

LOGARITHMS AND EXPONENTIAL EQUATIONS FOR AMC 12: SOLUTIONS

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1. SOLUTIONS

Beginner.

1. (1959 AHSME, Problem #11) The logarithm of 0.0625 to the base 2 is:
 (A) 0.025 (B) 0.25 (C) 5 (D) -4 (E) -2

Strategy. Express 0.0625 as a power of 2.

Solution. Since $0.0625 = 1/16 = 2^{-4}$, the definition of a logarithm gives $\log_2 0.0625 = -4$.

Answer: (D) -4 . □

2. (1993 AHSME, Problem #11) If $\log_2(\log_2(\log_2(x))) = 2$, then how many digits are in the base-ten representation for x ?
 (A) 5 (B) 7 (C) 9 (D) 11 (E) 13

Strategy. Undo the logarithms one at a time, starting with the outermost one.

Solution. Writing each equation in exponential form, starting with the outermost logarithm, gives

$$\log_2(\log_2 x) = 2^2 = 4, \quad \log_2 x = 2^4 = 16, \quad x = 2^{16} = 65536.$$

All three logarithms are defined, and 65536 has five digits.

Answer: (A) 5. □

3. (2010 AMC 12A, Problem #11) The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b ?
 (A) $\frac{7}{15}$ (B) $\frac{7}{8}$ (C) $\frac{8}{7}$ (D) $\frac{15}{8}$ (E) $\frac{15}{7}$

Strategy. Divide by 7^x to isolate a single exponential expression.

Solution. Since $7^{x+7} = 7^x \cdot 7^7$, the equation is $7^x \cdot 7^7 = 8^x$. Dividing by $7^x > 0$ gives

$$7^7 = \left(\frac{8}{7}\right)^x.$$

By the definition of a logarithm, $x = \log_{8/7} 7^7$, so the requested base is $8/7$.

Answer: (C) $\frac{8}{7}$. □

4. (2015 AMC 12B, Problem #8) What is the value of $(625^{\log_5 2015})^{\frac{1}{4}}$?
 (A) 5 (B) $\sqrt[4]{2015}$ (C) 625 (D) 2015 (E) $\sqrt[4]{5^{2015}}$

Strategy. Write 625 as a power of 5, then use the identity $b^{\log_b x} = x$.

Solution. Using $625 = 5^4$ and the power rule for exponents,

$$(625^{\log_5 2015})^{1/4} = 5^{4(\log_5 2015)/4} = 5^{\log_5 2015} = 2015.$$

Every base and argument is positive, so these operations are valid.

Answer: (D) 2015. □

Intermediate.

5. (2010 AMC 12B, Problem #12) For what value of x does

$$\begin{aligned} \log_{\sqrt{2}} \sqrt{x} + \log_2 x \\ + \log_4 (x^2) + \log_8 (x^3) \\ + \log_{16} (x^4) = 40? \end{aligned}$$

- (A) 8 (B) 16 (C) 32 (D) 256 (E) 1024

Strategy. Change every term to base 2.

Solution. The second term requires $x > 0$. On this domain,

$$\log_{\sqrt{2}} \sqrt{x} = \frac{\frac{1}{2} \log_2 x}{\frac{1}{2}} = \log_2 x, \quad \log_{2^k} (x^k) = \frac{k \log_2 x}{k} = \log_2 x$$

for $k = 2, 3, 4$. Thus all five terms equal $\log_2 x$, and $5 \log_2 x = 40$. It follows that $\log_2 x = 8$ and $x = 2^8 = 256$, which is positive.

Answer: (D) 256. □

6. (2021 AMC 12B, Problem #9) What is the value of the following expression?

$$\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}$$

- (A) 0 (B) 1 (C) $\frac{5}{4}$ (D) 2 (E) $\log_2 5$

Strategy. Use reciprocals to change every logarithm to base 2, then set $t = \log_2 5$.

Solution. Since $1/\log_{40} 2 = \log_2 40$ and $1/\log_{20} 2 = \log_2 20$, the expression is

$$(\log_2 80)(\log_2 40) - (\log_2 160)(\log_2 20).$$

Let $t = \log_2 5$. Factoring powers of 2 out of each argument gives

$$(t+4)(t+3) - (t+5)(t+2) = (t^2 + 7t + 12) - (t^2 + 7t + 10) = 2.$$

Answer: (D) 2. □

7. (2003 AMC 12B, Problem #17) If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$?

