SIMILAR TRIANGLES AND AREA RATIOS FOR AMC 10

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1. Similar Triangles

Two triangles are similar when one is a scaled copy of the other. This section defines similarity, shows that a line parallel to a side of a triangle cuts off a similar triangle, and gives three tests for similarity.

1.1. **Similarity.**

Definition 1.1.1. Triangles ABC and DEF are **similar**, written $\triangle ABC \sim \triangle DEF$, if

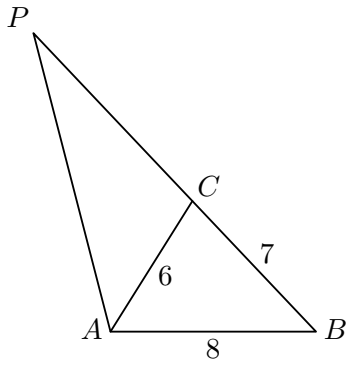
$$\angle A = \angle D, \qquad \angle B = \angle E, \qquad \angle C = \angle F \qquad \text{and} \qquad \frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA}.$$

The common value k of the three ratios is the **ratio of similarity**. The order of the letters matters: A corresponds to D, B to E and C to F.

To use a similarity, write the ratios of corresponding sides in the order given by the letters, as in the next example.

Example.

Problem 1.1.2 (1986 AHSME, Problem #16)**.** In $\triangle ABC$, $AB = 8$, $BC = 7$, $CA = 6$ and side BC is extended, as shown in the figure, to a point P so that $\triangle PAB$ is similar to $\triangle PCA$. The length of PC is



(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Strategy. The order of the letters in $\triangle PAB \sim \triangle PCA$ matches P with P, A with C and B with A. Write the equal ratios of corresponding sides, and use $PB = PC + 7$.

Solution. Let $x = PC$. The correspondence $P \to P$, $A \to C$, $B \to A$ gives

$$\frac{PA}{PC} = \frac{AB}{CA} = \frac{PB}{PA}, \qquad \text{where} \qquad \frac{AB}{CA} = \frac{8}{6} = \frac{4}{3}.$$

So $PA = \frac{4}{3}x$ and $PB = \frac{4}{3}PA = \frac{16}{9}x$. The point P lies on the extension of BC beyond C, as in the figure, so $PB = PC + CB = x + 7$. Hence

$$\frac{16}{9}x = x + 7, \qquad \frac{7}{9}x = 7, \qquad x = 9.$$

Then $PA = 12$ and $PB = 16$. As a check, the sides $PA = 12$, $AB = 8$, $BP = 16$ of triangle PAB are $\frac{4}{3}$ times the corresponding sides $PC = 9$, $CA = 6$, $AP = 12$ of triangle PCA.

Answer: (C) 9. □

1.2. **A Line Parallel to a Side.** The proofs in this subsection compare areas. Throughout, $[ABC]$ is the area of triangle ABC and $[ABCD]$ is the area of quadrilateral $ABCD$. The equal-heights theorem states that two triangles with equal heights have areas in the ratio of their bases.

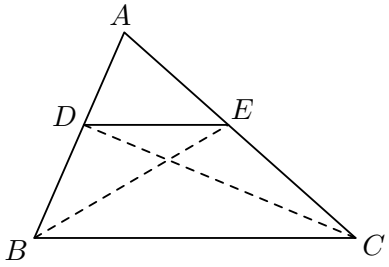
Theorem 1.2.1 (Equal areas between parallels)**.** *Let P and Q be points on the same side of line AB. Then $[ABP] = [ABQ]$ if and only if $PQ \parallel AB$.*

Proof. The triangles have the common base AB, so their areas are equal exactly when their heights to AB are equal, that is, when P and Q are at the same distance from line AB. The points on the same side of line AB as P and at the same distance from it as P form the line through P parallel to AB. So the areas are equal exactly when Q lies on this line, that is, when $PQ \parallel AB$. □

Theorem 1.2.2 (Side splitting)**.** *Let D and E be points of the sides AB and AC of triangle ABC such that $DE \parallel BC$. Then*

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC},$$

so $AD : DB = AE : EC$, and $\triangle ABC \sim \triangle ADE$ with ratio $\frac{AD}{AB}$.



Proof. Triangles DBE and DCE have the common base DE, and B and C lie on the same side of line DE with $BC \parallel DE$, so $[DBE] = [DCE]$ by Theorem 1.2.1. Triangles ADE and DBE have the common vertex E and their bases AD and DB on line AB, so $[ADE] : [DBE] = AD : DB$ by the equal-heights theorem. In the same way, with the common vertex D, $[ADE] : [DCE] = AE : EC$. The two left-hand sides are equal, so $AD : DB = AE : EC$, which is the same as $\frac{AD}{AB} = \frac{AE}{AC}$.

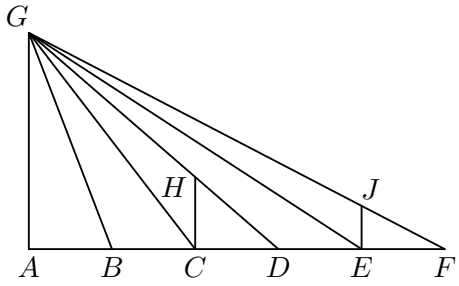
For the third ratio, let the line through E parallel to AB meet BC at F. Then $DBFE$ is a parallelogram, so $DE = BF$, and the first part, applied at the vertex C to the line $EF \parallel AB$, gives $\frac{CF}{CB} = \frac{CE}{CA}$. Hence

$$\frac{DE}{BC} = \frac{BF}{BC} = 1 - \frac{CF}{CB} = 1 - \frac{CE}{CA} = \frac{AE}{AC}.$$

Finally, $\angle ADE = \angle ABC$ and $\angle AED = \angle ACB$ are corresponding angles, and the angle at A is common, so $\triangle ABC \sim \triangle ADE$, with ratio $\frac{AD}{AB}$. □

Example.

