LOGARITHMS AND EXPONENTIAL EQUATIONS FOR AMC 12: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (2000 AMC 12, Problem #7) How many positive integers b have the property that $\log_b 729$ is a positive integer?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Strategy. By the definition of a logarithm, $\log_b 729 = n$ means $b^n = 729$. Since $729 = 3^6$, the base must be a power of 3 whose exponent divides 6.

Solution. A base must be positive and different from 1, so $b \geq 2$. Suppose $\log_b 729 = n$ for a positive integer n. By the definition of a logarithm, $b^n = 729 = 3^6$. Every prime factor of b divides $b^n = 3^6$, so b is a power of 3, say $b = 3^m$ with $m \geq 1$. Then $3^{mn} = 3^6$, and equal powers of 3 have equal exponents, so $mn = 6$. Hence m is one of the divisors 1, 2, 3, 6 of 6, which gives

$$\log_3 729 = 6, \qquad \log_9 729 = 3, \qquad \log_{27} 729 = 2, \qquad \log_{729} 729 = 1.$$

All four values are positive integers, so exactly 4 integers b have the property.

Answer: (E) 4. □

2. (1958 AHSME, Problem #20) If $4^x - 4^{x-1} = 24$, then $(2x)^x$ equals:

(A) $5\sqrt{5}$ **(B)** $\sqrt{5}$ **(C)** $25\sqrt{5}$ **(D)** 125 **(E)** 25

Strategy. Factor 4^{x-1} out of the left side, then write both sides of the resulting equation as powers of 2.

Solution. Since $4^x = 4 \cdot 4^{x-1}$, the left side is $4 \cdot 4^{x-1} - 4^{x-1} = 3 \cdot 4^{x-1}$. The equation becomes $3 \cdot 4^{x-1} = 24$, so $4^{x-1} = 8$. Writing both sides as powers of 2 gives

$$2^{2x-2} = 2^3.$$

Equal powers of 2 have equal exponents, so $2x - 2 = 3$ and $x = \frac{5}{2}$. Then $2x = 5$, and

$$(2x)^x = 5^{5/2} = 5^2 \cdot 5^{1/2} = 25\sqrt{5}.$$

As a check, $4^{5/2} - 4^{3/2} = 32 - 8 = 24$.

Answer: (C) $25\sqrt{5}$. □

3. (1982 AHSME, Problem #17) How many real numbers x satisfy the equation $3^{2x+2} - 3^{x+3} - 3^x + 3 = 0$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Strategy. Each term is a constant times a power of 3^x. Substitute $t = 3^x$, solve the quadratic in t, and keep only the positive roots.

Solution. Let $t = 3^x$, so $t > 0$. The exponent rules give $3^{2x+2} = 9(3^x)^2 = 9t^2$ and $3^{x+3} = 27t$, so the equation becomes

$$9t^2 - 27t - t + 3 = 9t^2 - 28t + 3 = (9t - 1)(t - 3) = 0.$$

Both roots, $t = \frac{1}{9}$ and $t = 3$, are positive. Each positive value of 3^x occurs for exactly one real x, so the equation has exactly two solutions: $3^x = \frac{1}{9}$ gives $x = -2$, and $3^x = 3$ gives $x = 1$. Both check: $3^{-2} - 3 - 3^{-2} + 3 = 0$ and $3^4 - 3^4 - 3 + 3 = 0$.

Answer: (C) 2. □

4. (2018 AMC 12B, Problem #7) What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdots \log_{21} 25 \cdot \log_{23} 27?$$

(A) 3 **(B)** $3\log_7 23$ **(C)** 6 **(D)** 9 **(E)** 10

Strategy. Each factor has the form $\log_n(n+4)$. Split the factors into two chains in which each argument is the base of the next factor, and collapse each chain with $(\log_a b)(\log_b c) = \log_a c$.

Solution. The bases are the odd numbers $3, 5, 7, \ldots, 23$, and the factor with base n is $\log_n(n+4)$. By the change-of-base formula,

$$(\log_a b)(\log_b c) = \frac{\log b}{\log a} \cdot \frac{\log c}{\log b} = \log_a c,$$

and the same cancellation shortens a longer chain to a single logarithm. The factors with bases $3, 7, 11, 15, 19, 23$ form one chain, and those with bases $5, 9, 13, 17, 21$ form another:

$$(\log_3 7)(\log_7 11)(\log_{11} 15)(\log_{15} 19)(\log_{19} 23)(\log_{23} 27) = \log_3 27 = 3,$$

$$(\log_5 9)(\log_9 13)(\log_{13} 17)(\log_{17} 21)(\log_{21} 25) = \log_5 25 = 2.$$

The two chains contain each of the 11 factors exactly once, so the product is $3 \cdot 2 = 6$.