- (A) $-\frac{1}{2}$ (B) 0 (C) $\frac{1}{2}$ (D) $\frac{3}{5}$ (E) 1

Strategy. Use the domains to show $x, y > 0$, then solve for their logarithms.

Solution. The argument x^2y must be positive, so $x \neq 0$ and $y > 0$. Then $xy^3 > 0$ forces $x > 0$. Therefore $u = \log x$ and $v = \log y$ are defined, and the product and power rules turn the two equations into

$$u + 3v = 1, \quad 2u + v = 1.$$

Adding the first equation to twice the second gives $5u + 5v = 3$, hence $u + v = 3/5$. The requested logarithm is $\log(xy) = u + v$.

Answer: (D) $\frac{3}{5}$. □

8. (2019 AMC 12A, Problem #12) Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $\left(\log_2 \frac{x}{y}\right)^2$?

(A) $\frac{25}{2}$ (B) 20 (C) $\frac{45}{2}$ (D) 25 (E) 32

Strategy. Set $u = \log_2 x$ and $v = \log_2 y$. Find $u + v$ and uv .

Solution. The condition $xy = 64$ gives $u + v = 6$. By change of base, $\log_y 16 = 4/v$, where $v \neq 0$ because $y \neq 1$. Thus $u = 4/v$ and $uv = 4$. Consequently

$$\left(\log_2 \frac{x}{y}\right)^2 = (u - v)^2 = (u + v)^2 - 4uv = 36 - 16 = 20.$$

The values $u, v = 3 \pm \sqrt{5}$ are both nonzero, so $x = 2^u$ and $y = 2^v$ are positive, differ from 1, and satisfy both conditions.

Answer: (B) 20. □

9. (2004 AMC 12A, Problem #16) The set of all real numbers x for which

$$\log_{2004}(\log_{2003}(\log_{2002}(\log_{2001} x)))$$

is defined is $\{x \mid x > c\}$. What is the value of c ?

(A) 0 (B) 2001^{2002} (C) 2002^{2003} (D) 2003^{2004} (E) $2001^{2002^{2003}}$

Strategy. The outermost argument must be positive. Translate that requirement inward one logarithm at a time.

Solution. All four bases exceed 1. The outermost logarithm is defined precisely when

$$\begin{aligned} \log_{2003}(\log_{2002}(\log_{2001} x)) &> 0, \\ \log_{2002}(\log_{2001} x) &> 1, \\ \log_{2001} x &> 2002, \\ x &> 2001^{2002}. \end{aligned}$$

Each step uses the fact that, for a base $b > 1$, $\log_b u > c$ exactly when $u > b^c$. The final condition $x > 2001^{2002}$ also makes every inner argument positive, so it describes the domain exactly.

Answer: (B) 2001^{2002} . □

10. (1984 AHSME, Problem #14) The product of all real roots of the equation $x^{\log_{10} x} = 10$ is

(A) 1 (B) -1 (C) 10 (D) 10^{-1} (E) none of these

Strategy. Set $t = \log_{10} x$ and take a logarithm of the equation.

Solution. The equation requires $x > 0$. Let $t = \log_{10} x$. By the power rule, the base-10 logarithm of the left side is $\log_{10}(x^t) = t \log_{10} x = t^2$, and that of the right side is 1. Thus

$$t^2 = 1,$$

so $t = 1$ or $t = -1$. Hence the roots are $x = 10$ and $x = 10^{-1}$. Both satisfy the original equation, since $10^1 = 10$ and $(10^{-1})^{-1} = 10$. Their product is $10 \cdot 10^{-1} = 1$.

Answer: (A) 1. □

Advanced.

11. (2015 AMC 12A, Problem #14) What is the value of a for which

$$\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \frac{1}{\log_4 a} = 1?$$

- (A) 9 (B) 12 (C) 18 (D) 24 (E) 36

Strategy. Replace each reciprocal logarithm by a logarithm to base a .

Solution. The expression requires $a > 0$ and $a \neq 1$. The reciprocal identity gives

$$\log_a 2 + \log_a 3 + \log_a 4 = 1,$$

so $\log_a(2 \cdot 3 \cdot 4) = \log_a 24 = 1$. Thus $a = 24$, which meets the domain restrictions and satisfies the equation.

