

LOGARITHMS AND EXPONENTIAL EQUATIONS FOR AMC 12

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1. EXPONENTS

An exponential equation has an unknown in an exponent. For example, $2^x = 8$ asks which power of 2 equals 8, so $x = 3$. When both sides can be written as powers of the same base, the equation reduces to an equation between the exponents. Logarithms solve the equations whose sides are not powers of a common base.

1.1. Exponent Rules. For a positive real number a and real exponents r and s ,

$$a^r a^s = a^{r+s}, \quad \frac{a^r}{a^s} = a^{r-s}, \quad (a^r)^s = a^{rs}.$$

In particular, $a^0 = 1$, $a^{-r} = 1/a^r$, and $a^{1/n} = \sqrt[n]{a}$ for a positive integer n . Thus $8^{2/3} = (\sqrt[3]{8})^2 = 4$, and $4^{-3/2} = 1/(\sqrt{4})^3 = 1/8$.

If $a > 1$, then a^x increases as x increases. If $0 < a < 1$, then a^x decreases as x increases. In both cases each positive value occurs exactly once, so

$$a^u = a^v \iff u = v \quad (a > 0, a \neq 1).$$

The condition $a \neq 1$ is necessary: $1^u = 1^v$ holds for every pair of exponents.

Example.

Problem 1.1.1. Solve $8^{x-1} = 4^{x+2}$.

Strategy. Write both sides as powers of 2.

Solution. Since $8 = 2^3$ and $4 = 2^2$, the equation becomes

$$2^{3x-3} = 2^{2x+4}.$$

Equal powers of 2 have equal exponents, so $3x - 3 = 2x + 4$ and $x = 7$. Substitution gives $8^6 = 2^{18} = 4^9$.

Answer: 7. □

1.2. Substitution. When several terms involve powers of the same expression, replace that expression with one variable. For example, $4^x = (2^x)^2$, so an equation involving 4^x and 2^x may become a quadratic in 2^x . The new variable is positive, so a negative root of the quadratic gives no solution.

Example.

Problem 1.2.1. Solve $4^x - 3 \cdot 2^x - 4 = 0$.

Strategy. Let $t = 2^x$. Then $4^x = t^2$ and $t > 0$.

Solution. The equation becomes

$$t^2 - 3t - 4 = (t - 4)(t + 1) = 0.$$

The roots are $t = 4$ and $t = -1$. Since $t = 2^x > 0$, only $t = 4$ is possible. Thus $2^x = 4$ and $x = 2$. The original equation checks: $16 - 12 - 4 = 0$.

Answer: 2. □

2. LOGARITHMS

2.1. Definition.

Definition 2.1.1. For $b > 0$, $b \neq 1$, and $x > 0$, the **logarithm of x to base b** , written $\log_b x$, is the unique real number y for which $b^y = x$. Equivalently,

$$\log_b x = y \iff b^y = x.$$

The number x is the **argument** of the logarithm. A logarithm without a written base, $\log x$, means $\log_{10} x$ in this handout.

For example, $\log_2 8 = 3$, $\log_3(1/9) = -2$, and $\log_{1/2} 8 = -3$. Logarithms can be negative even though their arguments must be positive. The base also must be positive and cannot equal 1.

The definition gives the identities

$$\log_b 1 = 0, \quad \log_b b = 1, \quad b^{\log_b x} = x, \quad \log_b(b^y) = y.$$

The last two identities say that taking a logarithm and raising the same base to a power undo one another. Parentheses matter: $\log_b(x^2) = 2\log_b x$ for $x > 0$, while $(\log_b x)^2$ is the square of the logarithm.

