POLYNOMIALS: FACTOR THEOREM, RATIONAL ROOTS, AND VIETA FOR AMC 12: SOLUTIONS

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1. Solutions

Beginner.

1. (1974 AHSME, Problem #4) What is the remainder when $x^{51}+51$ is divided by $x+1$?
(A) 0 **(B)** 1 **(C)** 49 **(D)** 50 **(E)** 51

Strategy. Since $x+1=x-(-1)$, the Remainder Theorem gives the remainder as a value of the polynomial.

Solution. By the Remainder Theorem, the remainder upon division by $x-(-1)$ is the value of $x^{51}+51$ at $x=-1$:

$$(-1)^{51}+51=-1+51=50.$$

Answer: (D) 50. □

2. (1970 AHSME, Problem #11) If two factors of $2x^3-hx+k$ are $x+2$ and $x-1$, the value of $|2h-3k|$ is:
(A) 4 **(B)** 3 **(C)** 2 **(D)** 1 **(E)** 0

Strategy. The roots 1 and -2 of the two factors are roots of the cubic. Each gives a linear equation in h and k; solve one for k and substitute into the other.

Solution. By the Factor Theorem, the factors $x-1$ and $x+2$ make 1 and -2 roots of $P(x)=2x^3-hx+k$. From $P(1)=2-h+k=0$ it follows that $k=h-2$. Substituting this into $P(-2)=-16+2h+k=0$ gives $3h-18=0$. Hence $h=6$ and $k=4$, so $2h$ and $3k$ are both 12, and $|2h-3k|=0$. As a check, $2x^3-6x+4=2(x-1)^2(x+2)$ has both factors.
Answer: (E) 0. □

3. (1999 AHSME, Problem #12) What is the maximum number of points of intersection of the graphs of two different fourth degree polynomial functions $y=p(x)$ and $y=q(x)$, each with leading coefficient 1?
(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 8

Strategy. Count the roots of $p-q$. It is not the zero polynomial, and its degree is less than 4 because p and q have the same leading term.

Solution. The graphs meet at the points $(x,p(x))$ with $p(x)=q(x)$, that is, at the roots of $p(x)-q(x)$, and different intersection points have different x-coordinates. Both polynomials are monic of degree 4, so $p-q$ has no x^4 term, and it is the zero polynomial or has degree at most 3. It is not the zero polynomial, because p and q are different. By the Roots and Degree theorem, $p-q$ has at most 3 roots, so the graphs meet in at most 3 points.

Three points are possible: $p(x)=x^4+x^3-x$ and $q(x)=x^4$ meet where $x^3-x=x(x-1)(x+1)=0$, that is, at $x=-1$, 0 and 1.
Answer: (C) 3. □

4. (2023 AMC 12B, Problem #6) When the roots of the polynomial

$$P(x)=(x-1)^1(x-2)^2(x-3)^3\cdots(x-10)^{10}$$

are removed from the number line, what remains is the union of 11 disjoint open intervals. On how many of these intervals is $P(x)$ positive?
(A) 3 **(B)** 7 **(C)** 6 **(D)** 4 **(E)** 5

Strategy. The sign of a product is determined by the number of negative factors. For x not a root, the factor $(x-k)^k$ is negative exactly when k is odd and $x<k$.

Solution. Let x be a number that is not a root. The factor $(x-k)^k$ is positive when $x>k$. When $x<k$, it is positive if k is even and negative if k is odd. So $P(x)$ is negative exactly when the number of odd k with $k>x$ is odd. On $(-\infty,1)$ this number is 5; on $(1,2)$ and $(2,3)$ it is 4; on $(3,4)$ and $(4,5)$, 3; on $(5,6)$ and $(6,7)$, 2; on $(7,8)$ and $(8,9)$, 1; and on $(9,10)$ and $(10,\infty)$, 0. Hence $P(x)>0$ exactly on $(1,2)$, $(2,3)$, $(5,6)$, $(6,7)$, $(9,10)$ and $(10,\infty)$, which are 6 of the 11 intervals.

Answer: (C) 6. □

5. (2004 AMC 12B, Problem #17) For some real numbers a and b, the equation

$$8x^3+4ax^2+2bx+a=0$$

has three distinct positive roots. If the sum of the base-2 logarithms of the roots is 5, what is the value of a?