Answer: (C) 6. □

5. (2023 AMC 12B, Problem #7) For how many integers n does the expression

$$\sqrt{\frac{\log(n^2) - (\log n)^2}{\log n - 3}}$$

represent a real number, where log denotes the base 10 logarithm?

(A) 900 **(B)** 3 **(C)** 902 **(D)** 2 **(E)** 901

Strategy. Find where the expression is defined, then set $t = \log n$ and determine the sign of the fraction under the square root.

Solution. The logarithm requires $n > 0$, and the denominator requires $\log n \neq 3$, that is, $n \neq 1000$. For $n > 0$ the power rule gives $\log(n^2) = 2\log n$, so with $t = \log n$ the expression under the square root is

$$\frac{2t - t^2}{t - 3} = \frac{t(2-t)}{t-3}.$$

The square root is a real number exactly when this fraction is nonnegative. Since n is a positive integer, $t \geq 0$. The numerator $t(2-t)$ is 0 at $t = 0$ and $t = 2$, positive for $0 < t < 2$, and negative for $t > 2$. The denominator is negative for $t < 3$ and positive for $t > 3$. Hence the fraction is nonnegative exactly when $t = 0$, $t = 2$, or $2 < t < 3$.

The values $t = 0$ and $t = 2$ give $n = 1$ and $n = 100$. Since the base 10 is greater than 1, the condition $2 < \log n < 3$ means $100 < n < 1000$, which holds for the 899 integers from 101 to 999. The total is $1 + 1 + 899 = 901$.

Answer: (E) 901. □

Intermediate.

6. (2022 AMC 12A, Problem #11) What is the product of all real numbers x such that the distance on the number line between $\log_6 x$ and $\log_6 9$ is twice the distance on the number line between $\log_6 10$ and 1?

(A) 10 **(B)** 18 **(C)** 25 **(D)** 36 **(E)** 81

Strategy. Write each distance as the absolute value of a difference. Combine the logarithms with the quotient and power rules, and then compare arguments.

Solution. The logarithm $\log_6 x$ requires $x > 0$. Since $1 = \log_6 6$ and $10 > 6$,

$$\log_6 10 - 1 = \log_6 \frac{10}{6} = \log_6 \frac{5}{3} > 0,$$

so twice the second distance is $2\log_6 \frac{5}{3} = \log_6 \frac{25}{9}$ by the power rule. By the quotient rule, $\log_6 x - \log_6 9 = \log_6 \frac{x}{9}$, and the condition becomes

$$\left|\log_6 \frac{x}{9}\right| = \log_6 \frac{25}{9}.$$

Thus $\log_6 \frac{x}{9}$ equals $\log_6 \frac{25}{9}$ or $-\log_6 \frac{25}{9} = \log_6 \frac{9}{25}$. Equal logarithms with the same base have equal arguments, so $\frac{x}{9} = \frac{25}{9}$ or $\frac{x}{9} = \frac{9}{25}$, which gives $x = 25$ or $x = \frac{81}{25}$. Both values are positive, and their product is $25 \cdot \frac{81}{25} = 81$.

Answer: (E) 81. □

7. (2022 AMC 12A, Problem #14) What is the value of

$$(\log 5)^3 + (\log 20)^3 + (\log 8)(\log 0.25)$$

where log denotes the base-ten logarithm?

(A) $\frac{3}{2}$ **(B)** $\frac{7}{4}$ **(C)** 2 **(D)** $\frac{9}{4}$ **(E)** 3

Strategy. Express every logarithm in terms of $a = \log 2$, using $\log 10 = 1$.

Solution. Let $a = \log 2$. Since $\log 10 = 1$, the product, quotient, and power rules give

$$\log 5 = \log \frac{10}{2} = 1 - a, \qquad \log 20 = \log(10 \cdot 2) = 1 + a,$$

$$\log 8 = \log 2^3 = 3a, \qquad \log 0.25 = \log 2^{-2} = -2a.$$

Expanding the cubes,

$$(1-a)^3 + (1+a)^3 = \left(1 - 3a + 3a^2 - a^3\right) + \left(1 + 3a + 3a^2 + a^3\right) = 2 + 6a^2.$$

Therefore the expression equals $2 + 6a^2 + (3a)(-2a) = 2$.