2.2. Product, Quotient, and Power Rules.

Proposition 2.2.1. If $b > 0$, $b \neq 1$, $u > 0$, $v > 0$, and r is real, then

$$\begin{aligned}\log_b(uv) &= \log_b u + \log_b v, \\ \log_b(u/v) &= \log_b u - \log_b v, \\ \log_b(u^r) &= r \log_b u.\end{aligned}$$

Proof. Write $u = b^s$ and $v = b^t$. The exponent rules give $uv = b^{s+t}$, $u/v = b^{s-t}$, and $u^r = b^{rs}$. Taking the base- b logarithm gives the three identities, since $s = \log_b u$ and $t = \log_b v$. $\square$

These rules do not split a sum: $\log_2(4 + 4) = 3$, whereas $\log_2 4 + \log_2 4 = 4$. Also, $\log_b(x^2)$ is defined for negative x , but $2\log_b x$ is not. The identity valid for every nonzero real x is $\log_b(x^2) = 2\log_b |x|$.

Example.

Problem 2.2.2 (1950 AHSME, Problem #25). The value of $\log_5 \frac{(125)(625)}{25}$ is equal to:

(A) 725 (B) 6 (C) 3125 (D) 5 (E) None of these answers

Strategy. Write the numerator and denominator as powers of 5.

Solution. Since $125 = 5^3$, $625 = 5^4$, and $25 = 5^2$,

$$\log_5 \frac{(125)(625)}{25} = \log_5(5^{3+4-2}) = \log_5(5^5) = 5.$$

Equivalently, the product and quotient rules give $3 + 4 - 2 = 5$ directly.

Answer: (D) 5. $\square$

2.3. Different Exponential Bases. When an exponential equation cannot be reduced to a common base, collect the terms involving the unknown exponent and take a logarithm. For $a, c > 0$ with $a \neq c$, the equation $a^x = Kc^x$ with $K > 0$ becomes $(a/c)^x = K$, so $x = \log_{a/c} K$.

Example.

Problem 2.3.1. Solve $3^{x+1} = 5^{x-1}$.

Strategy. Separate the fixed powers, then place all the factors with exponent x on one side.

Solution. Since $3^{x+1} = 3 \cdot 3^x$ and $5^{x-1} = 5^x/5$, multiplying by 5 gives $15 \cdot 3^x = 5^x$. Dividing by 3^x gives $(5/3)^x = 15$. Because $5/3 > 0$ and $5/3 \neq 1$, the definition of a logarithm gives the single solution $x = \log_{5/3} 15$.

Answer: $\log_{5/3} 15$. □

2.4. Change of Base.

Proposition 2.4.1. For positive bases $b, c \neq 1$ and a positive argument x ,

$$\log_b x = \frac{\log_c x}{\log_c b}.$$

Proof. Let $y = \log_b x$, so $b^y = x$. Taking the base- c logarithm gives $y \log_c b = \log_c x$. Since $b \neq 1$, the number $\log_c b$ is nonzero, and division gives the formula. □

Choose a common base that simplifies the arguments or the bases in the problem. Useful consequences are

$$\log_a b = \frac{1}{\log_b a}, \quad (\log_a b)(\log_b c) = \log_a c,$$

whenever the logarithms shown are defined. More generally, for $b > 0$, $b \neq 1$, $x > 0$, and real r, s with $r \neq 0$,

$$\log_{b^r}(x^s) = \frac{s}{r} \log_b x.$$

For example, $\log_4 8 = \frac{3}{2}$, and $\log_{\sqrt{2}} \sqrt{x} = \log_2 x$ for $x > 0$.

Examples.

Problem 2.4.2 (1961 AHSME, Problem #30). If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, then $\log_5 12$ equals:

(A) $\frac{a+b}{a+1}$ (B) $\frac{2a+b}{a+1}$ (C) $\frac{a+2b}{1+a}$ (D) $\frac{2a+b}{1-a}$ (E) $\frac{a+2b}{1-a}$

Strategy. Change the base to 10, then express 12 and 5 using 2, 3, and 10.