(A) -256 **(B)** -64 **(C)** -8 **(D)** 64 **(E)** 256

Strategy. A sum of logarithms is the logarithm of the product, and Vieta's formulas give the product of the roots.

Solution. Let the three distinct positive roots be r, s, t, so that $8x^3+4ax^2+2bx+a=8(x-r)(x-s)(x-t)$ by the Roots and Degree theorem. By the product rule for logarithms,

$$\log_2 r+\log_2 s+\log_2 t=\log_2(rst)=5,$$

so $rst=2^5=32$. By Vieta's formulas, $rst=-a/8$. Hence $a=-8\cdot 32=-256$.

Answer: (A) -256. □

Intermediate.

6. (2001 AMC 12, Problem #19) The polynomial $P(x)=x^3+ax^2+bx+c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. If the y-intercept of the graph of $y=P(x)$ is 2, what is b?

(A) -11 **(B)** -10 **(C)** -9 **(D)** 1 **(E)** 5

Strategy. Write each of the three equal quantities in terms of a, b, c. The graph meets the y-axis at $(0,c)$.

Solution. Let r, s, t be the zeros, so $P(x)=(x-r)(x-s)(x-t)$, and Vieta's formulas give $r+s+t=-a$ and $rst=-c$. The three equal quantities are therefore

$$\frac{r+s+t}{3}=-\frac{a}{3},\qquad rst=-c,\qquad P(1)=1+a+b+c.$$

The graph meets the y-axis at $(0,P(0))=(0,c)$, so $c=2$, and all three quantities equal -2. Then $-a/3=-2$ gives $a=6$, and $1+a+b+c=9+b=-2$ gives $b=-11$.

As a check, $P(x)=x^3+6x^2-11x+2$ takes the values -38, 2, -2, 12 at $x=-8$, 0, 1, 2, so it has three real zeros, one in each of the intervals $(-8,0)$, $(0,1)$ and $(1,2)$. Their mean is -2, their product is -2, and $P(1)=-2$.

Answer: (A) -11. □

7. (2025 AMC 12B, Problem #8) There are integers a and b such that the polynomial x^3-5x^2+ax+b has $4+\sqrt{5}$ as a root. What is $a+b$?

(A) 13 **(B)** 17 **(C)** 20 **(D)** 30 **(E)** 68

Strategy. Pair $4+\sqrt{5}$ with its conjugate, which is also a root. Then Vieta's formulas give the third root, and after that a and b.

Solution. By the Conjugate Roots theorem, x^3-5x^2+ax+b is divisible by $(x-4)^2-5 = x^2-8x+11$, whose roots are $4\pm\sqrt{5}$. The quotient is monic of degree 1, say $x-t$, so the roots of the cubic are $4+\sqrt{5}$, $4-\sqrt{5}$ and t. The two conjugates have sum 8 and product 11. By Vieta's formulas, the three roots add to 5, so $t=-3$; the sum of their products in pairs is a; and their product is $-b$. Hence

$$a = 11+8t = -13, \qquad b = -11t = 33,$$

and $a+b=20$. As a check, $(x+3)(x^2-8x+11) = x^3-5x^2-13x+33$.

Answer: (C) 20. □

8. (1969 AHSME, Problem #34) The remainder R obtained by dividing x^{100} by x^2-3x+2 is a polynomial of degree less than 2. Then R may be written as:

(A) $2^{100}-1$ **(B)** $2^{100}(x-1)-(x-2)$ **(C)** $2^{100}(x-3)$
(D) $x(2^{100}-1)+2(2^{99}-1)$ **(E)** $2^{100}(x+1)-(x+2)$

Strategy. Factor $x^2-3x+2=(x-1)(x-2)$. The remainder $R(x)=cx+d$ is found from its values at 1 and 2.

Solution. Write $x^{100} = (x-1)(x-2)Q(x)+cx+d$. Substituting $x=1$ and $x=2$ gives

$$c+d=1, \qquad 2c+d=2^{100}.$$

So $c=2^{100}-1$ and $d=1-c=2-2^{100}$, and

$$R(x) = (2^{100}-1)x+2-2^{100} = 2^{100}(x-1)-(x-2).$$

Answer: (B) $2^{100}(x-1)-(x-2)$. □

9. (2022 AMC 12A, Problem #15) The roots of the polynomial $10x^3-39x^2+29x-6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

(A) $\frac{24}{5}$ **(B)** $\frac{42}{5}$ **(C)** $\frac{81}{5}$ **(D)** 30 **(E)** 48

Strategy. The new volume is $(r+2)(s+2)(t+2)$, where r, s, t are the roots. Relate this product to the value of the polynomial at -2.