Answer: (C) 2. □

8. (2021 AMC 12A, Problem #14) What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

(A) 21 **(B)** $100\log_5 3$ **(C)** $200\log_3 5$ **(D)** 2,200 **(E)** 21,000

Strategy. Simplify each term with $\log_{b^r}(x^s) = \frac{s}{r}\log_b x$. The two sums become multiples of $\log_5 3$ and $\log_3 5$, which are reciprocals.

Solution. By the change-of-base formula, $\log_{b^r}(x^s) = \frac{s}{r}\log_b x$ for $r \neq 0$. In the first sum the base is 5^k and the argument is 3^{k^2}, so

$$\log_{5^k} 3^{k^2} = \frac{k^2}{k}\log_5 3 = k\log_5 3.$$

Since $1 + 2 + \cdots + 20 = \frac{20\cdot 21}{2} = 210$, the first sum is $210\log_5 3$. In the second sum every term is

$$\log_{9^k} 25^k = \frac{k}{k}\log_9 25 = \log_{3^2} 5^2 = \log_3 5,$$

so the second sum is $100\log_3 5$. By the reciprocal identity, $(\log_5 3)(\log_3 5) = 1$. Hence the product is

$$210 \cdot 100 \cdot (\log_5 3)(\log_3 5) = 21{,}000.$$

Answer: (E) 21,000. □

9. (1968 AHSME, Problem #29) Given the three numbers x, $y = x^x$, $z = x^{(x^x)}$, with $0.9 < x < 1.0$. Arranged in order of increasing magnitude, they are:

(A) x, z, y **(B)** x, y, z **(C)** y, x, z **(D)** y, z, x **(E)** z, x, y

Strategy. With $0 < x < 1$, the power x^s decreases as s increases. Put the exponents 0, x, y, and 1 in order, and the powers come out in the reverse order.

Solution. Since $0 < x < 1$, the power x^s decreases as s increases. The exponents satisfy $0 < x < 1$, so

$$x^1 < x^x < x^0, \qquad \text{that is,} \qquad x < y < 1.$$

Now $z = x^y$, and its exponent y lies between x and 1. Again because x^s decreases as s increases, $x^1 < x^y < x^x$, that is, $x < z < y$. In increasing order the numbers are x, z, y.

Answer: (A) x, z, y. □

10. (2009 AIME II, Problem #2) Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

Strategy. Each requested term is a given value raised to the same logarithm once more; for example, $a^{(\log_3 7)^2} = 27^{\log_3 7}$. Then use the definition of a logarithm in the form $3^{\log_3 7} = 7$.

Solution. All the bases are positive, so the exponent rule $(u^r)^s = u^{rs}$ applies. For the first term,

$$a^{(\log_3 7)^2} = \left(a^{\log_3 7}\right)^{\log_3 7} = 27^{\log_3 7} = \left(3^{\log_3 7}\right)^3 = 7^3 = 343,$$

because $3^{\log_3 7} = 7$ by the definition of a logarithm. In the same way,

$$b^{(\log_7 11)^2} = 49^{\log_7 11} = \left(7^{\log_7 11}\right)^2 = 11^2 = 121,$$

and

$$c^{(\log_{11} 25)^2} = \left(\sqrt{11}\right)^{\log_{11} 25} = \left(11^{\log_{11} 25}\right)^{1/2} = 25^{1/2} = 5.$$

The sum is $343 + 121 + 5 = 469$.

Answer: 469. □

Advanced.

11. (2017 AMC 12A, Problem #20) How many ordered pairs (a, b) such that a is a positive real number and b is an integer between 2 and 200, inclusive, satisfy the equation $(\log_b a)^{2017} = \log_b(a^{2017})$?

(A) 198 **(B)** 199 **(C)** 398 **(D)** 399 **(E)** 597

Strategy. For $a > 0$ the power rule turns the right side into $2017 \log_b a$. Substitute $t = \log_b a$, count the real values of t, and convert back to a.

Solution. Every integer b from 2 to 200 is a valid base, and $a > 0$, so the power rule gives $\log_b(a^{2017}) = 2017 \log_b a$. With $t = \log_b a$, the equation becomes

$$t^{2017} = 2017t, \qquad \text{that is,} \qquad t(t^{2016} - 2017) = 0.$$

So $t = 0$ or $t^{2016} = 2017$. Since 2016 is even, $t^{2016} = |t|^{2016}$, and the second equation holds exactly when $|t| = 2017^{1/2016}$. This gives three different real values of t: 0 and $\pm 2017^{1/2016}$.