Problem 1.2.3 (2002 AMC 10A, Problem #20)**.** Points A, B, C, D, E, and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line AF. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$ are parallel. Find $\frac{HC}{JE}$.



(A) $\frac{5}{4}$ **(B)** $\frac{4}{3}$ **(C)** $\frac{3}{2}$ **(D)** $\frac{5}{3}$ **(E)** 2

Strategy. Compare each of HC and JE with AG. The segment HC cuts triangle DAG parallel to its side AG, and JE cuts triangle FAG parallel to AG.

Solution. In triangle DAG, the points C and H lie on the sides DA and DG, and $CH \parallel AG$. By Theorem 1.2.2,

$$\frac{HC}{AG} = \frac{DC}{DA} = \frac{1}{3}.$$

In triangle FAG, the points E and J lie on the sides FA and FG, and $EJ \parallel AG$, so

$$\frac{JE}{AG} = \frac{FE}{FA} = \frac{1}{5}.$$

Dividing,

$$\frac{HC}{JE} = \frac{1/3}{1/5} = \frac{5}{3}.$$

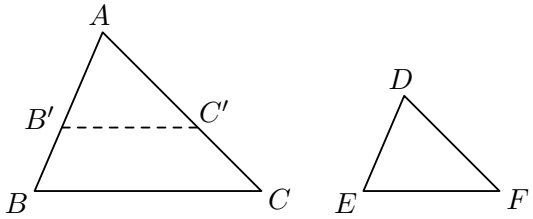
The answer does not depend on the position of G.

Answer: (D) $\frac{5}{3}$. □

1.3. **The Similarity Criteria.** To show that two triangles are similar, it is not necessary to check all six conditions of the definition.

Theorem 1.3.1 (Similarity criteria)**.** *Triangles ABC and DEF are similar, $\triangle ABC \sim \triangle DEF$, in each of the following cases.*

(a) (AA) $\angle A = \angle D$ and $\angle B = \angle E$.
(b) (SAS) $\angle A = \angle D$ and $\frac{DE}{AB} = \frac{DF}{AC}$.
(c) (SSS) $\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA}$.



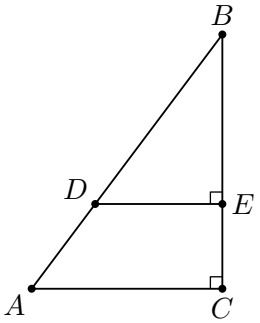
Proof. If $DE = AB$, the triangles are congruent by ASA, SAS or SSS respectively, so they are similar with ratio 1. Otherwise, exchanging the names of the two triangles if necessary, assume $DE < AB$. Let B' be the point of side AB with $AB' = DE$, and let the line through B' parallel to BC meet AC at C', as in the figure. By Theorem 1.2.2, $\triangle ABC \sim \triangle AB'C'$ with ratio $k = \frac{AB'}{AB} = \frac{DE}{AB}$. It remains to show that $\triangle AB'C'$ is congruent to $\triangle DEF$: then DEF has the angles of ABC, and its sides are k times as long.

(a) $\angle AB'C' = \angle B = \angle E$ (corresponding angles), $\angle A = \angle D$ and $AB' = DE$, so the triangles are congruent by ASA.
(b) $AC' = k \cdot AC = DF$, $\angle A = \angle D$ and $AB' = DE$, so they are congruent by SAS.
(c) $B'C' = k \cdot BC = EF$, $AC' = k \cdot AC = FD$ and $AB' = DE$, so they are congruent by SSS. □

Most applications use AA. The two equal angles usually come from parallel lines (corresponding or alternate interior angles), vertical angles, right angles, or an angle that the two triangles share.

Examples.

Problem 1.3.2 (1995 AHSME, Problem #8)**.** In triangle ABC, $\angle C = 90°$, $AC = 6$ and $BC = 8$. Points D and E are on $\overline{AB}$ and $\overline{BC}$, respectively, and $\angle BED = 90°$. If $DE = 4$, then $BD =$



(A) 5 **(B)** $\frac{16}{3}$ **(C)** $\frac{20}{3}$ **(D)** $\frac{15}{2}$ **(E)** 8

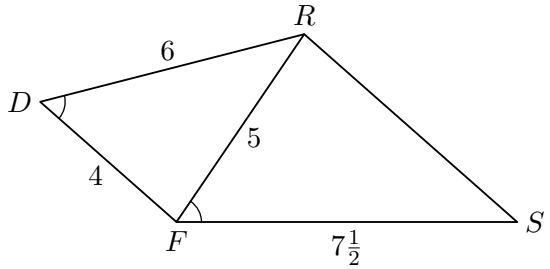
Strategy. Triangles BED and BCA both have a right angle, and they share the angle at B.

Solution. By the Pythagorean theorem, $AB = \sqrt{6^2 + 8^2} = 10$. Triangles BED and BCA have right angles at E and C and the common angle at B, so $\triangle BED \sim \triangle BCA$ by AA, with $B \to B$, $E \to C$, $D \to A$. Hence

$$\frac{BD}{BA} = \frac{ED}{CA}, \qquad \text{so} \qquad BD = 10 \cdot \frac{4}{6} = \frac{20}{3}.$$

Answer: (C) $\frac{20}{3}$. □

Problem 1.3.3 (1964 AHSME, Problem #29)**.** In this figure $\angle RFS = \angle FDR$, $FD = 4$ inches, $DR = 6$ inches, $FR = 5$ inches, and $FS = 7\frac{1}{2}$ inches. The length of RS, in inches, is:



(A) undetermined **(B)** 4 **(C)** $5\frac{1}{2}$ **(D)** 6 **(E)** $6\frac{1}{4}$

Strategy. Compare triangle DFR, with the angle at D, and triangle FRS, with the equal angle at F. Check whether the sides around these two angles are proportional.