Solution. Let $P(x)=10x^3-39x^2+29x-6$. Its roots r, s, t are the edges of the box, so $P(x)=10(x-r)(x-s)(x-t)$, and the new box has edges $r+2$, $s+2$, $t+2$. Substituting $x=-2$,

$$P(-2) = 10(-2-r)(-2-s)(-2-t) = -10(r+2)(s+2)(t+2).$$

Since $P(-2) = -80-156-58-6 = -300$, the new volume is $(r+2)(s+2)(t+2) = 300/10 = 30$.

As a check, testing the candidates from the Rational Root Theorem gives the roots $\frac{2}{5}$, $\frac{1}{2}$ and 3, so $P(x)=(5x-2)(2x-1)(x-3)$, and the new volume is $\frac{12}{5}\cdot\frac{5}{2}\cdot 5 = 30$.

Answer: (D) 30. □

10. (2023 AMC 12B, Problem #14) For how many ordered pairs (a,b) of integers does the polynomial x^3+ax^2+bx+6 have 3 distinct integer roots?

(A) 5 **(B)** 6 **(C)** 8 **(D)** 7 **(E)** 4

Strategy. Write the cubic as $(x-r)(x-s)(x-t)$ with distinct integers r, s, t; its constant term $-rst$ is 6. So $|r|$, $|s|$, $|t|$ are positive integers with product 6, and an odd number of r, s, t are negative.

Solution. Suppose the roots are distinct integers r, s, t. By the Roots and Degree theorem the polynomial is $(x-r)(x-s)(x-t)$, so Vieta's formulas give $rst=-6$, $a=-(r+s+t)$ and $b=rs+st+tr$. The absolute values $|r|$, $|s|$, $|t|$ multiply to 6, so they are $1,2,3$ or $1,1,6$.

- With $1,2,3$, the product is negative when one or three of the numbers are negative: $\{-1,2,3\}$, $\{1,-2,3\}$, $\{1,2,-3\}$ and $\{-1,-2,-3\}$.
- With $1,1,6$, the two numbers of absolute value 1 must be 1 and -1, because the roots are distinct, and the product is -6 only for $\{1,-1,6\}$.

Substituting each set into the formulas for a and b gives

$$(a,b)=(-4,1),\ (-2,-5),\ (0,-7),\ (6,11),\ (-6,-1),$$

no two of which are equal. Conversely, for each set, expanding $(x-r)(x-s)(x-t)$ gives a polynomial x^3+ax^2+bx+6 with three distinct integer roots. Hence the number of pairs is 5.

Answer: (A) 5. □

Advanced.

11. (2005 AIME II, Problem #13) Let $P(x)$ be a polynomial with integer coefficients that satisfies $P(17)=10$ and $P(24)=17$. Given that the equation $P(n)=n+3$ has two distinct integer solutions n_1 and n_2, find the product $n_1\cdot n_2$.

Strategy. For integers $m\neq n$, the difference $m-n$ divides $P(m)-P(n)$. Apply this to an integer solution n together with 17, and together with 24.

Solution. Let n be an integer with $P(n)=n+3$. Then $n\neq 17$, since $P(17)=10\neq 20$, and $n\neq 24$, since $P(24)=17\neq 27$. By the Integer Values corollary, $n-17$ divides

$$P(n)-P(17)=n-7=(n-17)+10,$$

so $n-17$ divides 10. In the same way, $n-24$ divides $P(n)-P(24)=n-14=(n-24)+10$, so $n-24$ divides 10. The divisors of 10 are ±1, ±2, ±5 and ±10, and $n-17$ is 7 more than $n-24$. The only pairs of divisors of 10 that differ by 7 are 2 and -5, and 5 and -2. So $n-17=2$ or $n-17=5$, that is, $n=19$ or $n=22$. The equation has two distinct integer solutions, so they are 19 and 22, and $n_1\cdot n_2=19\cdot 22=418$.