Solution. The numerator is $\log_{10} 12 = \log_{10}(2^2 \cdot 3) = 2a + b$. The denominator is $\log_{10} 5 = \log_{10}(10/2) = 1 - a$. Therefore

$$\log_5 12 = \frac{\log_{10} 12}{\log_{10} 5} = \frac{2a + b}{1 - a}.$$

Answer: (D) $\frac{2a + b}{1 - a}$. □

Problem 2.4.3 (1977 AHSME, Problem #18). If

$$y = (\log_2 3)(\log_3 4) \cdots (\log_n [n+1]) \cdots (\log_{31} 32),$$

then

(A) $4 < y < 5$ (B) $y = 5$ (C) $5 < y < 6$ (D) $y = 6$ (E) $6 < y < 7$

Strategy. Change every logarithm to the same base and cancel equal factors in adjacent fractions.

Solution. Changing to base 2 gives

$$y = \frac{\log_2 3}{\log_2 2} \cdot \frac{\log_2 4}{\log_2 3} \cdots \frac{\log_2 32}{\log_2 31} = \frac{\log_2 32}{\log_2 2} = 5.$$

Each intermediate logarithm appears once in a numerator and once in a denominator, so all of those factors cancel.

Answer: (B) $y = 5$. □

3. DOMAINS AND EQUATIONS

3.1. Domain Restrictions. The **domain** of an expression is the set of inputs for which it is defined as a real number. Every logarithm requires a positive argument and a positive base different from 1. For example, $\log_{x-1}(x+2)$ requires $x+2 > 0$, $x-1 > 0$, and $x-1 \neq 1$. Together these give $x > 1$, $x \neq 2$.

For nested logarithms, apply the restriction to every layer. If $b > 1$, then $\log_b u > 0$ exactly when $u > 1$. If $0 < b < 1$, then $\log_b u > 0$ exactly when $0 < u < 1$. A base between 0 and 1 reverses inequalities because its logarithm decreases as the argument increases.

More generally, for $b > 1$, $\log_b u < c$ exactly when $0 < u < b^c$. For $0 < b < 1$, $\log_b u < c$ exactly when $u > b^c$. Keep the condition $u > 0$ when changing a logarithmic inequality into an exponential one.

Example.

Problem 3.1.1. Find the domain of $\log_2(\log_{1/3} x)$.

Strategy. The outer logarithm requires its argument to be positive. Translate that condition using the inner base, which is less than 1.

Solution. First $x > 0$ is required. The outer logarithm also requires $\log_{1/3} x > 0$. Since $0 < 1/3 < 1$, this means $0 < x < 1$. Every x in this interval makes both logarithms defined.

Answer: $0 < x < 1$. □

3.2. Combining Logarithms. To solve a logarithmic equation, first record the domain. Then combine logarithms with the same base or change them to a common base. Equal logarithms with the same valid base have equal arguments. After solving the resulting equation, check each candidate against the original domain. Combining two logarithms can hide a restriction: the product of two negative arguments is positive, even though neither original logarithm exists.

Examples.

Problem 3.2.1 (1968 AHSME, Problem #23). If all the logarithms are real numbers, the equality

$$\begin{aligned} \log(x+3) + \log(x-1) \\ = \log(x^2 - 2x - 3) \end{aligned}$$

is satisfied for:

- (A) all real values of x (B) no real values of x (C) all real values of x except $x = 0$
 (D) no real values of x except $x = 0$ (E) all real values of x except $x = 1$

Strategy. Find the domain before combining the two logarithms on the left.

Solution. The first two arguments require $x > -3$ and $x > 1$. The third requires $(x-3)(x+1) > 0$. Together the restrictions give $x > 3$.

On this domain the product rule gives $(x+3)(x-1) = x^2 - 2x - 3$. Expanding the left side and canceling common terms gives $4x = 0$. The only candidate is $x = 0$, which is outside the domain. Thus no real value satisfies the original equation.