Solution. In triangle DFR the sides around the angle at D are $DF = 4$ and $DR = 6$. In triangle FRS the sides around the equal angle at F are $FR = 5$ and $FS = \frac{15}{2}$. Since

$$\frac{FR}{DF} = \frac{5}{4} \qquad \text{and} \qquad \frac{FS}{DR} = \frac{15/2}{6} = \frac{5}{4},$$

the triangles are similar by SAS, with $D \to F$, $F \to R$, $R \to S$. The side RS of triangle FRS corresponds to the side FR of triangle DFR, so

$$RS = \frac{5}{4} \cdot FR = \frac{5}{4} \cdot 5 = \frac{25}{4} = 6\tfrac{1}{4}.$$

The other pairing of the sides does not give a similarity, since $\frac{FR}{DR} = \frac{5}{6}$ while $\frac{FS}{DF} = \frac{15}{8}$.

Answer: (E) $6\frac{1}{4}$. □

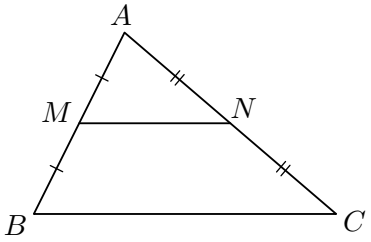
2. Midsegments and Trapezoids

Midpoints and parallel sides produce similar triangles with simple ratios. This section treats the segment joining two midpoints and the diagonals of a trapezoid.

2.1. The Midsegment.

A **midsegment** of a triangle is a segment joining the midpoints of two of its sides.

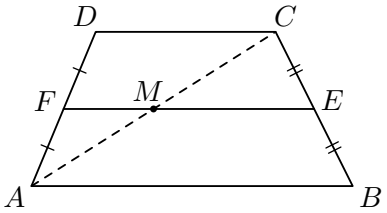
Theorem 2.1.1 (Midsegment)**.** *Let M and N be the midpoints of the sides AB and AC of triangle ABC. Then $MN \parallel BC$ and $MN = \frac{1}{2}BC$. Conversely, the line through M parallel to BC meets AC at its midpoint N.*



Proof. Since $\frac{AM}{AB} = \frac{AN}{AC} = \frac{1}{2}$ and the angle at A is common, $\triangle ABC \sim \triangle AMN$ by SAS, with ratio $\frac{1}{2}$. So $MN = \frac{1}{2}BC$ and $\angle AMN = \angle ABC$. These are corresponding angles for the lines MN and BC cut by AB, so $MN \parallel BC$. For the converse, if the line through M parallel to BC meets AC at N', then $\frac{AN'}{AC} = \frac{AM}{AB} = \frac{1}{2}$ by Theorem 1.2.2, so $N' = N$. □

A **trapezoid** is a quadrilateral with a pair of parallel sides, its **bases**; the other two sides are its **legs**. The segment joining the midpoints of the legs is the **midline** of the trapezoid.

Corollary 2.1.2 (Midline of a trapezoid)**.** *In trapezoid $ABCD$ with bases AB and DC, let F and E be the midpoints of the legs AD and BC. Then FE is parallel to the bases, $FE = \frac{1}{2}(AB+DC)$, and FE passes through the midpoints of both diagonals.*



Proof. Let M be the midpoint of the diagonal AC. By Theorem 2.1.1 in triangle ADC, $FM \parallel DC$ and $FM = \frac{1}{2}DC$; in triangle CAB, $ME \parallel AB$ and $ME = \frac{1}{2}AB$. The lines FM and ME both pass through M and are parallel to the bases, so they are the same line, and M lies between F and E. Hence $FE = FM + ME = \frac{1}{2}(AB + DC)$. The same argument with the diagonal BD shows that its midpoint also lies on FE. □

Example.

Problem 2.1.3 (1959 AHSME, Problem #22)**.** The line joining the midpoints of the diagonals of a trapezoid has length 3. If the longer base is 97, then the shorter base is:

(A) 94 **(B)** 92 **(C)** 91 **(D)** 90 **(E)** 89

Strategy. By Corollary 2.1.2, both midpoints lie on the midline of the trapezoid. Measure their distances from the midpoint of a leg.

Solution. Let the trapezoid be $ABCD$ with bases $AB = 97$ and $DC = x$, let F be the midpoint of the leg AD, and let M and N be the midpoints of the diagonals AC and BD. By Corollary 2.1.2, M and N lie on the midline through F. The segments FM and FN are midsegments of triangles ADC and DAB, so by Theorem 2.1.1

$$FM = \frac{x}{2} \qquad \text{and} \qquad FN = \frac{97}{2}.$$

Both points lie on the midline on the same side of F, so

$$MN = FN - FM = \frac{97 - x}{2}.$$

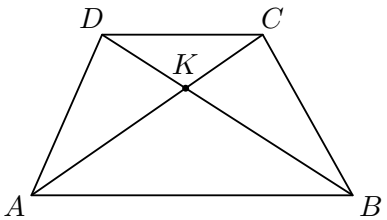
Setting this equal to 3 gives $97 - x = 6$, so $x = 91$.

Answer: (C) 91. □

2.2. **The Diagonals of a Trapezoid.**

Theorem 2.2.1 (Diagonals of a trapezoid)**.** *The diagonals AC and BD of trapezoid $ABCD$ with bases AB and DC meet at a point K with*

$$\frac{AK}{KC} = \frac{BK}{KD} = \frac{AB}{DC}.$$



Proof. Since $AB \parallel DC$, the alternate interior angles $\angle KAB$ and $\angle KCD$ are equal, and so are $\angle KBA$ and $\angle KDC$. By AA, $\triangle KAB \sim \triangle KCD$, so $\frac{AK}{CK} = \frac{BK}{DK} = \frac{AB}{CD}$. □

The theorem applies to any two parallel segments AB and DC joined crosswise by AC and BD, for example two segments on opposite sides of a rectangle or a square.

Examples.

Problem 2.2.2 (2008 AMC 10A, Problem #20)**.** Trapezoid $ABCD$ has bases AB and CD and diagonals intersecting at K. Suppose that $AB = 9$, $DC = 12$, and the area of $\triangle AKD$ is 24. What is the area of trapezoid $ABCD$?

