8. The smallest possible value of a is 315.

Answer: (B) 315. □

5. Problems

Beginner.

1. (1974 AHSME, Problem #4) What is the remainder when $x^{51}+51$ is divided by $x+1$?
(A) 0 **(B)** 1 **(C)** 49 **(D)** 50 **(E)** 51

2. (1970 AHSME, Problem #11) If two factors of $2x^3-hx+k$ are $x+2$ and $x-1$, the value of $|2h-3k|$ is:
(A) 4 **(B)** 3 **(C)** 2 **(D)** 1 **(E)** 0

3. (1999 AHSME, Problem #12) What is the maximum number of points of intersection of the graphs of two different fourth degree polynomial functions $y=p(x)$ and $y=q(x)$, each with leading coefficient 1?
(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 8

4. (2023 AMC 12B, Problem #6) When the roots of the polynomial

$$P(x)=(x-1)^1(x-2)^2(x-3)^3\cdots(x-10)^{10}$$

are removed from the number line, what remains is the union of 11 disjoint open intervals. On how many of these intervals is $P(x)$ positive?
(A) 3 **(B)** 7 **(C)** 6 **(D)** 4 **(E)** 5

5. (2004 AMC 12B, Problem #17) For some real numbers a and b, the equation

$$8x^3+4ax^2+2bx+a=0$$

has three distinct positive roots. If the sum of the base-2 logarithms of the roots is 5, what is the value of a?
(A) -256 **(B)** -64 **(C)** -8 **(D)** 64 **(E)** 256

Intermediate.

6. (2001 AMC 12, Problem #19) The polynomial $P(x)=x^3+ax^2+bx+c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. If the y-intercept of the graph of $y=P(x)$ is 2, what is b?
(A) -11 **(B)** -10 **(C)** -9 **(D)** 1 **(E)** 5

7. (2025 AMC 12B, Problem #8) There are integers a and b such that the polynomial x^3-5x^2+ax+b has $4+\sqrt{5}$ as a root. What is $a+b$?
(A) 13 **(B)** 17 **(C)** 20 **(D)** 30 **(E)** 68

8. (1969 AHSME, Problem #34) The remainder R obtained by dividing x^{100} by x^2-3x+2 is a polynomial of degree less than 2. Then R may be written as:
(A) $2^{100}-1$ **(B)** $2^{100}(x-1)-(x-2)$ **(C)** $2^{100}(x-3)$
(D) $x(2^{100}-1)+2(2^{99}-1)$ **(E)** $2^{100}(x+1)-(x+2)$

9. (2022 AMC 12A, Problem #15) The roots of the polynomial $10x^3-39x^2+29x-6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?
(A) $\frac{24}{5}$ **(B)** $\frac{42}{5}$ **(C)** $\frac{81}{5}$ **(D)** 30 **(E)** 48

10. (2023 AMC 12B, Problem #14) For how many ordered pairs (a, b) of integers does the polynomial $x^3 + ax^2 + bx + 6$ have 3 distinct integer roots?

(A) 5 **(B)** 6 **(C)** 8 **(D)** 7 **(E)** 4

Advanced.

11. (2005 AIME II, Problem #13) Let $P(x)$ be a polynomial with integer coefficients that satisfies $P(17) = 10$ and $P(24) = 17$. Given that the equation $P(n) = n + 3$ has two distinct integer solutions n_1 and n_2, find the product $n_1 \cdot n_2$.

12. (2010 AMC 12A, Problem #21) The graph of $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2$ lies above the line $y = bx + c$ except at three values of x, where the graph and the line intersect. What is the largest of those values?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

13. (2017 AMC 12B, Problem #23) The graph of $y = f(x)$, where $f(x)$ is a polynomial of degree 3, contains points $A(2, 4)$, $B(3, 9)$, and $C(4, 16)$. Lines AB, AC, and BC intersect the graph again at points D, E, and F, respectively, and the sum of the x-coordinates of D, E, and F is 24. What is $f(0)$?

(A) -2 **(B)** 0 **(C)** 2 **(D)** $\frac{24}{5}$ **(E)** 8

14. (2015 AIME I, Problem #10) Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

15. (1984 AIME, Problem #15) Determine $w^2 + x^2 + y^2 + z^2$ if

$$\begin{aligned}
\frac{x^2}{2^2-1} + \frac{y^2}{2^2-3^2} + \frac{z^2}{2^2-5^2} + \frac{w^2}{2^2-7^2} &= 1,\\
\frac{x^2}{4^2-1} + \frac{y^2}{4^2-3^2} + \frac{z^2}{4^2-5^2} + \frac{w^2}{4^2-7^2} &= 1,\\
\frac{x^2}{6^2-1} + \frac{y^2}{6^2-3^2} + \frac{z^2}{6^2-5^2} + \frac{w^2}{6^2-7^2} &= 1,\\
\frac{x^2}{8^2-1} + \frac{y^2}{8^2-3^2} + \frac{z^2}{8^2-5^2} + \frac{w^2}{8^2-7^2} &= 1.
\end{aligned}$$