Answer: (D) 24. □

12. (2018 AMC 12A, Problem #14) The solution to the equation $\log_{3x} 4 = \log_{2x} 8$, where x is a positive real number other than $\frac{1}{3}$ or $\frac{1}{2}$, can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

- (A) 5 (B) 13 (C) 17 (D) 31 (E) 35

Strategy. Change both sides to a common base and clear the nonzero denominators.

Solution. Using base 2, the given equation becomes

$$\frac{2}{\log_2(3x)} = \frac{3}{\log_2(2x)}.$$

Both denominators are nonzero by the given restrictions. Therefore

$$2 \log_2(2x) = 3 \log_2(3x), \quad \log_2((2x)^2) = \log_2((3x)^3).$$

Equal logarithms with the same base have equal arguments, so $4x^2 = 27x^3$. Since $x > 0$, divide by x^2 to get $x = 4/27$. This is neither $1/3$ nor $1/2$, so it is valid. The fraction is in lowest terms, and $p + q = 4 + 27 = 31$.

Answer: (D) 31. □

13. (2022 AMC 12B, Problem #16) Suppose x and y are positive real numbers such that

$$x^y = 2^{64}$$

and

$$(\log_2 x)^{\log_2 y} = 2^7.$$

What is the greatest possible value of $\log_2 y$?

- (A) 3 (B) 4 (C) $3 + \sqrt{2}$ (D) $4 + \sqrt{3}$ (E) 7

Strategy. Set $p = \log_2 x$ and $q = \log_2 y$. Take base-2 logarithms to obtain an equation in q .

Solution. Since $y > 0$, a base $x \leq 1$ would give $x^y \leq 1$. Because $x^y = 2^{64} > 1$, it follows that $x > 1$ and $p = \log_2 x > 0$. Taking the base-2 logarithm of the first equation gives $yp = 64$. Since $y = 2^q$,

$$p = 2^{6-q}, \quad \log_2 p = 6 - q.$$

The second equation is $p^q = 2^7$. Taking its base-2 logarithm gives

$$q \log_2 p = 7, \quad q(6 - q) = 7, \quad (q - 3)^2 = 2.$$

Thus $q = 3 \pm \sqrt{2}$. Both are possible: for either choice set $y = 2^q$ and $x = 2^{2^{6-q}}$, which are positive and satisfy both original equations. The larger value is $3 + \sqrt{2}$.

Answer: (C) $3 + \sqrt{2}$. □

14. (2014 AMC 12A, Problem #18) The domain of the function

$$f(x) = \log_{\frac{1}{2}} \left(\log_4 \left(\log_{\frac{1}{4}} \left(\log_{16} \left(\log_{\frac{1}{16}} x \right) \right) \right) \right)$$

is an interval of length $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

- (A) 19 (B) 31 (C) 271 (D) 319 (E) 511

Strategy. Work inward from the outermost logarithm, reversing inequalities at bases less than 1.

Solution. Set $u = \log_{1/16} x$ and $v = \log_{16} u$. The expression becomes $\log_{1/2}(\log_4(\log_{1/4} v))$. Starting with its outermost argument, it is defined exactly when

$$\log_4(\log_{1/4} v) > 0 \iff \log_{1/4} v > 1 \iff 0 < v < \frac{1}{4}.$$

Because $v = \log_{16} u$ and $16 > 1$, this becomes $1 < u < 16^{1/4} = 2$. Finally, $u = \log_{1/16} x$ and $1/16 < 1$, so

$$\left(\frac{1}{16}\right)^2 < x < \frac{1}{16}, \quad \text{or} \quad \frac{1}{256} < x < \frac{1}{16}.$$

These bounds make every argument positive. The interval has length $1/16 - 1/256 = 15/256$. Since 15 and 256 are relatively prime, $m + n = 15 + 256 = 271$.

Answer: (C) 271. □

2. ANSWER KEY

Problem	Letter	Value
1	D	-4
2	A	5
3	C	$\frac{8}{7}$
4	D	2015
5	D	256
6	D	2
7	D	$\frac{3}{5}$
8	B	20
9	B	2001^{2002}
10	A	1
11	D	24
12	D	31
13	C	$3 + \sqrt{2}$
14	C	271