(A) 92 **(B)** 94 **(C)** 96 **(D)** 98 **(E)** 100

Strategy. Theorem 2.2.1 gives the ratios in which K divides the diagonals. The equal-heights theorem then gives the areas of the other three triangles around K from $[AKD]$.

Solution. By Theorem 2.2.1, $AK : KC = BK : KD = AB : DC = 9 : 12 = 3 : 4$. The equal-heights theorem gives the areas of the other three triangles around K from $[AKD] = 24$:

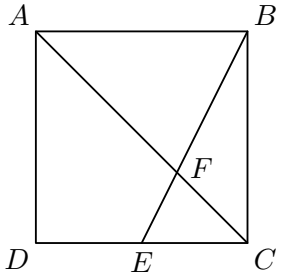
- triangles AKD and CKD have the common vertex D and bases AK and KC on line AC, so $[CKD] = \frac{4}{3} \cdot 24 = 32$;
- triangles AKD and AKB have the common vertex A and bases DK and KB on line BD, so $[AKB] = \frac{3}{4} \cdot 24 = 18$;
- triangles AKB and BKC have the common vertex B and bases AK and KC on line AC, so $[BKC] = \frac{4}{3} \cdot 18 = 24$.

The area of the trapezoid is $24 + 32 + 18 + 24 = 98$.

The two triangles next to the legs always have equal areas: $[ABD] = [ABC]$ by Theorem 1.2.1, since $DC \parallel AB$, and removing $[ABK]$ from both gives $[AKD] = [BKC]$.

Answer: (D) 98. □

Problem 2.2.3 (2018 AMC 8, Problem #22)**.** Point E is the midpoint of side $\overline{CD}$ in square $ABCD$, and $\overline{BE}$ meets diagonal $\overline{AC}$ at F. The area of quadrilateral $AFED$ is 45. What is the area of $ABCD$?



(A) 100 **(B)** 108 **(C)** 120 **(D)** 135 **(E)** 144

Strategy. The segments AB and EC are parallel and are joined crosswise by AC and BE, so Theorem 2.2.1 locates F on AC. Then write $[AFED]$ as a fraction of the square.

Solution. Let s be the side of the square, so its area is s^2. The quadrilateral $ABCE$ is a trapezoid with bases $AB = s$ and $EC = \frac{s}{2}$, and its diagonals AC and BE meet at F. By Theorem 2.2.1,

$$\frac{AF}{FC} = \frac{AB}{EC} = 2, \qquad \text{so} \qquad FC = \tfrac{1}{3}\, AC.$$

Triangles CFD and CAD have the common vertex D and their bases CF and CA on line AC, so $[CFD] = \frac{1}{3}[CAD] = \frac{1}{3} \cdot \frac{s^2}{2} = \frac{s^2}{6}$. The segment FE is a median of triangle CFD, and a median

divides a triangle into two triangles of equal area, so $[CFE] = \frac{1}{2}[CFD] = \frac{s^2}{12}$. The quadrilateral $AFED$ is triangle ACD without triangle CFE, so

$$[AFED] = \frac{s^2}{2} - \frac{s^2}{12} = \frac{5s^2}{12} = 45,$$

and $s^2 = 108$.

Answer: (B) 108. □

3. The Altitude to the Hypotenuse

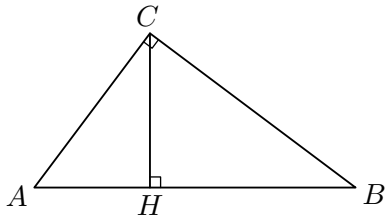
The altitude to the hypotenuse of a right triangle cuts it into two right triangles, and both are similar to the original one.

Theorem 3.1 (Altitude to the hypotenuse)**.** *Let ABC be a right triangle with the right angle at C, and let H be the foot of the altitude from C. Then*

$$\triangle ACB \sim \triangle AHC \sim \triangle CHB,$$

and consequently

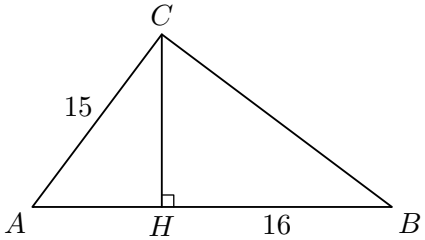
$$CH^2 = AH \cdot HB, \qquad AC^2 = AH \cdot AB, \qquad BC^2 = BH \cdot BA.$$



Proof. The angles at A and B are acute, so H lies on the side AB. Triangles ACB and AHC have right angles at C and H and share the angle at A, so they are similar by AA. In the same way, $\triangle ACB \sim \triangle CHB$, since these share the angle at B. From $\triangle AHC \sim \triangle ACB$, $\frac{AH}{AC} = \frac{AC}{AB}$, so $AC^2 = AH \cdot AB$. From $\triangle CHB \sim \triangle ACB$, $\frac{BH}{BC} = \frac{BC}{BA}$, so $BC^2 = BH \cdot BA$. Both AHC and CHB are scaled copies of ACB, so they are similar to each other, $\triangle AHC \sim \triangle CHB$, and $\frac{AH}{CH} = \frac{CH}{HB}$, that is, $CH^2 = AH \cdot HB$. □

Examples.

Problem 3.2 (1984 AHSME, Problem #17)**.** A right triangle ABC with hypotenuse AB has side $AC = 15$. Altitude CH divides AB into segments AH and HB, with $HB = 16$. The area of $\triangle ABC$ is



(A) 120 **(B)** 144 **(C)** 150 **(D)** 216 **(E)** $144\sqrt{5}$

Strategy. Theorem 3.1 relates the leg AC to the segment AH next to it: $AC^2 = AH \cdot AB$.