Such a polynomial exists: $P(x)=-x^2+42x-415$ has $P(17)=10$ and $P(24)=17$, and $P(n)=n+3$ becomes $n^2-41n+418=(n-19)(n-22)=0$.

Answer: 418. □

12. (2010 AMC 12A, Problem #21) The graph of $y=x^6-10x^5+29x^4-4x^3+ax^2$ lies above the line $y=bx+c$ except at three values of x, where the graph and the line intersect. What is the largest of those values?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Strategy. Subtract the line. The difference is a monic polynomial of degree 6 that is positive except at three numbers, where it is 0. A root of odd multiplicity would give a sign change, so each of the three roots is at least double, and the degree forces each to be exactly double.

Solution. Let $D(x)=x^6-10x^5+29x^4-4x^3+ax^2-bx-c$, the polynomial minus the line. Then $D(x)>0$ except at three numbers p, q, r, where $D(x)=0$. By the Sign near a Root

theorem, a root of odd multiplicity would make D change sign, so each of p, q, r has even multiplicity, at least 2. By the Factorization with Multiplicities theorem, the multiplicities add up to at most $\deg D = 6$, so each of them is exactly 2, and since D is monic,

$$D(x) = (x-p)^2(x-q)^2(x-r)^2 = C(x)^2,$$

where $C(x) = (x-p)(x-q)(x-r)$ is a monic cubic, say $C(x) = x^3 + ux^2 + vx + w$. Expanding,

$$C(x)^2 = x^6 + 2ux^5 + (u^2+2v)x^4 + (2w+2uv)x^3 + (v^2+2uw)x^2 + 2vwx + w^2.$$

Comparing coefficients with D gives $2u = -10$, $u^2 + 2v = 29$ and $2w + 2uv = -4$, so $u = -5$, $v = 2$ and $w = 8$. The cubic $C(x) = x^3 - 5x^2 + 2x + 8$ has the root -1 and factors as $(x+1)(x-2)(x-4)$. So the graph and the line meet at $x = -1$, 2 and 4, and the largest of these is 4. (The remaining coefficients give $a = -76$, $b = -32$ and $c = -64$.)

Answer: (A) 4. □

13. (2017 AMC 12B, Problem #23) The graph of $y = f(x)$, where $f(x)$ is a polynomial of degree 3, contains points $A(2,4)$, $B(3,9)$, and $C(4,16)$. Lines AB, AC, and BC intersect the graph again at points D, E, and F, respectively, and the sum of the x-coordinates of D, E, and F is 24. What is $f(0)$?

(A) -2 **(B)** 0 **(C)** 2 **(D)** $\dfrac{24}{5}$ **(E)** 8

Strategy. Compare f with x^2, which takes the same values as f at 2, 3 and 4, so the cubic $f(x) - x^2$ has these three roots. For each line, $f(x)$ minus the line has two known roots, and its third root is the x-coordinate of the new point.

Solution. Since $f(2) = 4$, $f(3) = 9$ and $f(4) = 16$, the polynomial $f(x) - x^2$ has the roots 2, 3, 4. It has degree 3 and the same leading coefficient $k \neq 0$ as f, so

$$f(x) = x^2 + k(x-2)(x-3)(x-4).$$

Line AB has slope $\frac{9-4}{3-2} = 5$, so it is $y = 5x - 6$, and $x^2 - (5x-6) = (x-2)(x-3)$. Hence

$$f(x) - (5x-6) = (x-2)(x-3)\big(k(x-4)+1\big),$$

whose third root, the x-coordinate of D, is $4 - 1/k$. In the same way, line AC is $y = 6x - 8$ with $x^2 - (6x-8) = (x-2)(x-4)$, so E has x-coordinate $3 - 1/k$, and line BC is $y = 7x - 12$ with $x^2 - (7x-12) = (x-3)(x-4)$, so F has x-coordinate $2 - 1/k$. Therefore

$$\left(4 - \frac{1}{k}\right) + \left(3 - \frac{1}{k}\right) + \left(2 - \frac{1}{k}\right) = 9 - \frac{3}{k} = 24,$$

so $k = -1/5$, and $f(0) = k(-2)(-3)(-4) = -24k = 24/5$. With this k, the points D, E, F have x-coordinates 9, 8, 7, which differ from those of A, B, C.