For a fixed base b, each real t comes from exactly one positive a, namely $a = b^t$, and different values of t give different values of a. Hence each of the 199 integers b gives exactly 3 pairs, for a total of $3 \cdot 199 = 597$.

Answer: (E) 597. □

12. (2022 AIME II, Problem #4) There is a positive real number x not equal to either $\frac{1}{20}$ or $\frac{1}{2}$ such that

$$\log_{20x}(22x) = \log_{2x}(202x).$$

The value $\log_{20x}(22x)$ can be written as $\log_{10}\left(\frac{m}{n}\right)$, where m and n are relatively prime positive integers. Find $m + n$.

Strategy. Call the common value k and rewrite both logarithms in exponential form. Dividing the two equations removes x.

Solution. Let $k = \log_{20x}(22x) = \log_{2x}(202x)$. The bases $20x$ and $2x$ are positive and different from 1, because $x > 0$ and $x \neq \frac{1}{20}, \frac{1}{2}$. By the definition of a logarithm,

$$(20x)^k = 22x \qquad \text{and} \qquad (2x)^k = 202x.$$

Dividing the first equation by the second gives

$$\left(\frac{20x}{2x}\right)^k = \frac{22x}{202x}, \qquad \text{that is,} \qquad 10^k = \frac{11}{101}.$$

Therefore $\log_{10} \frac{m}{n} = k = \log_{10} \frac{11}{101}$, and equal logarithms have equal arguments, so $\frac{m}{n} = \frac{11}{101}$. Since 101 is prime and does not divide 11, this fraction is in lowest terms. Thus $m = 11$, $n = 101$, and $m + n = 112$.

Answer: 112. □

13. (1988 AHSME, Problem #26) Suppose that p and q are positive numbers for which

$$\log_9(p) = \log_{12}(q) = \log_{16}(p + q).$$

What is the value of $\frac{q}{p}$?

(A) $\frac{4}{3}$ **(B)** $\frac{1}{2}(1 + \sqrt{3})$ **(C)** $\frac{8}{5}$ **(D)** $\frac{1}{2}(1 + \sqrt{5})$ **(E)** $\frac{16}{9}$

Strategy. Let t be the common value of the three logarithms and write p, q, and $p + q$ as powers. Dividing by 9^t leaves only powers of $\frac{4}{3}$.

Solution. Let t be the common value. By the definition of a logarithm,

$$p = 9^t, \qquad q = 12^t, \qquad p + q = 16^t,$$

so $9^t + 12^t = 16^t$. Dividing by $9^t > 0$ gives

$$1 + \left(\frac{12}{9}\right)^t = \left(\frac{16}{9}\right)^t.$$

Let $r = (4/3)^t$, so $r > 0$. Since $(12/9)^t = r$ and $(16/9)^t = ((4/3)^2)^t = r^2$, the equation becomes

$$1 + r = r^2, \qquad \text{that is,} \qquad r^2 - r - 1 = 0.$$

By the quadratic formula, $r = (1 \pm \sqrt{5})/2$. Only the root with the plus sign is positive, so $r = (1 + \sqrt{5})/2$. Finally,

$$\frac{q}{p} = \frac{12^t}{9^t} = \left(\frac{4}{3}\right)^t = r = \frac{1}{2}\left(1 + \sqrt{5}\right).$$

This value occurs: for $t = \log_{4/3} r$, the positive numbers $p = 9^t$ and $q = 12^t$ satisfy $p + q = 16^t$, so all three logarithms equal t.

Answer: (D) $\frac{1}{2}(1 + \sqrt{5})$. □

14. (2009 AMC 12A, Problem #24) The tower function of twos is defined recursively as follows: $T(1) = 2$ and $T(n + 1) = 2^{T(n)}$ for $n \geq 1$. Let $A = (T(2009))^{T(2009)}$ and $B = (T(2009))^A$. What is the largest integer k such that

$$\underbrace{\log_2 \log_2 \log_2 \ldots \log_2}_{k} B$$

is defined?

(A) 2009 **(B)** 2010 **(C)** 2011 **(D)** 2012 **(E)** 2013

Strategy. Each logarithm needs a positive argument. Compute the first logarithms of B exactly with $\log_2 T(m) = T(m - 1)$, then show that each later value lies just above a smaller tower value, until the values drop below 1.