Solution. Let $x = AH$, so $AB = x + 16$. By Theorem 3.1, $AC^2 = AH \cdot AB$:

$$225 = x(x + 16), \qquad x^2 + 16x - 225 = 0, \qquad (x - 9)(x + 25) = 0.$$

Since $x > 0$, $AH = 9$ and $AB = 25$. By Theorem 3.1 again, $CH^2 = AH \cdot HB = 9 \cdot 16 = 144$, so $CH = 12$, and

$$[ABC] = \tfrac{1}{2} \cdot AB \cdot CH = \tfrac{1}{2} \cdot 25 \cdot 12 = 150.$$

As a check, $BC^2 = BH \cdot BA = 16 \cdot 25 = 400$, so $BC = 20$, and $\tfrac{1}{2} \cdot AC \cdot BC = \tfrac{1}{2} \cdot 15 \cdot 20 = 150$.

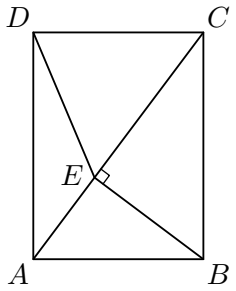
Answer: (C) 150. □

Problem 3.3 (2017 AMC 10B, Problem #15)**.** Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. What is the area of $\triangle ADE$?

(A) 1 **(B)** $\frac{42}{25}$ **(C)** $\frac{28}{15}$ **(D)** 2 **(E)** $\frac{54}{25}$

Strategy. In the right triangle ABC, the segment BE is the altitude to the hypotenuse AC, which gives AE. Triangles ADE and ADC have the same height from D.

Solution. Triangle ABC has a right angle at B and hypotenuse $AC = \sqrt{3^2 + 4^2} = 5$, and BE is its altitude to the hypotenuse, as in the figure.



By Theorem 3.1, $AB^2 = AE \cdot AC$, so

$$AE = \frac{AB^2}{AC} = \frac{9}{5}.$$

Triangles ADE and ADC have the common vertex D and their bases AE and AC on line AC, so by the equal-heights theorem

$$[ADE] = \frac{AE}{AC} \cdot [ADC] = \frac{9}{25} \cdot \frac{1}{2} \cdot 3 \cdot 4 = \frac{54}{25}.$$

Answer: (E) $\frac{54}{25}$. □

4. Areas of Similar Figures

When all lengths are multiplied by k, areas are multiplied by k^2. This section proves this for triangles and for convex polygons.

4.1. **Similar Triangles.**

Theorem 4.1.1 (Areas of similar triangles)**.** *If $\triangle ABC \sim \triangle DEF$ with ratio k, then each altitude of DEF is k times the corresponding altitude of ABC, and $[DEF] = k^2\,[ABC]$.*

Proof. Let AH and DH' be the altitudes from A and D. If the angle at B is right, then $H = B$, $H' = E$ and $DH' = DE = k \cdot AB = k \cdot AH$. Otherwise the right triangles ABH and DEH' have equal acute angles at B and E: these are $\angle B$ and $\angle E$ themselves, or their supplements when the foot lies on the extension of the side beyond B. So $\triangle ABH \sim \triangle DEH'$ by AA, and $DH' = k \cdot AH$ because $DE = k \cdot AB$. The same holds for the other two altitudes. Hence

$$[DEF] = \tfrac{1}{2} \cdot EF \cdot DH' = \tfrac{1}{2}\,(k \cdot BC)(k \cdot AH) = k^2\,[ABC]. \qquad \square$$

Conversely, if two similar triangles have areas in the ratio r, their ratio of similarity is $\sqrt{r}$.

Example.

Problem 4.1.2 (1953 AHSME, Problem #26)**.** The base of a triangle is 15 inches. Two lines are drawn parallel to the base, terminating in the other two sides, and dividing the triangle into three equal areas. The length of the parallel closer to the base is:

(A) $5\sqrt{6}$ inches **(B)** 10 inches **(C)** $4\sqrt{3}$ inches **(D)** 7.5 inches **(E)** none of these

Strategy. The parallel closer to the base cuts off, at the top vertex, a triangle similar to the whole triangle with two thirds of its area.

Solution. Let the triangle be ABC with base $BC = 15$, and let the parallel closer to the base meet AB and AC at D and E. Above DE lie two of the three equal parts, so $[ADE] = \frac{2}{3}[ABC]$. By Theorem 1.2.2, $\triangle ABC \sim \triangle ADE$ with some ratio k, so $DE = k \cdot BC$. By Theorem 4.1.1, $[ADE] = k^2[ABC]$, so $k^2 = \frac{2}{3}$ and

$$DE = k \cdot BC = 15\sqrt{\tfrac{2}{3}} = \frac{15\sqrt{6}}{3} = 5\sqrt{6}.$$

In the same way, the other parallel cuts off a triangle with one third of the area, and its length is $15\sqrt{\frac{1}{3}} = 5\sqrt{3}$.

Answer: (A) $5\sqrt{6}$ inches. $\square$

4.2. **Similar Polygons.**

Definition 4.2.1. Two polygons are **similar** if their vertices can be matched in order so that corresponding angles are equal and each side of the second polygon is k times the corresponding side of the first, for one number k, the **ratio of similarity**.

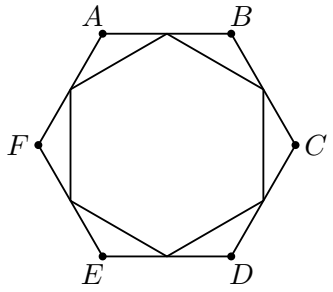
For example, two regular polygons with the same number of sides are similar: their angles are all equal, and every pair of corresponding sides has the ratio of the two side lengths.

Theorem 4.2.2 (Areas of similar polygons)**.** *If two convex polygons are similar with ratio k, then each diagonal of the second is k times the corresponding diagonal of the first, and the area of the second is k^2 times the area of the first.*

Proof. The diagonals from one vertex cut the first polygon into triangles, and the corresponding diagonals cut the second polygon into corresponding triangles. Take the triangles in order, starting with one that has two sides of the polygon. Corresponding triangles have two sides in the ratio k and equal angles between them, so they are similar by SAS, and the diagonal that each shares with the next triangle is in the ratio k. At the other end of that diagonal, the angle of the next triangle is the angle of the polygon there minus the angle of the triangle before it, a difference of two corresponding angles, so the next pair of triangles is again similar by SAS, and so on. Hence every diagonal from the chosen vertex is in the ratio k, and since any vertex can be chosen, every diagonal is. Each pair of triangles has areas in the ratio k^2 by Theorem 4.1.1, and adding gives the same ratio for the polygons. $\square$

Example.