Answer: (D) $\dfrac{24}{5}$. □

14. (2015 AIME I, Problem #10) Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

Strategy. Sort the six numbers by the sign of f there. A cubic has at most three roots, and $f(x) - 12$ and $f(x) + 12$ share their x^3 and x^2 coefficients, so Vieta's formulas give the two sets of roots the same sum.

Solution. Each of 1, 2, 3, 5, 6, 7 is a root of $f(x)-12$ or of $f(x)+12$. Both have degree 3, so each has at most three roots, and the six numbers split three and three. Let k be the leading coefficient of f. The polynomials $f(x)-12$ and $f(x)+12$ have the same leading coefficient k and the same coefficient of x^2, so by Vieta's formulas their roots have the same sum. That sum is half of $1+2+3+5+6+7=24$, so it is 12. The only triples of these numbers with sum 12 are $\{1,5,6\}$ and $\{2,3,7\}$.

Suppose first that

$$f(x)-12=k(x-1)(x-5)(x-6), \qquad f(x)+12=k(x-2)(x-3)(x-7).$$

Subtracting the first equation from the second gives

$$24=k\big[(x^3-12x^2+41x-42)-(x^3-12x^2+41x-30)\big]=-12k,$$

so $k=-2$, and $f(0)=12+k(-1)(-5)(-6)=12+60=72$. In the other case the two triples exchange roles, and the same computation gives $k=2$ and

$$f(0)=12+2(-2)(-3)(-7)=-72.$$

Either way, $|f(0)|=72$.

Answer: 72. □

15. (1984 AIME, Problem #15) Determine $w^2+x^2+y^2+z^2$ if

$$\begin{aligned}
\frac{x^2}{2^2-1}+\frac{y^2}{2^2-3^2}+\frac{z^2}{2^2-5^2}+\frac{w^2}{2^2-7^2}&=1,\\
\frac{x^2}{4^2-1}+\frac{y^2}{4^2-3^2}+\frac{z^2}{4^2-5^2}+\frac{w^2}{4^2-7^2}&=1,\\
\frac{x^2}{6^2-1}+\frac{y^2}{6^2-3^2}+\frac{z^2}{6^2-5^2}+\frac{w^2}{6^2-7^2}&=1,\\
\frac{x^2}{8^2-1}+\frac{y^2}{8^2-3^2}+\frac{z^2}{8^2-5^2}+\frac{w^2}{8^2-7^2}&=1.
\end{aligned}$$

Strategy. Read the four equations as one equation in t, taken at $t=4$, 16, 36, 64. Clearing denominators gives a polynomial in t with these four roots. Compare its coefficients.

Solution. The four equations say that

$$\frac{x^2}{t-1}+\frac{y^2}{t-9}+\frac{z^2}{t-25}+\frac{w^2}{t-49}=1$$

for $t=4$, 16, 36 and 64. Multiplying by $(t-1)(t-9)(t-25)(t-49)$, which is not 0 for these values of t, shows that 4, 16, 36, 64 are roots of

$$\begin{aligned}
F(t)=&(t-1)(t-9)(t-25)(t-49)\\
&-x^2(t-9)(t-25)(t-49)-y^2(t-1)(t-25)(t-49)\\
&-z^2(t-1)(t-9)(t-49)-w^2(t-1)(t-9)(t-25).
\end{aligned}$$

The subtracted terms have degree at most 3, so F is monic of degree 4, and by the Roots and Degree theorem $F(t)=(t-4)(t-16)(t-36)(t-64)$. The coefficient of t^3 in F is $-(1+9+25+49)-(x^2+y^2+z^2+w^2)$, and in $(t-4)(t-16)(t-36)(t-64)$ it is $-(4+16+36+64)$. Comparing the two,

$$w^2+x^2+y^2+z^2=120-84=36.$$

Answer: 36. □

2. Answer Key

Problem	Letter	Value
1	D	50
2	E	0
3	C	3
4	C	6
5	A	-256
6	A	-11
7	C	20
8	B	$2^{100}(x-1)-(x-2)$
9	D	30
10	A	5
11		418
12	A	4
13	D	$24/5$
14		72
15		36