Solution. Write L_j for the number obtained from B by taking $\log_2$ a total of j times. Since $T(m) = 2^{T(m-1)}$, the definition of a logarithm gives $\log_2 T(m) = T(m - 1)$ for $m \geq 2$. By the power rule,

$$L_1 = \log_2 \left(T(2009)^A\right) = A\,T(2008) \qquad \text{and} \qquad \log_2 A = T(2009)\,T(2008),$$

so by the product rule

$$L_2 = \log_2 A + \log_2 T(2008) = T(2009)\,T(2008) + T(2007).$$

Let $P = T(2009)\,T(2008)$. Then $P < L_2 < 2P$ because $0 < T(2007) < P$, and $\log_2 P = T(2008) + T(2007)$, so

$$T(2008) + T(2007) < L_3 < T(2008) + T(2007) + 1.$$

Since $2^m \geq m + 1$ for every positive integer m, $T(2007) + 1 \leq 2^{T(2007)} = T(2008)$. This gives $T(2008) < L_3 < 2\,T(2008)$, and one more logarithm gives

$$T(2007) < L_4 < T(2007) + 1.$$

Now suppose $T(m) < L < T(m)+1$ for some $m \geq 2$. Because $T(m) \geq 1$, also $T(m)+1 \leq 2\,T(m)$, so $\log_2 L$ lies between $\log_2 T(m) = T(m-1)$ and $\log_2(2\,T(m)) = T(m-1)+1$. Applying this step 2006 times, starting from L_4, gives

$$T(1) < L_{2010} < T(1)+1, \qquad \text{that is,} \qquad 2 < L_{2010} < 3.$$

Then $1 < L_{2011} < \log_2 3 < 2$, so $0 < L_{2012} < 1$. Thus $L_{2013} = \log_2 L_{2012}$ is defined and negative. All of $L_1, \ldots, L_{2012}$ are positive, so the first 2013 logarithms are defined, but the next one would need $L_{2013} > 0$. The largest k is 2013.

Answer: (E) 2013. □

15. (2017 AIME II, Problem #7) Find the number of integer values of k in the closed interval $[-500, 500]$ for which the equation $\log(kx) = 2\log(x+2)$ has exactly one real solution.

Strategy. Record the domain, $kx > 0$ and $x > -2$, before combining. On the domain the equation is equivalent to $kx = (x+2)^2$, so count the roots of this quadratic that lie in the domain, separately for $k < 0$ and $k > 0$.

Solution. The equation requires $kx > 0$ and $x + 2 > 0$; in particular $k \neq 0$. On this domain the power rule gives $2\log(x+2) = \log\left((x+2)^2\right)$, and equal logarithms have equal arguments. So the solutions are exactly the numbers x in the domain with

$$kx = (x+2)^2, \qquad \text{that is,} \qquad x^2 + (4-k)x + 4 = 0.$$

Both cases below use a fact about a quadratic f with leading coefficient 1: if $f(c) < 0$, then f has exactly one root less than c and exactly one root greater than c, and f is negative between its roots and positive outside them.

Case $k < 0$. Here $kx > 0$ forces $x < 0$, so the domain is $-2 < x < 0$. Let $f(x) = x^2 + (4-k)x + 4$. Then $f(-2) = 2k < 0$, so f has one root less than -2 and one root greater than -2. Since $f(0) = 4 > 0$, the number 0 is not between the roots, and since $0 > -2$, it lies to the right of both. So the larger root lies between -2 and 0, while the smaller one is outside the domain. The equation has exactly one solution for each of the 500 values $k = -500, \ldots, -1$.

Case $k > 0$. Here the domain is $x > 0$. Since $(x+2)^2 - 8x = (x-2)^2$, the equation $kx = (x+2)^2$ is equivalent to

$$(x-2)^2 = (k-8)x.$$

If $0 < k < 8$, the right side is negative for $x > 0$ and the left side is not, so there is no solution. If $k = 8$, the only solution is $x = 2$, which lies in the domain. If $k > 8$, let $g(x) = (x-2)^2 - (k-8)x$. Then $g(2) = -2(k-8) < 0$ and $g(0) = 4 > 0$, so g has one root between 0 and 2 and another root greater than 2. Both lie in the domain, so there are two solutions. Among the positive values, only $k = 8$ works.

The total is $500 + 1 = 501$.

Answer: 501. □

2. Answer Key

Problem	Letter	Value
1	E	4
2	C	$25\sqrt{5}$
3	C	2
4	C	6
5	E	901
6	E	81
7	C	2
8	E	21,000
9	A	x, z, y
10		469
11	E	597
12		112
13	D	$\frac{1}{2}(1+\sqrt{5})$
14	E	2013
15		501