Problem 4.2.3 (1996 AHSME, Problem #19)**.** The midpoints of the sides of a regular hexagon $ABCDEF$ are joined to form a smaller hexagon. What fraction of the area of $ABCDEF$ is enclosed by the smaller hexagon?



(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{3}}{3}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{\sqrt{3}}{2}$

Strategy. Show that the smaller hexagon is also regular, so that the two hexagons are similar, and find the ratio of their sides with a midsegment.

Solution. Let s be the side of $ABCDEF$, and let M and N be the midpoints of AB and BC. In triangle ABC, the segment MN is a midsegment, so $MN = \frac{1}{2}AC$ by Theorem 2.1.1. Triangle ABC is isosceles with $BA = BC = s$ and $\angle ABC = 120°$. The altitude from B bisects AC and the angle at B, and the right triangle it cuts off has a $30°$ angle at A, so the altitude is $\frac{s}{2}$ and, by the Pythagorean theorem,

$$\frac{1}{2}AC = \sqrt{s^2 - \frac{s^2}{4}} = \frac{\sqrt{3}}{2}\,s.$$

Hence $MN = \frac{\sqrt{3}}{2}s$, and in the same way every side of the smaller hexagon has length $\frac{\sqrt{3}}{2}s$.

The angles of the smaller hexagon are equal as well. Triangle MBN is isosceles with vertex angle $120°$, so $\angle BNM = 30°$; in the same way the next side of the smaller hexagon makes a $30°$ angle with NC. So the angle of the smaller hexagon at N is $180° - 30° - 30° = 120°$, and the same holds at every vertex. The smaller hexagon is regular, so $ABCDEF$ and the smaller hexagon are similar with ratio $\frac{\sqrt{3}}{2}$, and by Theorem 4.2.2 the fraction is

$$\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}.$$

Answer: (D) $\frac{3}{4}$. □

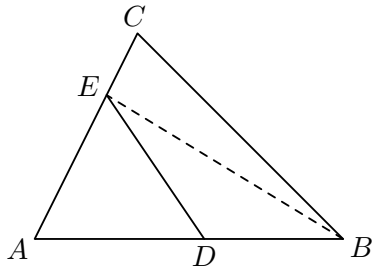
5. Area Ratios in a Triangle

Segments from the vertices and lines parallel to the sides cut a triangle into pieces whose areas are related by simple ratios. This section gives a formula for two triangles that share an angle and two ways to use parallel lines.

5.1. **Triangles with a Common Angle.**

Theorem 5.1.1 (Common angle)**.** *If D lies on ray AB and E lies on ray AC, then*

$$\frac{[ADE]}{[ABC]} = \frac{AD}{AB} \cdot \frac{AE}{AC}.$$



Proof. Triangles ADE and ABE have the common vertex E and their bases AD and AB on one line, so $\frac{[ADE]}{[ABE]} = \frac{AD}{AB}$ by the equal-heights theorem. Triangles ABE and ABC have the common vertex B and their bases AE and AC on one line, so $\frac{[ABE]}{[ABC]} = \frac{AE}{AC}$. Multiplying the two equations gives the formula. □

When $DE \parallel BC$, both factors equal the ratio of similarity k of Theorem 1.2.2, and the formula gives k^2, as in Theorem 4.1.1.

Example.

Problem 5.1.2 (2005 AMC 10A, Problem #25)**.** In $\triangle ABC$ we have $AB = 25$, $BC = 39$, and $AC = 42$. Points D and E are on AB and AC respectively, with $AD = 19$ and $AE = 14$. What is the ratio of the area of triangle ADE to the area of the quadrilateral $BCED$?

(A) $\frac{266}{1521}$ **(B)** $\frac{19}{75}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{19}{56}$ **(E)** 1

Strategy. Triangles ADE and ABC share the angle at A, so Theorem 5.1.1 gives $[ADE]$ as a fraction of $[ABC]$. The quadrilateral is the rest of the triangle.

Solution. By Theorem 5.1.1,

$$\frac{[ADE]}{[ABC]} = \frac{AD}{AB} \cdot \frac{AE}{AC} = \frac{19}{25} \cdot \frac{14}{42} = \frac{19}{75}.$$

The quadrilateral $BCED$ is triangle ABC without triangle ADE, so $[BCED] = \frac{56}{75}[ABC]$, and

$$\frac{[ADE]}{[BCED]} = \frac{19}{56}.$$

The length $BC = 39$ is not needed.

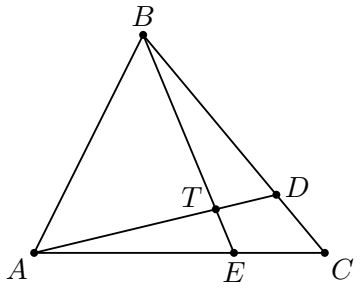
Answer: (D) $\frac{19}{56}$. □

5.2. **Parallels and Cevians.** A line parallel to a side of a triangle cuts off a similar triangle, and Theorem 1.2.2 turns it into ratios of lengths. To find the ratio in which two cevians divide each other or the opposite side, draw a line through a point of the figure parallel to one of the cevians. The new similar triangles connect the given ratios to the unknown one.

Lines through an interior point parallel to the three sides cut off three small triangles, each similar to the whole triangle. The second example shows how their ratios of similarity are related.

Examples.