Answer: (B) no real values of x . □

Problem 3.2.2 (1997 AHSME, Problem #17). A line $x = k$ intersects the graph of $y = \log_5 x$ and the graph of $y = \log_5(x+4)$. The distance between the points of intersection is 0.5. Given that $k = a + \sqrt{b}$, where a and b are integers, what is $a + b$?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Strategy. The points have the same horizontal coordinate. Their vertical separation is the difference of the two logarithms.

Solution. Both logarithms require $k > 0$. Since the base is greater than 1 and $k+4 > k$, the upper point has height $\log_5(k+4)$. Therefore

$$\log_5(k+4) - \log_5 k = \frac{1}{2}, \quad \log_5 \frac{k+4}{k} = \frac{1}{2}.$$

Converting to exponential form gives $(k+4)/k = \sqrt{5}$, so $k = 4/(\sqrt{5}-1) = \sqrt{5}+1$. This value is positive, so it is allowed. Thus $a = 1$, $b = 5$, and $a + b = 6$.

Answer: (A) 6. □

3.3. Logarithmic Substitution. If the same logarithm occurs several times, replace it with one variable. For example, setting $t = \log_b x$ gives $x = b^t$. There is one positive value of x for each real t , so solve for t and then convert back. In a system, setting $u = \log_b x$ and $v = \log_b y$ changes products into sums and powers into multiples.

Taking a logarithm also removes an unknown from an exponent. For positive x , the equation $x^{\log_b x} = c$ becomes $(\log_b x)^2 = \log_b c$. This step uses the power rule and requires $c > 0$.

Example.

Problem 3.3.1 (1962 AHSME, Problem #28). The set of x -values satisfying the equation

$$x^{\log_{10} x} = \frac{x^3}{100}$$

consists of:

(A) $\frac{1}{10}$, only (B) 10, only (C) 100, only (D) 10 or 100, only (E) more than two real numbers

Strategy. Set $t = \log_{10} x$ and take the base-10 logarithm of both sides.

Solution. The original equation requires $x > 0$. Both sides are then positive, so taking logarithms is reversible. The equation becomes

$$(\log_{10} x)^2 = 3 \log_{10} x - 2,$$

or $t^2 - 3t + 2 = (t-1)(t-2) = 0$. Thus $t = 1$ or $t = 2$, giving $x = 10$ or $x = 100$. Both are positive and satisfy the original equation: $10^1 = 10^3/100$ and $100^2 = 100^3/100$.

Answer: (D) 10 or 100, only. □

3.4. Unknown Bases. When the unknown is in the base of a logarithm, the base must be positive and different from 1. The definition changes $\log_x a = c$ into $x^c = a$, and the reciprocal identity $\log_x a = 1/\log_a x$ moves the unknown from the base into the argument. A power equation $x^r = a$ with $x > 0$, $a > 0$, and $r \neq 0$ has the single solution $x = a^{1/r}$. For example, $x^{3/2} = 8$ gives $x = 8^{2/3} = 4$.

Example.

Problem 3.4.1. Solve $\log_x 2 + \log_2 x = \frac{5}{2}$.

Strategy. Write $\log_x 2$ as $1/\log_2 x$ and set $t = \log_2 x$.

Solution. The base x requires $x > 0$ and $x \neq 1$, so $t = \log_2 x$ is a nonzero real number. The equation becomes

$$t + \frac{1}{t} = \frac{5}{2}.$$

Multiplying by $2t$ gives $2t^2 - 5t + 2 = (2t - 1)(t - 2) = 0$, so $t = 2$ or $t = \frac{1}{2}$. These give $x = 2^2 = 4$ and $x = 2^{1/2} = \sqrt{2}$. Both are positive and different from 1. They satisfy the equation: $\log_4 2 + \log_2 4 = \frac{1}{2} + 2$ and $\log_{\sqrt{2}} 2 + \log_2 \sqrt{2} = 2 + \frac{1}{2}$.

Answer: 4 and $\sqrt{2}$. □

4. PROBLEMS

Beginner.