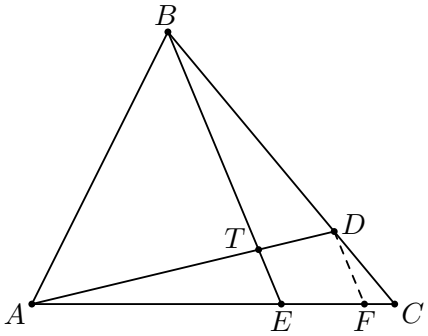
Problem 5.2.1 (2004 AMC 10B, Problem #20)**.** In $\triangle ABC$ points D and E lie on $\overline{BC}$ and $\overline{AC}$, respectively. If $\overline{AD}$ and $\overline{BE}$ intersect at T so that $\frac{AT}{DT} = 3$ and $\frac{BT}{ET} = 4$, what is $\frac{CD}{BD}$?



(A) $\frac{1}{8}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{3}{10}$ **(D)** $\frac{4}{11}$ **(E)** $\frac{5}{12}$

Strategy. Draw the line through D parallel to BE. It creates two triangles cut by a parallel line: one at the vertex A, which uses the ratio $AT : TD$, and one at the vertex C, which contains CD and CB.

Solution. Let the line through D parallel to BE meet AC at F, as in the figure, and let $TE = t$. Then $BT = 4t$ and $BE = 5t$.



In triangle ADF, the points T and E lie on the sides AD and AF, and $TE \parallel DF$. By Theorem 1.2.2,

$$\frac{TE}{DF} = \frac{AT}{AD} = \frac{3}{4}, \qquad \text{so} \qquad DF = \frac{4}{3}\,t.$$

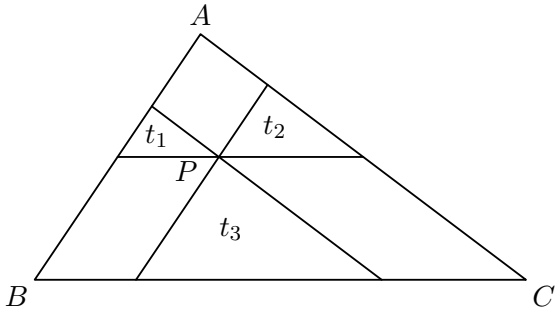
In triangle CBE, the points D and F lie on the sides CB and CE, and $DF \parallel BE$. By Theorem 1.2.2 again,

$$\frac{CD}{CB} = \frac{DF}{BE} = \frac{4t/3}{5t} = \frac{4}{15}.$$

So $CD : DB = 4 : 11$, and $\frac{CD}{BD} = \frac{4}{11}$.

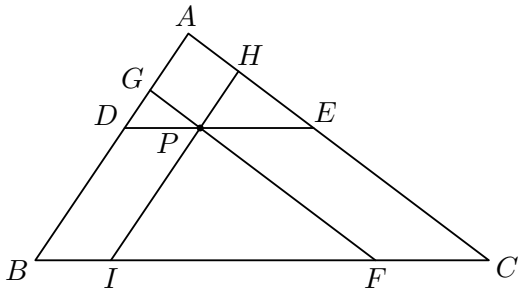
Answer: (D) $\frac{4}{11}$. □

Problem 5.2.2 (1984 AIME, Problem #3)**.** A point P is chosen in the interior of $\triangle ABC$ so that when lines are drawn through P parallel to the sides of $\triangle ABC$, the resulting smaller triangles, t_1, t_2, and t_3 in the figure, have areas 4, 9, and 49, respectively. Find the area of $\triangle ABC$.



Strategy. Each small triangle is similar to ABC. Follow one side of ABC to show that their ratios of similarity add up to 1, and then use Theorem 4.1.1.

Solution. Name the points where the three lines meet the sides, as in the figure below: the line through P parallel to BC meets AB at D and AC at E; the line through P parallel to CA meets AB at G and BC at F; the line through P parallel to AB meets BC at I and CA at H.



The small triangles are $t_1 = PDG$, $t_2 = PEH$ and $t_3 = PIF$. The sides of each are parallel to the sides of ABC, so its angles equal those of ABC (corresponding angles), and it is similar to ABC by AA. Let k_1, k_2 and k_3 be the ratios of the sides of t_1, t_2 and t_3 to the corresponding sides of ABC.

Follow the side BC from B to C: it consists of BI, IF and FC. The quadrilateral $BIPD$ has both pairs of opposite sides parallel, so it is a parallelogram and $BI = DP$; in the same way $FCEP$ is a parallelogram and $FC = PE$. In the three small triangles, DP, PE and IF are the sides parallel to BC, so $DP = k_1\,BC$, $PE = k_2\,BC$ and $IF = k_3\,BC$. Hence

$$BC = BI + IF + FC = DP + PE + IF = (k_1 + k_2 + k_3)\,BC, \qquad \text{so} \qquad k_1 + k_2 + k_3 = 1.$$

Let $K = [ABC]$. By Theorem 4.1.1 the small triangles have areas k_1^2K, k_2^2K and k_3^2K, so the square roots of their areas are $k_1\sqrt{K}$, $k_2\sqrt{K}$ and $k_3\sqrt{K}$, which add up to $\sqrt{K}$. Therefore

$$\sqrt{K} = \sqrt{4} + \sqrt{9} + \sqrt{49} = 2 + 3 + 7 = 12, \qquad K = 144.$$

The sum, and so the answer, does not depend on which small triangle has which area.

Answer: 144. □

6. Problems

Beginner.

1. (1951 AHSME, Problem #29) Of the following sets of data, the only one that does not determine the shape of a triangle is:

(A) The ratio of two sides and the included angle
(B) The ratios of the three altitudes **(C)** The ratios of the three medians
(D) The ratio of the altitude to the corresponding base **(E)** Two angles

2. (1965 AHSME, Problem #12) A rhombus is inscribed in triangle ABC in such a way that one of its vertices is A and two of its sides lie along AB and AC. If $AC = 6$ inches, $AB = 12$ inches, and $BC = 8$ inches, the side of the rhombus, in inches, is:

(A) 2 **(B)** 3 **(C)** $3\frac{1}{2}$ **(D)** 4 **(E)** 5

3. (1972 AHSME, Problem #18) Let $ABCD$ be a trapezoid with the measure of base AB twice that of base DC, and let E be the point of intersection of the diagonals. If the measure of diagonal AC is 11, then that of segment EC is equal to:

(A) $3\frac{2}{3}$ **(B)** $3\frac{3}{4}$ **(C)** 4 **(D)** $3\frac{1}{2}$ **(E)** 3

4. (1958 AHSME, Problem #19) The sides of a right triangle are a and b and the hypotenuse is c. A perpendicular from the vertex divides c into segments r and s, adjacent respectively to a and b. If $a : b = 1 : 3$, then the ratio of r to s is:

(A) $1 : 3$ **(B)** $1 : 9$ **(C)** $1 : 10$ **(D)** $3 : 10$ **(E)** $1 : \sqrt{10}$

5. (1962 AHSME, Problem #16) Given rectangle R_1 with one side 2 inches and area 12 square inches. Rectangle R_2 with diagonal 15 inches is similar to R_1. Expressed in square inches the area of R_2 is:

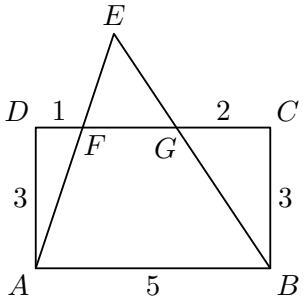
(A) $\frac{9}{2}$ **(B)** 36 **(C)** $\frac{135}{2}$ **(D)** $9\sqrt{10}$ **(E)** $\frac{27\sqrt{10}}{4}$

Intermediate.

6. (1971 AHSME, Problem #28) Nine lines parallel to the base of a triangle divide the other sides each into 10 equal segments and the area into 10 distinct parts. If the area of the largest of these parts is 38, then the area of the original triangle is:

(A) 180 **(B)** 190 **(C)** 200 **(D)** 210 **(E)** 240

7. (2003 AMC 10B, Problem #20) In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on $\overline{CD}$ so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. Find the area of $\triangle AEB$.



(A) 10 **(B)** $\frac{21}{2}$ **(C)** 12 **(D)** $\frac{25}{2}$ **(E)** 15

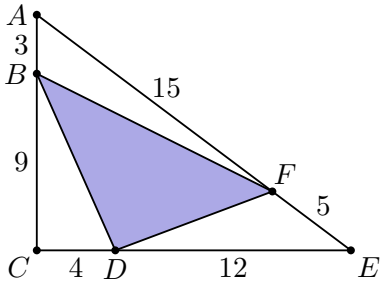
8. (2016 AMC 10A, Problem #19) In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C, and point F between E and C are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q, respectively. The ratio $BP : PQ : QD$ can be written as $r : s : t$ where the greatest common factor of r, s, and t is 1. What is $r + s + t$?

(A) 7 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 20

9. (2005 AMC 10B, Problem #23) In trapezoid $ABCD$ we have $\overline{AB}$ parallel to $\overline{DC}$, E as the midpoint of $\overline{BC}$, and F as the midpoint of $\overline{DA}$. The area of $ABEF$ is twice the area of $FECD$. What is $\frac{AB}{DC}$?

(A) 2 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 8

10. (2004 AMC 10B, Problem #18) In right triangle $\triangle ACE$, we have $AC = 12$, $CE = 16$, and $EA = 20$. Points B, D, and F are located on AC, CE, and EA, respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the ratio of the area of $\triangle BDF$ to that of $\triangle ACE$?



(A) $\frac{1}{4}$ **(B)** $\frac{9}{25}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{11}{25}$ **(E)** $\frac{7}{16}$

Advanced.

11. (2009 AMC 10A, Problem #17) Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. Segment EF is constructed through B so that $EF \perp DB$, and A and C lie on DE and DF, respectively. What is EF?

(A) 9 **(B)** 10 **(C)** $\frac{125}{12}$ **(D)** $\frac{103}{9}$ **(E)** 12

12. (1966 AHSME, Problem #32) Let M be the midpoint of side AB of triangle ABC. Let P be a point on AB between A and M, and let MD be drawn parallel to PC and intersecting BC at D. If the ratio of the area of triangle BPD to that of triangle ABC is denoted by r, then:

(A) $\frac{1}{2} < r < 1$ depending upon the position of P

(B) $r = \frac{1}{2}$ independent of the position of P

(C) $\frac{1}{2} \leq r < 1$ depending upon the position of P

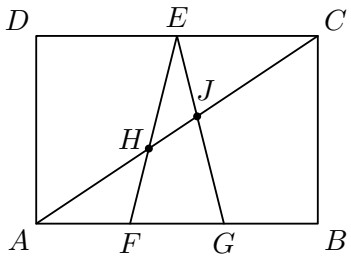
(D) $\frac{1}{3} < r < \frac{2}{3}$ depending upon the position of P

(E) $r = \frac{1}{3}$ independent of the position of P

13. (2009 AMC 10A, Problem #23) Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals AC and BD intersect at E, $AC = 14$, and $\triangle AED$ and $\triangle BEC$ have equal areas. What is AE?

(A) $\frac{9}{2}$ **(B)** $\frac{50}{11}$ **(C)** $\frac{21}{4}$ **(D)** $\frac{17}{3}$ **(E)** 6

14. (2001 AMC 12, Problem #22) In rectangle $ABCD$, points F and G lie on $\overline{AB}$ so that $AF = FG = GB$ and E is the midpoint of $\overline{DC}$. Also, $\overline{AC}$ intersects $\overline{EF}$ at H and $\overline{EG}$ at J. The area of rectangle $ABCD$ is 70. Find the area of triangle EHJ.



(A) $\frac{5}{2}$ **(B)** $\frac{35}{12}$ **(C)** 3 **(D)** $\frac{7}{2}$ **(E)** $\frac{35}{8}$

15. (2021 AMC 10A, Problem #17) Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. Given that $OP = 11$, the length of AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

(A) 65 **(B)** 132 **(C)** 157 **(D)** 194 **(E)** 215