- (1959 AHSME, Problem #11) The logarithm of 0.0625 to the base 2 is:
(A) 0.025 (B) 0.25 (C) 5 (D) -4 (E) -2
- (1993 AHSME, Problem #11) If $\log_2(\log_2(\log_2(x))) = 2$, then how many digits are in the base-ten representation for x ?
(A) 5 (B) 7 (C) 9 (D) 11 (E) 13
- (2010 AMC 12A, Problem #11) The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b ?
(A) $\frac{7}{15}$ (B) $\frac{7}{8}$ (C) $\frac{8}{7}$ (D) $\frac{15}{8}$ (E) $\frac{15}{7}$
- (2015 AMC 12B, Problem #8) What is the value of $(625^{\log_5 2015})^{\frac{1}{4}}$?
(A) 5 (B) $\sqrt[4]{2015}$ (C) 625 (D) 2015 (E) $\sqrt[4]{5^{2015}}$

Intermediate.

- (2010 AMC 12B, Problem #12) For what value of x does

$$\begin{aligned} &\log_{\sqrt{2}} \sqrt{x} + \log_2 x \\ &\quad + \log_4 (x^2) + \log_8 (x^3) \\ &\quad + \log_{16} (x^4) = 40? \end{aligned}$$

(A) 8 (B) 16 (C) 32 (D) 256 (E) 1024
- (2021 AMC 12B, Problem #9) What is the value of the following expression?

$$\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}$$

(A) 0 (B) 1 (C) $\frac{5}{4}$ (D) 2 (E) $\log_2 5$
- (2003 AMC 12B, Problem #17) If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$?
(A) $-\frac{1}{2}$ (B) 0 (C) $\frac{1}{2}$ (D) $\frac{3}{5}$ (E) 1
- (2019 AMC 12A, Problem #12) Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $\left(\log_2 \frac{x}{y}\right)^2$?
(A) $\frac{25}{2}$ (B) 20 (C) $\frac{45}{2}$ (D) 25 (E) 32
- (2004 AMC 12A, Problem #16) The set of all real numbers x for which

$$\log_{2004}(\log_{2003}(\log_{2002}(\log_{2001} x)))$$
is defined is $\{x \mid x > c\}$. What is the value of c ?
(A) 0 (B) 2001^{2002} (C) 2002^{2003} (D) 2003^{2004} (E) $2001^{2002^{2003}}$

10. (1984 AHSME, Problem #14) The product of all real roots of the equation $x^{\log_{10} x} = 10$ is
(A) 1 (B) -1 (C) 10 (D) 10^{-1} (E) none of these

Advanced.

11. (2015 AMC 12A, Problem #14) What is the value of a for which

$$\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \frac{1}{\log_4 a} = 1?$$

- (A) 9 (B) 12 (C) 18 (D) 24 (E) 36
12. (2018 AMC 12A, Problem #14) The solution to the equation $\log_{3x} 4 = \log_{2x} 8$, where x is a positive real number other than $\frac{1}{3}$ or $\frac{1}{2}$, can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?
(A) 5 (B) 13 (C) 17 (D) 31 (E) 35
13. (2022 AMC 12B, Problem #16) Suppose x and y are positive real numbers such that

$$x^y = 2^{64}$$

and

$$(\log_2 x)^{\log_2 y} = 2^7.$$

What is the greatest possible value of $\log_2 y$?

- (A) 3 (B) 4 (C) $3 + \sqrt{2}$ (D) $4 + \sqrt{3}$ (E) 7
14. (2014 AMC 12A, Problem #18) The domain of the function

$$f(x) = \log_{\frac{1}{2}} \left(\log_4 \left(\log_{\frac{1}{4}} \left(\log_{16} \left(\log_{\frac{1}{16}} x \right) \right) \right) \right)$$

is an interval of length $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

- (A) 19 (B) 31 (C) 271 (D) 319 (E) 511